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Volterra operators and semigroups in weighted Banach spaces of analytic functions

Manuela Basallote · Manuel D. Contreras · Carmen Hernández-Mancera · María J. Martín · Pedro J. Paúl

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Abstract We characterize the boundedness, compactness and weak compactness of Volterra operators $V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)\,d\zeta$ acting between different weighted spaces of type H_v^∞ in terms of the symbol function g, for the case when v is a quasi-normal weight, a notion weaker than normality. Then we apply the characterization of compactness to analyze the behavior of semigroups of composition operators on H_v^∞.

Keywords Integral operator · Volterra operator · Cesàro operator · Boundedness · Compactness · Semigroups of analytic functions · Weighted spaces of analytic functions

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M. Basallote · M. D. Contreras · C. Hernández-Mancera (✉) · P. J. Paúl
Departamento de Matemática Aplicada II, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Seville, Spain
e-mail: cmancera@us.es

M. Basallote
e-mail: mabas@esi.us.es

M. D. Contreras
e-mail: contreras@us.es
URL: http://personal.us.es/contreras

P. J. Paúl
e-mail: piti@us.es

M. J. Martín
Departamento de Matemáticas (Módulo 17, Edificio de Ciencias), Universidad Autónoma de Madrid, 28049 Madrid, Spain
e-mail: mariaj.martin@uam.es
URL: http://www.uam.es/mariaj.martin

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1 Introduction

Given a function g in the space $H(\mathbb{D})$ of complex analytic functions in the unit disk, the familiar term in the integration by part formula

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)\,d\zeta \qquad (z \in \mathbb{D}) \tag{1}$$

defines a linear operator V_g on $H(\mathbb{D})$ called the *Volterra operator with symbol* g. For $g(z) = z$ we have that V_g is the integration operator, and for $g(z) = \log(1/(1-z))$ we obtain the Cesàro operator.

The Volterra operator V_g was introduced by Pommerenke in [31] to study exponentials of BMOA functions; he proved that V_g is bounded on the Hardy space H^2 if, and only if, g is a BMOA function. This important result has motivated a number of interesting characterizations of the boundedness and compactness of V_g acting between different types of spaces of analytic functions. To mention only a few, Aleman and Siskakis [5] extended this result to the Hardy spaces H^p $(1 \le p < \infty)$ and proved that V_g is compact on H^p if, and only if, g is in the VMOA class. Analogous results on some general weighted Bergman spaces were given by these authors in [6] and also by Pau and Peláez [29]. Other relevant papers are [1,4,21,24,25,40,41], and [43]. Similar results in higher dimensions have been also given by Stević (see [38] and references therein). The reader is referred to the nice survey about the origins of Volterra operator, its relevance, and connections with other areas of mathematics, written by Aleman [3].

The first objective of this paper is to study the boundedness and compactness of the Volterra operator V_g acting between weighted Banach spaces of analytic functions H_v^∞ in terms of the symbol g and the involved weights.

Let us recall at this point that a *weight* v is a non-negative continuous function in $\mathbb{D}$ that depends only on the radius $r = |z|$ and is decreasing. The *weighted Banach spaces* H_v^∞ *and* H_v^0 are defined by

$$H_v^\infty := \left\{ f \in H(\mathbb{D}) \colon \|f\|_{H_v^\infty} := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\} \tag{2}$$

and

$$H_v^0 := \left\{ f \in H_v^\infty \colon \lim_{|z|\to 1} v(z)|f(z)| = 0 \right\}. \tag{3}$$

These spaces, which are natural spaces in the sense that norm convergence implies uniform convergence on compact subsets of $\mathbb{D}$, are also known in the literature as *growth spaces* and are a particular case of mixed norm spaces; see Sect. 2 for some historical remarks and the properties we shall use in this paper.

Closely related to H_v^∞ are the *Bloch-type spaces* $\mathcal{B}_v^\infty$. A function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}_v^\infty$ whenever $f' \in H_v^\infty$. Bloch-type spaces are Banach spaces of analytic functions endowed with the norm

$$\|f\|_{\mathcal{B}_v^\infty} := |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)|. \tag{4}$$

An analytic function f belongs to the *little Bloch space* $\mathcal{B}_v^0$ if $f \in \mathcal{B}_v$ and

$$\lim_{|z|\to 1} v(z)|f'(z)| = 0.$$

When the weight v is $v_\alpha(|z|) = (1-|z|^2)^\alpha$, $\alpha > 0$, the spaces $H_{v_\alpha}^\infty$ and $\mathcal{B}_{v_\alpha}^\infty$ are the *standard weighted Hardy spaces* and *standard Bloch spaces*, respectively, and are usually denoted by H_α^∞ and $\mathcal{B}_\alpha^\infty$. In particular, $\mathcal{B}_1^\infty = \mathcal{B}$ is the classical Bloch space and $\mathcal{B}_1^0 = \mathcal{B}_0$ is the little Bloch space.

A key point in our study of the boundedness and compactness of the Volterra operator acting between two spaces of type H_v^∞ is the relationship between the growth of a function and the growth of its derivative. It was proved by Hardy and Littlewood in 1932 (see [22, Theorem 39] or [20, p. 80]) that, in some cases, the growth of an analytic function in the unit disk $\mathbb{D}$ determines, and is determined by, the growth of its derivative. Namely, for all $\beta > 0$ we have $H_\beta^\infty = \mathcal{B}_{\beta+1}^\infty$. This was extended by Lusky [27], who proved that if the weight v is normal (a well-known class of weights introduced by Shields and Williams [33,34]), then $H_{v(r)}^0 = \mathcal{B}_{(1-r^2)v(r)}^0$ and, by duality, $H_{v(r)}^\infty = \mathcal{B}_{(1-r^2)v(r)}^\infty$. Inspired by this result, we say that a weight v is *quasi-normal* if $H_{v(r)}^0 = \mathcal{B}_{(1-r^2)v(r)}^0$.

Sections 3 and 4 are devoted to our first objective: the study the boundedness and compactness of the Volterra operator $V_g : H_{v_1}^\infty \to H_{v_2}^\infty$ in terms of its symbol g for the case when v_2 is a quasi-normal weight. Our main result in these two sections is Theorem 2 where we characterize the compactness and weak compactness of V_g.

It turns out that this characterization is useful, via the connection between Volterra operators and semigroups of composition operators [10], to study the maximal subspace of a semigroup of composition operators in the weighted spaces H_v^∞; this is done in Sect. 5. Recall that a semigroup of analytic functions $(\varphi_t)_{t\geq 0}$ mapping $\mathbb{D}$ into itself generates a semigroup of composition operators on a Banach space of analytic functions X when the composition operators $C_t(f(z)) := f(\varphi_t(z))$ form a semigroup of bounded operators in X. This semigroup is said to be *strongly continuous* if for all $f \in X$, we have

$$\lim_{t\to 0^+} \|C_t(f) - f\|_X = 0.$$

When the semigroup is not strongly continuous on X, one looks for the maximal closed subspace of X, denoted by $[(\varphi_t), X]$, on which (φ_t) generates a strongly continuous semigroup of composition operators. The existence of such a maximal subspace, as well as analytical descriptions of it, was obtained in [10]. In Theorem 3 of Sect. 5 we prove that if the functions of the semigroup fix a point in the unit disk, then $[(\varphi_t), H_v^\infty]$ always contains the little space H_v^0 and, when v is quasi-normal, it never coincides with the big space H_v^∞; that is, the semigroup of operators is never strongly continuous on H_v^∞. Our final result about semigroups, Corollary 5, characterizes when the maximal subspace $[(\varphi_t), H_v^\infty]$ coincides with H_v^0 in terms of the infinitesimal generator of the semigroup.

