Nonlinear stochastic resonance with subthreshold rectangular pulses

Jesús Casado-Pascual,* José Gómez-Ordóñez, and Manuel Morillo
Física Teórica, Universidad de Sevilla, Apartado de Correos 1065, Sevilla 41080, Spain
(Received 30 December 2003; published 2 June 2004)

We analyze the phenomenon of nonlinear stochastic resonance (SR) in noisy bistable systems driven by pulsed time periodic forces. The driving force contains, within each period, two pulses of equal constant amplitude and duration but opposite signs. Each pulse starts every half period and its duration is varied. For *subthreshold* amplitudes, we study the dependence of the output signal-to-noise ratio and the SR gain on the noise strength and the relative duration of the pulses. We find that the SR gains can reach values larger than unity, with maximum values showing a nonmonotonic dependence on the duration of the pulses.

DOI: 10.1103/PhysRevE.69.067101 PACS number(s): 02.50.−r, 05.40.−a, 05.10.Gg

In recent work [1,2], we have carried out a detailed analytical and numerical study of the nonlinear response of a noisy bistable system subject to a subthreshold time periodic driving force. The force considered in those works is such that it remains constant within the duration of each half period while switching its sign every half period. We have focused our attention on the analysis of the dependence of the output signal-to-noise ratio (SNR) and the corresponding stochastic resonance (SR) gain on the noise strength. The analytical study was based on a two-state approximation amenable to exact treatment [1]. We showed that, for subthreshold input signals of sufficiently long periods, the phenomenon of SR can be accompanied by SR gains larger than unity. This is a genuine characterization of nonlinearity, as SR gains larger than unity are strictly forbidden within a linear response description [3–5]. The analytical results were corroborated by numerical simulations [2].

The linear and nonlinear regimes of SR have been studied experimentally in a recent paper by Mantegna *et al.* [6]. These authors analyzed the response of a tunnel diode driven by a sinusoidal signal of varying amplitude and frequency. In the last few years, Gingl and collaborators [7–9] have carried out analog simulations of noisy bistable systems subject to nonsinusoidal driving forces of the type given in Eq. (1),

$$F(t)=\begin{cases} A: & 0\leqslant t<t_c \\ -A: & \dfrac{T}{2}\leqslant t<\dfrac{T}{2}+t_c \\ 0: & \text{otherwise} \end{cases} \tag{1}$$

with a Fourier expansion $F(t)=\Sigma_m F_m \exp(im\Omega t)$, where

$$F_m=\frac{1}{T}\int_0^T dt\, F(t)e^{-im\Omega t} \tag{2}$$

and $\Omega=2\pi/T$. It is convenient to introduce the parameter $r=2t_c/T$, measuring the fraction of a period during which this driving force has a nonvanishing value (the parameter r in the present paper corresponds exactly to what Gingl and coworkers term "duty cycle"). In Refs. [7–9], the SNR and the SR gain for subthreshold amplitude input signals with $r\leqslant 0.3$ were studied. These authors find SR gains larger than unity, and, also, that increasing the r value lowers the SNR gain. They rationalize their observations by noting that the input SNR increases as r increases, while the output SNR is less sensitive to the value of r. The case studied by us in Refs. [1,2] corresponds to the largest possible value of the parameter r, namely, $r=1$. It seems therefore of interest to extend our analysis to input signals with $r<1$ in order to compare with the predictions of Gingl *et al.*

Let us consider a system characterized by a single degree of freedom x, whose dynamics (in dimensionless units) is governed by the Langevin equation

$$\dot{x}(t)=-U'(x(t),t)+\xi(t), \tag{3}$$

where $\xi(t)$ is a Gaussian white noise of zero mean with $\langle\xi(t)\xi(s)\rangle=2D\delta(t-s)$, and $-U'(x,t)$ represents the force stemming from the time-dependent, archetype bistable potential