The paper finishes with a section devoted to offer an understanding of the notion of quasi-normal weight, analyzing when $H_v^0 \subseteq \mathcal{B}_{(1-r^2)v(r)}^0$ and $\mathcal{B}_{(1-r^2)v(r)}^0 \subseteq H_v^0$ in terms of intrinsic properties of the weight v.

2 Preliminaries: weighted Banach spaces of analytic functions

In this section we review some of the properties of the weighted Banach spaces H_v^∞ and H_v^0, defined by (2) and (3) above, that we will use in this paper. To the best of our knowledge, these spaces were first studied by Rubel and Shields [32]. A good reference for their properties is [8]. Of course, many of these properties depend on the weight, so we start by recalling different types of weights which are often considered in the literature.

The weight is called *typical* if $\lim_{|z|\to 1} v(z) = 0$. If v is typical, then $(H_v^0)^{**} = H_v^\infty$ and the polynomials are dense in H_v^0. The case when $\lim\sup_{|z|\to 1} v(z) > 0$ is usually excluded because we have that H_v^∞ is isomorphic to H^∞ and $H_v^0 = \{0\}$.

A weight v is said to be *analytic* if $v(z) = 1/f(|z|)$ for some $f \in H(\mathbb{D})$ that takes real values on $[0, 1)$ and is such that $|f(z)| \leq f(|z|)$ in the unit disk.

Many results on weighted spaces must be formulated in terms of the *associated weight*

$$\widetilde{v}(z) := \frac{1}{\sup\{|f(z)|\colon\ f \in H_v^\infty,\ \|f\|_{H_v^\infty} \leq 1\}} \tag{5}$$

the supremum being, in fact, a maximum. The associated $\widetilde{v}$ is also a weight, satisfies $v(z) \leq \tilde{v}(z)$ for all $z \in \mathbb{D}$, and has the key property that if we take $\tilde{v}$ instead of v, neither the spaces H_v^∞ and H_v^0 nor the norm $\|\cdot\|_{H_v^\infty}$ change. We will use that if v is typical, then $\widetilde{v}$ is also typical and we have

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)|\colon\ f \in H_v^0,\ \|f\|_{H_v^\infty} \leq 1\}}.$$

A weight v is called *essential* if there exists a constant $C \geq 1$ such that

$$v(z) \leq \widetilde{v}(z) \leq Cv(z) \qquad \text{for all } z \in \mathbb{D}.$$

It is well-known that if v is analytic, then $v = \tilde{v}$ and, in particular, it is essential; this property allows us to give plenty of examples of essential weights.

Example 1 (a) Take $f_\alpha(z) = (1 - z^2)^{-\alpha}$ $(0 < \alpha < \infty)$, then the corresponding analytic weights $v_\alpha(z) = (1 - |z|^2)^\alpha$ are sometimes called *standard weights*.

(b) Take $\beta > 0$ and $f(z) = \exp\{1/(1-z^2)^\beta\}$ to obtain the weights $v_{\exp,\beta}(z) = \exp\{-1/(1-|z|^2)^\beta\}$.

(c) Finally, the analytic functions $f(z) = [1 - \log(1 - z^2)]^{-\gamma}$, with $\gamma < 0$, produce the essential weights $v_{\log,\gamma}(z) = [1 - \log(1 - |z|^2)]^\gamma$.

As we mentioned in the Introduction, a fundamental tool in our study is the result due to Hardy and Littlewood (see [22, Theorem 39] or [20, p. 80]) on the relationship between the growth of a function and the growth of its derivative in the unit disk $\mathbb{D}$. Namely, if $\beta > 0$, then the hypotheses

$$f(z) = O\left(\frac{1}{(1 - |z|^2)^\beta}\right)$$

and

$$f'(z) = O\left(\frac{1}{(1 - |z|^2)^{\beta+1}}\right)$$

are equivalent or, in terms of standard weighted Hardy and Bloch spaces, $H_\beta^\infty = \mathcal{B}_{\beta+1}^\infty$. This was generalized by Lusky [27]; to state his result we need the next definition.

Definition 1 Following Shields and Williams [34], we say that

(a) the weight v satisfies *property (U)* if there exists a positive number α such that the function $r \to v(r)/(1 - r)^\alpha$ is almost increasing;

(b) the weight v satisfies *property (L)* if there exists a positive number β such that the function $r \to v(r)/(1-r)^\beta$ is almost decreasing;
(c) the weight v is *normal* if it satisfies both properties (U) and (L).

In [18, Lemma 1], Domański and Lindström proved that a weight v satisfies property (U) if, and only if, $\inf_n \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$, and that v satisfies property (L) if, and only if, there exists a natural number k such that $\limsup_n \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1$.

Using these equivalences for the weights introduced in Example 1, one can easily deduce that any weight v_α is normal; that any weight $v_{\log,\gamma}$ satisfies property (U) but it is never normal; and that any weight $v_{\exp,\beta}$ satisfies property (L) but it is never normal. Other normal weights are, for instance, $v_{\alpha,\log,\gamma}(r) := (1-r^2)^\alpha[1-\log(1-r^2)]^\gamma$ (where $\alpha > 0$ and $\gamma < 0$) and $v_{\log\log,\gamma}(r) := \min\{1, \log^\gamma(1-\log(r))\}$ where $\gamma > 0$.

There is a technical characterization of weights with property (L) given by Shields and Williams [34, Lemma 2] that will be used three times in what follows; namely that v has property (L) if, and only if,

$$\sup_{0<r<1}\left(v(r)\int_0^r \frac{ds}{v(s)(1-s^2)}\right) < +\infty.$$

(We must warn the reader that Shields and Williams use weights $\psi(x)$ defined in the positive real line that they translate into a weight in the unit disk via the change of variable $r = (1-1/x)^{-1}$; one must perform this change in order to obtain the condition as it is written above.)

Lusky's extension of Hardy and Littlewood's result mentioned above can be rewritten, using these equivalences, as follows [27, Theorem 3.1].

Theorem A *Assume that the weight v has property (U). Then v has property (L) if, and only if, $H^0_{v(r)} = \mathcal{B}^0_{(1-r^2)v(r)}$. In particular, if v is a normal weight, then $H^0_{v(r)} = \mathcal{B}^0_{(1-r^2)v(r)}$ and, by duality, $H^\infty_{v(r)} = \mathcal{B}^\infty_{(1-r^2)v(r)}$.*

3 Boundedness of the Volterra operators

In this section, we study the boundedness of the Volterra operators $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ and $V_g : H^0_{v_1} \to H^0_{v_2}$. Some of our results in this section extend previous results obtained by Hu [24] for the case when V_g is defined from H^∞_v into itself and v is a normal weight. Lusky's Theorem A tells us that if v is normal then $H^\infty_{v(r)} = \mathcal{B}^\infty_{(1-r^2)v(r)}$ and this equality is an important technical tool in this context due to the following simple observation: Denote by $M_{g'}$ the multiplication operator defined by $M_{g'}(f)(z) := f(z)g'(z)$, then $V_g(f) \in H^\infty_{v(r)} = B^\infty_{(1-r^2)v(r)}$ if, and only if, $M_{g'}(f) \in H^\infty_{(1-r^2)v(r)}$ and both elements have comparable norms. This motivates our following definition, which will be widely used along the paper.

Definition 2 We say that a weight v is *quasi-normal* if $H^0_{v(r)} = \mathcal{B}^0_{(1-r^2)v(r)}$. Note that any quasi-normal weight is typical and that for quasi-normal weights one has $H^\infty_{v(r)} = \mathcal{B}^\infty_{(1-r^2)v(r)}$.

This is a rather ad-hoc technical definition, of course, and it would be nice to have a characterization of quasi-normal weights in terms of the properties of the weight as a function. We shall devote the final section of our paper to this question.

The following lemma will be used in the proof of Theorem 1 below. Although the arguments are straightforward, we include the proof for the sake of completeness.

Lemma 1 *Let v_1 and v_2 be typical weights such that $V_g : H^0_{v_1} \to H^0_{v_2}$ is bounded. Then $V_g^{**} = V_g$ and, therefore, $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is bounded.*

Proof If $V_g : H^0_{v_1} \to H^0_{v_2}$ is bounded, then $V_g^* : (H^0_{v_2})^* \to (H^0_{v_1})^*$ and $V_g^{**} : H^\infty_{v_1} \to H^\infty_{v_2}$ are bounded as well.

Consider the elements $\delta_z \in \left(H^0_{v_2}\right)^*$ defined by $\delta_z(f) := f(z)$. The span of such functions is dense on $\left(H^0_{v_2}\right)^*$ and for all $f \in H^0_{v_1}$ we have that

$$\langle V_g^*(\delta_z), f\rangle = \langle \delta_z, V_g(f)\rangle = \int_0^z f(\zeta)g'(\zeta)\, d\zeta.$$

Now, for $f \in H^\infty_{v_1}$ we have

$$\langle (V_g)^{**}(f), \delta_z\rangle = \langle f, V_g^*(\delta_z)\rangle.$$

Since the functions $f_r(z) = f(rz)$ converge to f as $r \to 1$ in the weak-$*$ topology, it follows that $\langle f_r, x^*\rangle \to \langle f, x^*\rangle$ for all $x^* \in (H^0_{v_1})^*$. Hence,

$$\langle f, V_g^*(\delta_z)\rangle = \lim_{r\to 1}\langle f_r, V_g^*(\delta_z)\rangle = \lim_{r\to 1}\langle V_g(f_r), \delta_z\rangle = \lim_{r\to 1} \textstyle\int_0^z f_r(\zeta)g'(\zeta)\, d\zeta.$$

Finally, since $f_r \to f$ uniformly on compact subsets in the unit disk, we obtain

$$\lim_{r\to 1} \int_0^z f_r(\zeta)g'(\zeta) = \int_0^z f(\zeta)g'(\zeta)\, d\zeta = \langle V_g(f), \delta_z\rangle$$

and it follows that $(V_g)^{**} = V_g$. □

Remark 1 To simplify the notation, we shall denote $w_2(r) := v_2(r)(1 - r^2)$ throughout.

Theorem 1 *Let v_1 and v_2 be two weights such that v_2 is quasi-normal. Then, the following conditions are equivalent:*

(a) *$V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is bounded.*
(b) $\sup_{z\in\mathbb{D}} \frac{v_2(z)}{\widetilde{v_1}(z)}(1 - |z|^2)|g'(z)| < \infty$.
If, in addition, v_1 is a typical weight, then both (a) *and* (b) *are equivalent to*
(c) *$V_g : H^0_{v_1} \to H^0_{v_2}$ is bounded.*

Proof To see that (a) implies (b), note first that the inclusion operator $I : H^\infty_{v_2} \to \mathcal{B}^\infty_{w_2}$ is bounded because v_2 is quasi-normal. Since $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is bounded by hypothesis, we obtain that the multiplication operator $M_{g'} : H^\infty_{v_1} \to H^\infty_{w_2}$ is bounded. This implies, by [15, Proposition 4.1], that

$$\sup_{z\in\mathbb{D}} \frac{v_2(z)}{\widetilde{v_1}(z)}(1 - |z|^2)|g'(z)| < \infty.$$

To see that (b) implies (a), we start by proving that $V_g : H^\infty_{\widetilde{v_1}} \to \mathcal{B}^\infty_{w_2}$ is bounded: Take $f \in H^\infty_{\widetilde{v_1}}$, then

$$\begin{aligned}\|V_g(f)\|_{\mathcal{B}^\infty_{w_2}} &= \sup_{z\in\mathbb{D}} v_2(z)\,(1 - |z|^2)\,|f(z)|\,|g'(z)| \\ &= \sup_{z\in\mathbb{D}} \frac{v_2(z)}{\widetilde{v_1}(z)}\,(1 - |z|^2)\widetilde{v_1}(z)|f(z)|\,|g'(z)| \\ &\le \left(\sup_{z\in\mathbb{D}} \frac{v_2(z)}{\widetilde{v_1}(z)}\,(1 - |z|^2)|g'(z)|\right)\|f\|_{H^\infty_{\widetilde{v_1}}}.\end{aligned}$$

Since neither $H_{v_1}^\infty$ nor the norm $\|\cdot\|_{H_{v_1}^\infty}$ change if we replace v_1 by $\widetilde{v_1}$, we have that $V_g : H_{v_1}^\infty \to \mathcal{B}_{w_2}^\infty$ is bounded. Finally, use that $H_{v_2}^\infty = \mathcal{B}_{w_2}^\infty$ because v_2 is quasi-normal.

Assume now that v_1 is typical. Since v_2 being quasi-normal is also typical, we can apply Lemma 1 to obtain that (c) implies (a).

To see that (b) implies (c), take $f \in H_{\widetilde{v_1}}^0$. Using that (b) holds, we obtain

$$\lim_{|z|\to 1} w_2(|z|)|\left(V_g(f)\right)'(z)| = \lim_{|z|\to 1} v_2(z)\,(1-|z|^2)\,|f(z)|\,|g'(z)|$$
$$\leq \left(\sup_{z\in\mathbb{D}} \frac{v_2(z)}{\widetilde{v_1}(z)}\,(1-|z|^2)|g'(z)|\right) \lim_{|z|\to 1} \widetilde{v_1}(z)\,|f(z)| = 0.$$

This finishes the proof of the theorem. □

Remark 2 Danikas and Siskakis proved in [16] that the Cesàro operator, that is, V_g for $g(z) = \log(1/(1-z))$, is bounded from H^∞ into BMOA.

On the other hand, note that for $v_1 \equiv 1$ in the unit disk, we obtain from Theorem 1 that V_g is bounded from H^∞ into the Banach space of analytic functions H_v^∞ provided that v is quasi-normal. However, in this case, H_v^∞ contains the Bloch space and, therefore, it contains BMOA as well. Thus, for the particular case $v_1 \equiv 1$, Theorem 1 follows as a simple consequence of [16, Theorem 1].

Since we may replace v_2 by its associate weight $\widetilde{v_2}$ without changing neither the weighted space nor its norm, in the case when $v_1 = v_2$, we obtain the following corollary. The equivalence between (a) and (c) was proved by Hu [24] for normal weights.

Corollary 1 *Let v be a quasi-normal weight. Then, the following are equivalent:*

(a) *the Volterra operator V_g is bounded on H_v^∞,*
(b) *the Volterra operator V_g is bounded on H_v^0,*
(c) *the symbol g belongs to the Bloch space.*

Our second corollary below says that in the settings of analytic weights with property (U), the boundedness of Volterra operator is equivalent to property (L).