$$U(x,t)=\frac{x^4}{4}-\frac{x^2}{2}-F(t)x, \tag{4}$$

with $F(t)$ given by Eq. (1). The one-time correlation function is defined as

$$C(\tau)=\frac{1}{T}\int_0^T dt\,\langle x(t+\tau)x(t)\rangle_\infty. \tag{5}$$

It can be written exactly as the sum of two contributions: a coherent part $C_{coh}(\tau)$, which is periodic in τ with period T, and an incoherent part $C_{incoh}(\tau)$, which decays to 0 for large values of τ and reflects the correlation of the output fluctuations about its average (the noisy part of the output). The coherent part $C_{coh}(\tau)$ is given by [11,12]

$$C_{coh}(\tau)=\frac{1}{T}\int_0^T dt\,\langle x(t+\tau)\rangle_\infty\langle x(t)\rangle_\infty, \tag{6}$$

and $C_{incoh}(\tau)$ is obtained from the difference of Eqs. (5) and (6). In the expressions above, the susbcript indicates that the

*Email address: jcasado@us.es
URL: http://numerix.us.es

 ©2004 The American Physical Society

averages are to be evaluated in the limit $t\to\infty$.

The SNR of a random signal measures the signal strength relative to the background noise. The output SNR that is generally considered takes into account the response at the fundamental harmonics. Due to the nature of our problem, it seems convenient to extend the analysis to the overtones. Following the procedure in Refs. [10–12], we calculate the output SNR for the nth overtone, $R_{out}^{(n)}, (n=0,1,\ldots)$, in terms of Fourier transforms of the coherent and incoherent parts of $C(\tau)$ as

$$R_{out}^{(n)} = \frac{\lim_{\epsilon\to 0+}\int_{(2n+1)\Omega-\epsilon}^{(2n+1)\Omega+\epsilon} d\omega \widetilde{C}(\omega)}{\widetilde{C}_{incoh}[(2n+1)\Omega]}, \quad (7)$$

where $\widetilde{H}(\omega)$ denotes the Fourier cosine transform of $H(\tau)$, i.e., $\widetilde{H}(\omega)=(2/\pi)\int_0^\infty d\tau\, H(\tau)\cos(\omega\tau)$. The periodicity of the coherent part gives rise to delta peaks in the spectrum. Actually, the coherent part can be expressed in terms of the Fourier components of the average output as

$$\widetilde{C}_{coh}(\omega) = 2\sum_{n=0}^{\infty} \delta(\omega-(2n+1)\Omega)|M_{2n+1}|^2, \quad (8)$$

where

$$M_{2n+1} = \frac{1}{T}\int_0^T dt\, \langle x(t)\rangle_\infty e^{-i(2n+1)\Omega t}. \quad (9)$$

Only the odd harmonics contribute due to symmetry considerations. The only contribution to the numerator in Eq. (7) stems from the coherent part of the correlation function. Thus we write

$$R_{out}^{(n)} = \frac{Q_u^{(n)}}{Q_l^{(n)}}, \quad (10)$$

where

$$Q_u^{(n)} = \frac{2}{T}\int_0^T d\tau\, C_{coh}(\tau)\cos[(2n+1)\Omega\tau] = 2|M_{2n+1}|^2 \quad (11)$$

and

$$Q_l^{(n)} = \frac{2}{\pi}\int_0^\infty d\tau\, C_{incoh}(\tau)\cos[(2n+1)\Omega\tau]. \quad (12)$$

For an input signal $F(t)+\xi(t)$ we have

$$R_{in}^{(n)} = \frac{2|F_{2n+1}|^2}{\frac{2}{\pi}D} = \frac{\left(\frac{2A}{(2n+1)\pi}\right)^2[1-\cos(2n+1)\pi r]}{\frac{2}{\pi}D}. \quad (13)$$

The SR gains at the different overtones, $G^{(n)}$, are defined as the ratio of the SNR of the output at a given overtone over that of the input at the same overtone; namely,