Corollary 2 *Assume that v is an analytic weight satisfying property (U). Then, the following conditions are equivalent:*

(a) *The weight v satisfies property (L) (that is, v is normal).*
(b) *The Volterra operator V_g is bounded on H_v^∞ for all $g \in \mathcal{B}$.*
(c) *V_g is bounded on H_v^∞ for*

$$g(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Proof By Corollary 1, we only need to prove that (c) implies (a). So assume that (c) holds. By the very definition of analytic weight, there is a holomorphic function in the unit disk f such that $\|f\|_{H_v^\infty} = 1$ and $f(r)v(r) = 1$ for all $0 < r < 1$. Thus,

$$\|V_g\| \geq \|V_g(f)\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)\left|\int_0^z f(\zeta)\,g'(\zeta)\,d\zeta\right| \geq \sup_{0<r<1}\left(v(r)\int_0^r \frac{ds}{v(s)\,(1-s^2)}\right).$$

That is, $\sup_{0<r<1}\left(v(r)\int_0^r \frac{ds}{v(s)\,(1-s^2)}\right) < +\infty$, hence v has property (L) by Shields and Williams's characterization [34, Lemma 2] mentioned in Sect. 2. □

4 Compactness of Volterra operators

Recall that if X and Y are Banach spaces and $T : X \to Y$ is a linear operator, then T is *compact* if for every bounded sequence $\{x_n\} \subset X$, the sequence $\{T(x_n)\}$ has a norm convergent subsequence and T is *weakly compact* if for every bounded sequence $\{x_n\} \subset X$, the sequence $\{T(x_n)\}$ has a weakly convergent subsequence. Every compact operator is weakly compact, but the converse is not true in general. In this section we show that both notions of compactness coincide for Volterra operators between different weighted spaces when the second weight is quasi-normal. We will need the following lemma, that can be proved by a standard argument.

Lemma 2 *Let v_1, v_2 be weights such that the operator $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is bounded. Then V_g is weakly compact (resp. compact) if, and only if, for any bounded sequence $\{f_n\}$ in $H^\infty_{v_1}$ that converges to zero uniformly on compact subsets of the unit disk, we have that $\{V_g(f_n)\}$ converges weakly to zero (resp. converges to zero in the norm topology of $H^\infty_{v_1}$).*

We are now ready to state our main result about Volterra operators.

Theorem 2 *Let v_1, v_2 be weights such that v_2 is quasi-normal. Then, the following conditions are equivalent.*

(a) $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ *is compact.*
(b) $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ *is weakly compact.*
(c) $\lim_{|z|\to 1} \frac{v_2(z)}{\widetilde{v_1}(z)}(1-|z|^2)|g'(z)| = 0.$
(d) $V_g : H^\infty_{v_1} \to H^0_{v_2}$ *is bounded.*
If, in addition, v_1 is a typical weight, then the above conditions are equivalent to
(e) $V_g : H^0_{v_1} \to H^0_{v_2}$ *is compact.*
(f) $V_g : H^0_{v_1} \to H^0_{v_2}$ *is weakly compact.*

Proof Since v_2 is quasi-normal, we have that $H^\infty_{v_2} = B^\infty_{w_2}$. Hence, as we pointed out above, $V_g(f) \in H^\infty_{v_2} = B^\infty_{w_2}$ if, and only if, $M_{g'}(f) \in H^\infty_{w_2}$ and both elements have comparable norms. Thus the compactness (resp. weak compactness) of $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is equivalent to the compactness (resp. weak compactness) of $M_{g'} : H^\infty_{v_1} \to H^\infty_{w_2}$. But on these spaces $M_{g'}$ is weakly compact if, and only if, it is compact (see [15, Theorem 5.2]). Thus, (a) and (b) are equivalent.

We prove now that (b) implies (c). Since $V_g : H^\infty_{v_1} \to H^\infty_{v_2}$ is weakly compact and the inclusion operator $I : H^\infty_{v_2} \to \mathcal{B}_{w_2}$ is bounded, we have that the multiplication operator $M_{g'} : H^\infty_{v_1} \to H^\infty_{w_2}$ is weakly compact. Using [15, Theorem 5.2], we obtain that $M_{g'} : H^\infty_{v_1} \to H^\infty_{w_2}$ is, in fact, compact. This implies, by [15, Corollary 4.3], that

$$\lim_{|z|\to 1} \frac{v_2(z)}{\widetilde{v_1}(z)}(1-|z|^2)|g'(z)| = 0.$$

Let us prove now that (c) implies (d). Since $H^0_{v_2} = \mathcal{B}^0_{w_2}$, the operator $V_g : H^\infty_{v_1} \to H^0_{v_2}$ is bounded if, and only if, the multiplication operator $M_{g'} : H^\infty_{v_1} \to H^0_{w_2}$ is bounded. Note that for all $f \in H^\infty_{v_1}$,

$$\begin{aligned}\lim_{|z|\to 1}(1-|z|^2)v_2(z)|M_{g'}(f)| &= \lim_{|z|\to 1}(1-|z|^2)v_2(z)|f(z)||g'(z)| \\ &= \lim_{|z|\to 1} \frac{v_2(z)}{\widetilde{v_1}(z)}(1-|z|^2)\widetilde{v_1}(z)|f(z)||g'(z)| \\ &\le \left(\lim_{|z|\to 1} \frac{v_2(z)}{\widetilde{v_1}(z)}(1-|z|^2)|g'(z)|\right) ||f(z)||_{H^\infty_{v_1}}.\end{aligned}$$

Therefore, using (c), we see that $M_{g'}(H_{v_1}^\infty) \subset H_{w_2}^0$ which is equivalent to (d).

Let us see now that (d) implies (b). We will make use of the following useful characterization of weak compactness (see [19, p. 482]): *"Let $T : X \to Y$ be a bounded linear operator between two Banach spaces X and Y. Then, T is weakly compact if, and only if, $T^{**}(X^{**}) \subset Y$."*

Thus, if (d) holds, then $V_g^{**} : (H_{v_1}^\infty)^{**} \to (H_{v_2}^0)^{**}$ is bounded. Since v_2 is typical, we obtain that $(H_{v_2}^0)^{**} = H_{v_2}^\infty$. Hence $V_g^{**} : (H_{v_1}^\infty)^{**} \to H_{v_2}^\infty$ is bounded and this implies that $V_g : H_{v_1}^\infty \to H_{v_2}^\infty$ is weakly compact. Since $V_g(H_{v_1}^\infty) \subset H_{v_2}^0$, we also have the weak compactness of $V_g : H_{v_1}^\infty \to H_{v_2}^0$.

To prove the second group of equivalences, assume that v_1 is a typical weight. Bearing in mind that an operator is compact if, and only if, so is its bi-adjoint, it follows that (e) implies (a). If (a), hence (d), holds, we get that $V_g : H_{v_1}^0 \to H_{v_2}^0$ is bounded. Using again that its bi-adjoint is compact, we obtain (e).

Finally, being clear that (e) implies (f), the proof that (f) implies (c) can be done following the same steps as in the proof that (b) implies (c) above but using, in this case, [15, Corollary 4.5] instead. □

Corollary 3 *Let v be a quasi-normal weight. Then, the following conditions are equivalent.*

(a) *V_g is compact on H_v^0.*
(b) *V_g is weakly compact on H_v^0.*
(c) *V_g is compact on H_v^∞.*
(d) *V_g is weakly compact on H_v^∞.*
(e) *$V_g(H_v^\infty) \subset H_v^0$.*
(f) *$g \in \mathcal{B}_0$.*

The equivalence between statements (c) and (f) in Corollary 3 was proved by Hu [24] under the assumption that the weight is normal.

Corollary 4 *Let v be a quasi-normal weight. Then, the following conditions are equivalent.*

(a) *$V_g : H^\infty \to H_v^\infty$ is compact.*
(b) *$V_g : H^\infty \to H_v^\infty$ is weakly compact.*
(c) $\lim_{|z|\to 1} v(z)(1-|z|^2)|g'(z)| = 0$.
(d) *$V_g : H^\infty \to H_v^0$ is bounded.*

5 Semigroups of analytic functions

A (one-parameter) *semigroup of analytic functions* (φ_t) is a continuous homomorphism $\Phi : t \to \Phi(t) = \varphi_t$ from the additive semigroup of non-negative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$. In other words, (φ_t) consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ for which the following three conditions hold:

1. φ_0 is the identity in $\mathbb{D}$,
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
3. $\varphi_t(z) \to z$, as $t \to 0$, for all $z \in \mathbb{D}$.

Good references for the properties of semigroups listed below are the books by Abate [2] and Shoikhet [35]. It is worth pointing out that (3) can be replaced by uniform convergence on compact sets in the unit disk. If (φ_t) is a semigroup, then each map φ_t is univalent. The *infinitesimal generator* of (φ_t) is the function

$$G(z) := \lim_{t\to 0} \frac{\varphi_t(z) - z}{t}, \qquad \text{for } z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of the unit disk. Therefore, G is analytic in $\mathbb{D}$. Moreover, G satisfies

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z}, \qquad \text{for all } z \in \mathbb{D} \quad \text{and} \quad t \geq 0.$$

Further, G has a representation

$$G(z) = (\overline{b}z - 1)(z - b)p(z)$$

in the unit disk, where $b \in \overline{\mathbb{D}}$ and $p \in H(\mathbb{D})$ has positive real part. If G is not identically null, the couple (b, p) is uniquely determined from (φ_t) and the point b is the common Denjoy-Wolff point of all the functions of the semigroup. This fact plays a crucial role in the dynamical behavior of the semigroup (see [35,37]).

As we mentioned in the Introduction, a semigroup of analytic functions (φ_t) defines, via

$$C_t : f \in H(\mathbb{D}) \to C_t(f) := f \circ \varphi_t \in H(\mathbb{D}),$$

a semigroup of composition operators (C_t) on the space $H(\mathbb{D})$ of analytic functions in the unit disk. This connection between composition operators and operator semigroups opens the possibility of studying spectral properties, operator ideal properties, or dynamical properties of the semigroup of operators (C_t) in terms of the theory of functions. The paper [7] can be considered as the starting point in this direction.

A semigroup of analytic functions (C_t) on a Banach space of analytic functions X is said to be *strongly continuous* on X if, for all $f \in X$, we have

$$\lim_{t\to 0} \|C_t(f) - f\|_X = 0.$$

Thus, the crucial step to show that (φ_t) generates a strongly continuous semigroup of operators in X is to pass from the pointwise convergence $\lim_{t\to 0} (f \circ \varphi_t)(z) = f(z)$ in $\mathbb{D}$ to the convergence in the norm of X.

An important branch of current research is the study of the behavior of the semigroup (C_t) acting on a given Banach space of analytic functions X. Classical choices of X treated in the literature are the Hardy spaces H^p, the disk algebra $A(\mathbb{D})$, the Bergman spaces A^p, the Dirichlet space $\mathcal{D}$, and the chain of spaces Q_p and $Q_{p,0}$ which have been introduced recently and that include the spaces BMOA, Bloch, and their "little" analogues. See [42,44] for definitions and basic facts of the mentioned spaces, and [36,37,39] for composition semigroups on these spaces.

Very briefly, the state of the art is the following:

- Every semigroup of analytic functions generates a strongly continuous semigroup of operators on the Hardy spaces H^p $(1 \leq p < \infty)$, on the Bergman spaces A^p $(1 \leq p < \infty)$, on the Dirichlet space, and on the spaces VMOA and little Bloch.
- No non-trivial semigroup generates a strongly continuous semigroup of operators on either H^∞ or the Bloch space.
- There are plenty of semigroups (but not all) which generate strongly continuous semigroups of operators on the disk algebra. Indeed, they can be characterized in different analytical terms [14].

- Semigroups of composition operators in the framework of BMOA and Bloch spaces have been recently studied in [9,10].

When the semigroup is not strongly continuous on X, we are interested in the maximal closed subspace of X on which (φ_t) generates a strongly continuous semigroup (C_t) of composition operators. The existence of such a maximal subspace, as well as analytical descriptions of it, was obtained in [10]. Namely, let (φ_t) be a semigroup of analytic functions and X a Banach space of analytic functions such that the composition operators $C_t : X \to X$ are uniformly bounded. Then there exists a closed subspace $[(\varphi_t), X]$ of X such that (φ_t) generates a strongly continuous semigroup of operators on $[(\varphi_t), X]$ and that it is maximal, in the sense that any other subspace of X with this property is contained in it. If X contains the constant functions, then $[(\varphi_t), X]$ has the following two descriptions:

$$[(\varphi_t), X] = \{f \in X : \lim_{t\to 0} \|f \circ \varphi_t - f\|_X = 0\} = \overline{\{f \in X : Gf' \in X\}},$$

where G is the infinitesimal generator of the semigroup.

This section is devoted to studying the maximal subspace of (C_t) on the spaces H_v^∞ and H_v^0. To start with, let us mention that, in general, composition operators are not bounded on H_v^0. It was proven in [11] that given an analytic self-map of the unit disk φ, the composition operator C_φ with symbol φ is bounded on H_v^∞ (resp. H_v^0) for every weight v if, and only if, C_φ is bounded on H_v^∞ (resp. H_v^0) for every typical weight v and if, and only if, there is $r < 1$ such that $|\varphi(z)| \le |z|$ for every $z \in \mathbb{D}$ with $|z| \ge r$. Thus, by assuming that $\varphi(0) = 0$, we have that the composition operator with symbol φ is always bounded on both H_v^∞ and on H_v^0. Moreover, under this assumption, it is also known that the composition operator is a contraction; that is, $\|C_\varphi\| \le 1$ (both on H_v^∞ and on H_v^0). These properties will be used in the proof of the following theorem.

Theorem 3 *Assume that v is a typical weight and let (φ_t) be a non-trivial semigroup of analytic functions such that $\varphi_t(0) = 0$ for all t. Then,*

$$H_v^0 \subseteq [(\varphi_t), H_v^\infty].$$

That is, the semigroup of composition operators (C_t) is strongly continuous on H_v^0.

Moreover, if v is quasi-normal, then (C_t) is never strongly continuous on H_v^∞, that is

$$[(\varphi_t), H_v^\infty] \neq H_v^\infty.$$

Proof We begin by showing that the semigroup of composition operators (C_t) is strongly continuous on H_v^0. Since the set of polynomials is dense in H_v^0 and $\|C_t\| \le 1$, it is enough to show that $\lim_{t\to 0} \|\varphi_t(z)^n - z^n\|_{H_v^\infty} = 0$ for all $n = 0, 1, 2, \ldots$ Fixed n and given $\varepsilon > 0$, there is $R < 1$ such that $v(z) < \varepsilon/2$ for all $|z| \ge R$. Also, since φ_t converges to the identity map on compact sets, as $t \to 0$, we have that there is $t_0 > 0$ such that $|\varphi_t(z)^n - z^n| \le \varepsilon/M$ for all $t < t_0$ and for all $|z| \le R$, where $M = \sup_{|z|<1} |v(z)|$. Therefore, $\|\varphi_t(z)^n - z^n\|_{H_v^\infty} \le \varepsilon$ for all $t < t_0$.

Now, we want to show that the maximal space $[(\varphi_t), H_v^\infty]$ cannot be the whole of H_v^∞ when v is quasi-normal. This will be proved in two steps. First, we show that any strongly continuous operator semigroup on H_v^∞ is uniformly continuous, and conclude that its infinitesimal generator is a bounded operator. In the second step, we show that for composition semigroups (C_t), this implies that the infinitesimal generator is the null function. Therefore, the semigroup must be the trivial one.

Step 1. Each strongly continuous operator semigroup in H_v^∞ is uniformly continuous. We are going to use a theorem of Lotz [26, Theorem 3] which says that if X is a Grothendieck space with the Dunford-Pettis property, then each strongly continuous semigroup of operators on X is, in fact, uniformly continuous (we refer the reader to [26] for the definition of Grothendieck spaces and the Dunford-Pettis property).

It is well-known that H_v^∞ is isomorphic to either the space ℓ_∞ of bounded sequences of complex numbers or to H^∞ (see [28]) and also that ℓ_∞ and H^∞ are Grothendieck spaces with the Dunford-Pettis property (see [17, Chapter VII, Exercises 1 and 12] for l_∞ and [12] and [13, Corollary 5.4] for H^∞). As a direct consequence, it follows that H_v^∞ is a Grothendieck space with the Dunford-Pettis property and Lotz's theorem can be applied.

Step 2. If (φ_t) is a semigroup with generator G and the induced semigroup of composition operators (C_t) is strongly continuous on H_v^∞, then $G \equiv 0$. To prove this, let Γ denote the infinitesimal generator of (C_t). From Step 1, Γ is a bounded operator. Now, for each $f \in H_v^\infty$, we have

$$\Gamma(f) = \lim_{t\to 0} \frac{f\circ\varphi_t - f}{t},$$

being the convergence in the norm of H_v^∞ and, therefore, uniformly on compact subsets of the unit disk. Thus, for $z \in \mathbb{D}$,

$$\Gamma(f)(z) = \lim_{t\to 0} \frac{f(\varphi_t(z)) - f(z)}{t} = \frac{\partial f(\varphi_t(z))}{\partial t}|_{t=0} = G(z) f'(z),$$

so that $\Gamma(f) = Gf'$ for each $f \in H_v^\infty$.

Since we are assuming that the weight v is quasi-normal, we have that $H_v^\infty = \mathcal{B}_w^\infty$, where $w(z) = v(z)(1-|z|^2)$ in the unit disk. Take $f \in H_w^\infty$, then its primitive $F(z) := \int_0^z f(\xi)\,d\xi$ is in $\mathcal{B}_w^\infty = H_v^\infty$. The boundedness of Γ implies that $GF' = Gf \in H_v^\infty$ or, in other words, that the multiplication operator $M_G : H_w^\infty \to H_v^\infty$ is bounded.

To finish the proof, for each $\zeta \in \mathbb{D}$ consider a function $f_\zeta \in H_{\widetilde{v}}^\infty$ (with norm one) such that $|f_\zeta(\zeta)|\,\widetilde{v}(|\zeta|) = 1$. Define $F_\zeta(z) = f_\zeta(z)/(1-\overline{\zeta}z)$. Since

$$\|F_\zeta\|_{H_w^\infty} = \sup_{|z|<1} v(z)(1-|z|^2)\tfrac{|f_\zeta(z)|}{|1-\overline{\zeta}z|} \le \sup_{|z|<1} \widetilde{v}(z)\tfrac{1-|z|^2}{1-|z|}|f_\zeta(z)| \le 2\|f_\zeta\|_{H_{\widetilde{v}}^\infty} = 2,$$

we have that the functions F_ζ belong to H_w^∞ for all $\zeta \in \mathbb{D}$. Thus, each GF_ζ must be in $H_{\widetilde{v}}^\infty$ and $\|GF_\zeta\|_{H_{\widetilde{v}}^\infty} \le 2\|M_G\|$ for all ζ. This implies that

$$2\|M_G\| \ge \sup_{|z|<1} \widetilde{v}(z)|G(z)|\frac{|f_\zeta(z)|}{|1-\overline{\zeta}z|} \ge \widetilde{v}(\zeta)\frac{|G(\zeta)f_\zeta(\zeta)|}{1-|\zeta|^2} = \frac{|G(\zeta)|}{1-|\zeta|^2}.$$

Therefore, by the maximum modulus principle, $G \equiv 0$ in the unit disk. This completes the proof. □

Now we are ready to connect what we have studied in the previous sections of this paper about Volterra operators with the maximal subspace $[(\varphi_t), H_v^\infty]$. The key is the next result from [10].

Theorem B *Let (φ_t) be a semigroup that fixes the origin and with infinitesimal generator $G(z) = -zp(z)$ (where* $\mathrm{Re}\{p\} > 0$*). For z in the unit disk, define $g(z) = \int_0^z \frac{1}{p(\xi)}d\xi$. Let X be a Banach space of analytic functions with the following properties:*

(i) *X contains the constant functions,*
(ii) *for each $b \in \mathbb{D}$, $f \in X \iff \frac{f(z)-f(b)}{z-b} \in X$, and*
(iii) *if (C_t) is the induced semigroup on X then* $\sup_{t\in[0,1]} \|C_t\|_X < \infty$.

Then,

$$[(\varphi_t), X] = \overline{X \cap (V_g(X) \oplus \mathbb{C})}.$$

Corollary 5 *Let v be a quasi-normal weight. Let (φ_t) be a semigroup that fixes the origin and with generator $G(z) = -zp(z)$ (where $\mathrm{Re}\{p\} > 0$). Take $g(z) = \int_0^z \frac{1}{p(\xi)} d\xi$. Then, $[(\varphi_t), H_v^\infty] = H_v^0$ if, and only if, g belongs to the little Bloch space.*

Proof It is clear that the space H_v^∞ satisfies properties (i), (ii), and (iii) in Theorem B. Thus, the maximal subspace $[(\varphi_t), H_v^\infty]$ depends on the range of V_g.

Now, the well-known growth estimate

$$|1/p(z)| \leq C\frac{1+|z|}{1-|z|} \qquad (|z| < 1)$$

for the function $1/p$ with positive real part implies that $(1-|z|)|g'(z)| \leq 2C$ (see [30, page 40]). Therefore, $g \in \mathcal{B}$. By Corollary 1, the operator V_g is always bounded and

$$[(\varphi_t), H_v^\infty] = \overline{V_g(H_v^\infty) \oplus \mathbb{C}}.$$

The proof finishes by using Corollary 3. □

6 Quasi-normal weights and the differentiation operator

The main hypothesis we have imposed on weights in the target spaces is that they are quasi-normal, that is, the equality $H_v^0 = \mathcal{B}_w^0$ (where we recall that $w(r) = (1-r^2)v(r)$). This is an ad hoc technical definition, of course, and we devote this final section to characterize quasi-normal weights in terms of their properties as functions. So far we have two characterizations, in both cases for weights having property (U). Namely, Lusky's Theorem A saying that if v has property (U), then v is quasi-normal if, and only if, it satisfies property (L); and our Corollary 2 stating that for analytic weights with property (U), normality is equivalent to the boundedness of the Volterra operator on H_v^∞.

The equality $H_v^0 = \mathcal{B}_w^0$ means, in terms of the differentiation operator $D(f) = f'$, that $D : H_v^0 \to H_w^0$ is bounded and onto. This provides a characterization of property (U) that was originally given in [23]. For the sake of completeness, we will offer its proof with a slightly different formulation using $\tilde{v}$.

Proposition 1 *Let v be a differentiable weight which is essential and typical. Suppose that $\widetilde{v}$ is differentiable (this hypothesis is satisfied, for example, if v is analytic). Then the following conditions are equivalent:*

(a) *$D : H_v^0 \to H_w^0$ is bounded.*
(b) *$D : H_v^\infty \to H_w^\infty$ is bounded.*
(c) $\sup\limits_{r\in[0,1)} \left(-\frac{(1-r^2)\widetilde{v}'(r)}{\widetilde{v}(r)}\right) < \infty.$
(d) *$\frac{\widetilde{v}(r)}{(1-r)^\alpha}$ is increasing on $[r_0, 1)$ for some $\alpha > 0$ and $r_0 > 0$.*
(e) *v satisfies property (U).*

Proof By duality, it follows that (a) implies (b).

To see that (b) implies (c), fix an arbitrary $r_0 \in (0, 1)$. By the definition of associated weight, there exists a function $h \in H_v^\infty$ with norm $\|h\|_{H_v^\infty} = 1$ such that $h(r_0)\widetilde{v}(r_0) = 1$.

Consider now the differentiable function $g(r) := |h(r)|^2 = h(r)\overline{h(r)}$, for $r \in (0,1)$. It is clear that $g'(r) = 2h(r)\,\mathrm{Re}\,h'(r)$ and, in particular, $g'(r_0) = 2h(r_0)\,\mathrm{Re}\,h'(r_0)$. Since r_0 is a global maximum point of the function $g(r)\widetilde{v}(r)^2$, we deduce

$$g'(r_0) = -2\frac{g(r_0)\widetilde{v}'(r_0)}{\widetilde{v}(r_0)} \geq 0.$$

Since (b) holds and v is an essential weight, taking $w^*(r) := \widetilde{v}(r)(1-r^2)$, we have that the differentiation operator $D : H_v^\infty \to H_{w^*}^\infty$ is also bounded. Then

$$\mathrm{Re}\,h'(r_0)w^*(r_0) \leq \left|h'(r_0)\right| w^*(r_0) \leq \left\|h'\right\|_{H_{w^*}^\infty} \leq \|D\|\,\|h\|_{H_v^\infty} = \|D\|\,h(r_0)\widetilde{v}(r_0).$$

Finally,

$$-\frac{\widetilde{v}'(r_0)}{\widetilde{v}^2(r_0)}w^*(r_0) = \frac{g'(r_0)w^*(r_0)}{2g(r_0)\widetilde{v}(r_0)} = \frac{h(r_0)\,\mathrm{Re}\,h'(r_0)w^*(r_0)}{h(r_0)^2\widetilde{v}(r_0)} \leq \|D\|$$

which implies (c).

Let us see that (c) and (d) are equivalent: take the function $f(r) = \frac{\widetilde{v}(r)}{(1-r)^\alpha}$, for $r \in [0,1)$. Then,

$$f'(r) = \frac{\widetilde{v}(r)}{(1-r)^{\alpha+1}}\left[\frac{(1-r)\,\widetilde{v}'(r)}{\widetilde{v}(r)} + \alpha\right],$$

which shows clearly the equivalence between (c) and (d).

Since v is essential, it is clear that (d) implies property (U). Finally, from [18, Lemma 1(a)] and [27, Theorem 3.1] it follows that (e) implies (a). □

Remark 3 The equivalence between (a) and (b) in the proposition above is true for any typical weight. Indeed, if we assume that $D : H_v^\infty \to H_w^\infty$ is bounded and $f \in H_v^0$, we can take a sequence of polynomials $\{p_n\}$ such that, for each non-negative integer n, we have that $\|p_n - f\|_{H_v^\infty} \leq 1/n$. Therefore, $||p_n' - f'||_{H_w^\infty} \leq C/n$, for all n and for some constant C (which does not depend on n). In particular, f' belongs to the closure of the polynomials in H_w^∞. Hence, $f' \in H_w^0$. Thus, $D : H_v^0 \to H_w^0$ is bounded.

Corollary 6 *Let v be a weight such that $\tilde{v}$ is differentiable. Then $\tilde{v}$ is normal if, and only if, it is quasi-normal.*

Proof Let us recall that the weight $\tilde{v}$ is essential. If $\tilde{v}$ is quasi-normal, we have that $D : H_{\tilde{v}}^0 \to H_{w^*}^0$ is bounded, where $w^*(r) := \widetilde{v}(r)(1-r^2)$. Then, by Proposition 1, the weight $\tilde{v}$ satisfies property (U). By Theorem A, we have that it is normal. □

We finish this section studying when $\mathcal{B}_w^\infty$ is a subset of H_v^∞. This will be related to the properties of a new weight that we list in the following lemma.

Lemma 3 *Let v be a typical weight. Define*

$$u(r) = \frac{1}{1 + \displaystyle\int_0^r \frac{ds}{v(s)\,(1-s^2)}}.$$

Then:

(a) *The weight u is typical.*
(b) *$\mathcal{B}_w^0 \subseteq H_u^0$ and $\mathcal{B}_w^\infty \subseteq H_u^\infty$.*

(c) *$H_u^0 \subseteq H_v^0$ if, and only if, $H_u^\infty \subseteq H_v^\infty$ and also if, and only if, there is a constant C such that $v \leq C\tilde{u}$.*
(d) *If v has property (L), then $\mathcal{B}_w^0 \subseteq H_v^0$ and $\mathcal{B}_w^\infty \subseteq H_v^\infty$.*

Proof (a) Since v is bounded from above, there is a constant $M > 0$ such that

$$\int_0^r \frac{ds}{v(s)\,(1-s^2)} \geq \frac{1}{M}\int_0^r \frac{ds}{1-s^2} \to \infty \quad \text{as} \quad r \to 1.$$

Thus, u is typical.
(b) Assume that $f \in \mathcal{B}_w^\infty$, then

$$|f(z) - f(0)| \leq \int_0^z |f'(\zeta)|\,|d\zeta| \leq \int_0^{|z|} \frac{K}{v(|\zeta|)\,(1-|\zeta|^2)}\,|d\zeta|,$$

for some constant K. Therefore, $f - f(0)$, and hence f, belongs to H_u^∞. If $f \in \mathcal{B}_w^0$, then it belongs to the closure of the polynomials in $\mathcal{B}_w^\infty$. Since the inclusion map from $\mathcal{B}_w^\infty$ to H_u^∞ is bounded, the function f also belongs to the closure of the polynomials in H_u^∞. Thus, $f \in H_u^0$.
(c) This is a particular case of [15, Propositions 3.1 and 3.2].
(d) By Shields and Williams's characterization [34, Lemma 2], it follows that if the weight v satisfies property (L), then there is $C > 0$ such that $v \leq Cu$. Since $u \leq \tilde{u}$, from (c) we deduce that $H_u^0 \subseteq H_v^0$. Finally, by (b), we have that $\mathcal{B}_w^0 \subseteq H_v^0$ and $\mathcal{B}_w^\infty \subseteq H_v^\infty$. □

Combining the arguments above, we obtain a characterization of the normality of v in terms of the weight u.

Proposition 2 *Let v be an essential and typical weight and write*

$$u(r) = \frac{1}{1 + \displaystyle\int_0^r \frac{ds}{v(s)\,(1-s^2)}}.$$

The following conditions are equivalent:

(a) *v is normal.*
(b) *$H_v^0 = H_u^0$.*
(c) *$H_v^\infty = H_u^\infty$.*

Proof The normality of the weight v and the preceding lemma imply the chain of inclusions

$$H_v^0 \subseteq \mathcal{B}_w^0 \subseteq H_u^0.$$

Since v has property (L), arguing as in the proof of Lemma 3, we have that $H_u^0 \subseteq H_v^0$. Thus, $H_v^0 = H_u^0$. Similarly, we conclude that (c) is satisfied.

Assume now that (b) is satisfied. Since $H_v^0 \subseteq H_u^0$, by [15, Proposition 3.2], there is a constant C such that $u \leq C\tilde{v}$ and, since v is essential, there is another constant K such that $u \leq Kv$. Hence,

$$\sup_{r\in(0,1)} \frac{-(1-r^2)u'(r)}{u(r)} = \sup_{r\in(0,1)} \frac{u(r)}{v(r)} < +\infty.$$

Arguing as in the proof of "(c) implies (d)" in Proposition 1, we obtain that u satisfies property (U). In particular, u is essential [34, Lemma 1]. Since $H_v^0 = H_u^0$, using again [15, Proposition 3.2], we deduce that $\tilde{u}$ and $\tilde{v}$ are equivalent. Essentiality of both weights implies that v and u are equivalent too. Thus, v also has property (U). We use again Shields and Williams's characterization [34, Lemma 2] to obtain that v has property (L) (because v and u are equivalent) and we conclude that v is normal.

By using the same arguments as above but using [15, Proposition 3.1] instead of [15, Proposition 3.2], we obtain that (c) implies (a). □

References

1. Abakumov, E., Doubtsiv, E.: Reverse estimates in growth spaces, to appear in Math. Z. **271**, 399–413 (2012)
2. Abate, M.: Iteration Theory of Holomorphic Maps on Taut Manifolds. Mediterranean Press, Rende (1989)
3. Aleman, A.: A class of integral operators on spaces of analytic functions. In: Topics in Complex Analysis and Operator Theory, pp. 3–30. Univ. Málaga, Málaga (2007)
4. Aleman, A., Cima, J.A.: An integral operator on H^p and Hardy's inequality. J. Anal. Math. **85**, 157–176 (2001)
5. Aleman, A., Siskakis, A.G.: An integral operator on H^p. Complex Var. Theory Appl. **28**, 149–158 (1995)
6. Aleman, A., Siskakis, A.G.: Integration operators on Bergman spaces. Indiana Univ. Math. J. **46**, 337–356 (1997)
7. Berkson, E., Porta, H.: Semigroups of analytic functions and composition operators. Mich. Math. J. **25**, 101–115 (1978)
8. Bierstedt, K.D., Bonet, J., Taskinen, J.: Associated weights and spaces of holomorphic functions. Studia Math. **127**, 137–168 (1998)
9. Blasco, O., Contreras, M.D., Díaz-Madrigal, S., Martínez, J., Siskakis, A.G.: Semigroups of composition operators in BMOA and the extension of a theorem of Sarason. Int. Equ. Oper. Theory **61**, 45–62 (2008)
10. Blasco, O., Contreras, M.D., Díaz-Madrigal, S., Martínez, J., Papadimitrakis, M., Siskakis, A.G.: Semigroups of composition operators and integral operators in spaces of analytic functions. Ann. Acad. Sci. Fenn. Math. **38**, 67–89 (2013)
11. Bonet, J., Domański, P., Lindström, M., Taskinen, J.: Composition operators between weighted Banach spaces of analytic functions. J. Aust. Math. Soc. (Ser. A) **64**, 101–118 (1998)
12. Bourgain, J.: H^∞ is a Grothendieck space. Studia Math. **75**, 193–216 (1983)
13. Bourgain, J.: New Banach space properties of the disc algebra and H^∞. Acta Math. **152**, 1–48 (1984)
14. Contreras, M.D., Díaz-Madrigal, S.: Fractional iteration in the disk algebra: prime ends and composition operators. Revista Mat. Iberoamericana **21**, 911–928 (2005)
15. Contreras, M.D., Hernández-Díaz, A.G.: Weighted composition operators in weighted Banach spaces of analytic functions. J. Aust. Math. Soc. (Ser. A) **69**, 41–60 (2000)
16. Danikas, N., Siskakis, A.G.: The Cesàro operator on bounded analytic functions. Analysis **13**, 295–299 (1993)
17. Diestel, J.: Sequences and Series in Banach Spaces. Springer-Verlag, New York (1984)
18. Domański, P., Lindström, M.: Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions. Ann. Polon. Math. **89**, 233–264 (2002)
19. Dunford, N., Schwartz, J.T.: Linear Operators I. Interscience Publishers. Wiley, New York (1958)
20. Duren, P.L.: Theory of H^p spaces, Academic Press, New York (1970) (Reprinted by Dover, Mineola, NY (2000))
21. Galanopoulos, P., Girela, D., Peláez, J.A.: Multipliers and integration operators on Dirichlet spaces. Trans. Am. Math. Soc. **363**(4), 1855–1886 (2011)
22. Hardy, G.H., Littlewood, J.E.: Some properties of fractional integrals. II. Math. Z. **34**(1), 403–439 (1932)
23. Harutyunyan, A., Lusky, W.: On the boundedness of the differentiation operator between weighted spaces of holomorphic functions. Studia Math. **184**, 233–247 (2008)
24. Hu, Z.: Extended Cesàro operators on mixed-norm spaces. Proc. Am. Math. Soc. **131**(7), 2171–2179 (2003)
25. Li, S., Stević, S.: Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces. J. Math. Anal. Appl. **345**, 40–52 (2008)
26. Lotz, H.P.: Uniform convergence of operators on L^∞ and similar spaces. Math. Z. **190**, 207–220 (1985)
27. Lusky, W.: Growth conditions for harmonic and holomorphic functions. In: Dierolf, S. et al (eds.) Functional Analysis (Trier,1994), pp. 281–291. de Gruyter (1996)
28. Lusky, W.: On the isomorphism classes of weighted spaces of harmonic and holomorphic functions. Studia Math. **175**, 19–45 (2006)

29. Pau, J., Peláez, J.A.: Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights. J. Funct. Anal. **259**(10), 2727–2756 (2010)
30. Pommerenke, Ch.: Univalent Functions. Vandenhoeck & Ruprecht, Göttingen (1975)
31. Pommerenke, Ch.: Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszilation. Comment. Math. Helv. **52**, 591–502 (1977)
32. Rubel, L.A., Shields, A.L.: The second duals of certain spaces of analytic functions. J. Aust. Math. Soc. **11**, 276–280 (1970)
33. Shields, A.L., Williams, D.L.: Bounded projections, duality, and multipliers in spaces of analytic functions. Trans. Am. Math. Soc. **162**, 287–302 (1971)
34. Shields, A.L., Williams, D.L.: Bounded projections and the growth of harmonic conjugates in the unit disc. Mich. Math. J. **29**, 3–25 (1982)
35. Shoikhet, D.: Semigroups in Geometrical Function Theory. Kluwer Academic Publishers, Dordrecht (2001)
36. Siskakis, A.G.: Semigroups of composition operators on the Dirichlet space. Result Math. **30**, 165–173 (1996)
37. Siskakis, A.G.: Semigroups of composition operators on spaces of analytic functions, a review. In: Contemporary Mathematics, vol. 213, pp. 229–252. American Mathematical Society, Providence (1998)
38. Stević, S.: Extended Cesàro operators between mixed-norm spaces and Bloch-type spaces in the unit ball. Houst. J. Math. **36**, 84–858 (2010)
39. Wirths, K.J., Xiao, J.: Recognizing $Q_{p,0}$ functions as per Dirichlet space structure. Bull. Belg. Math. Soc. **8**, 47–59 (2001)
40. Wolf, E.: Volterra composition operators between weighted Bergman spaces and weighted Bloch type spaces. Collect. Math. **61**, 57–63 (2010)
41. Wolf, E.: Products of Volterra type operators and composition operators between weighted Bergman spaces of infinite order and weighted Bloch type spaces. Georgian Math. J. **17**, 621–627 (2010)
42. Xiao, J.: Holomorphic Q Classes, Lecture Notes in Mathematics, 1767. Springer-Verlag, Berlin (2001)
43. Yoneda, R.: Integration operators on weighted Bloch spaces. Nihonkai Math. J. **12**, 123–133 (2001)
44. Zhu, K.: Operator Theory in Function spaces. Marcel Dekker, Inc., New York (1990)