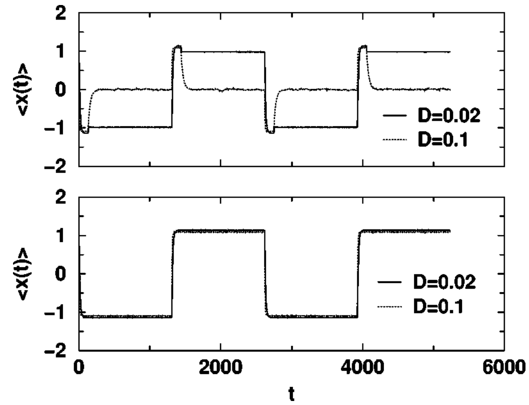

FIG. 1. Time behavior of $\langle x(t)\rangle_\infty$. The input signal has a subthreshold amplitude $A=0.35$, a fundamental frequency $\Omega=2\pi/T=0.0024$, and $r=0.1$ (upper panel), $r=1$ (lower panel) (all quantities in dimensionless units).

$$G^{(n)} = \frac{R_{out}^{(n)}}{R_{in}^{(n)}}. \quad (14)$$

$G^{(0)}$ values larger than 1 have been obtained in driven nondynamical systems [7], in stochastic resonators with static nonlinearities driven by square pulses [13], or in noisy bistable systems driven by superthreshold input sinusoidal frequencies [14]. The existence of SR gains with values larger than 1 indicates a truly nonlinear SR effect.

Although the two-state approximation introduced in Ref. [1] can, in principle, be extended to analyze systems driven by input signals with $r<1$, the analytical expressions obtained are too cumbersome to be of practical value. Thus, in the present work, we rely on the numerical treatment of the Langevin equation, Eq. (3), following the procedure detailed in Ref. [3].

The distortion of the output with respect to an input sinusoidal signal in a nonlinear regime has been discussed in Ref. [4], and in Refs. [2,15,16] for multifrequency inputs. The behavior of $\langle x(t)\rangle_\infty$ for an input signal of the type given in Eq. (1) with subthreshold amplitude $A=0.35$ and $\Omega=2\pi/T=0.0024$ is depicted in Fig. 1 for $r=0.1$ (short pulses) and $r=1$ (rectangular signals) and two representative noise values, $D=0.02$ and $D=0.1$. It is clear that for short driving pulses, the shape of the average output is distorted with respect to that of the input for low values of the noise strength, while the degree of distortion is minor as D is sufficiently large. On the other hand, for a rectangular signal ($r=1$), the shape of the output is similar to that of the input in the range of D considered but with a much larger amplitude. As noticed in Ref. [2], a rectangular signal can also be distorted at sufficiently low values of D. Below we will see that SR gains larger than 1 are possible for very strong and very weak distortion.

In Fig. 2, we depict the behavior of the SNR at the fundamental frequency $R_{out}^{(0)}$ with the noise strength D for input signals of the type given in Eq. (1), with subthreshold amplitude $A=0.35$, fundamental frequency $\Omega=0.0024$, and $r=0.1,0.4,0.7,0.95,0.98,1$. For all values of r, the nonmonotonic behavior of the SNR with D, typical of SR, is obtained.

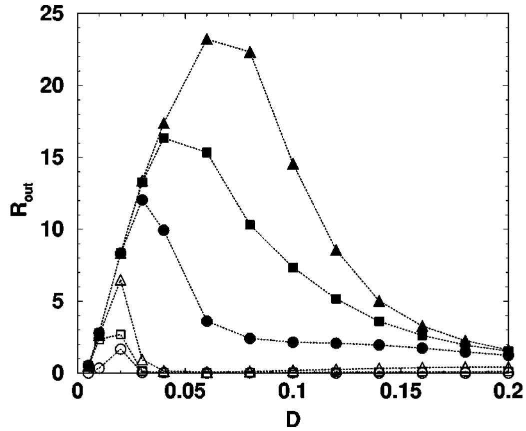

FIG. 2. The output signal-to-noise ratio $R_{out}^{(0)}$ vs the noise strength D for $r=0.1$ (circles), $r=0.4$ (squares), $r=0.7$ (triangles), $r=0.95$ (filled circles), $r=0.98$ (filled squares), and $r=1.0$ (filled triangles). The input signal has a subthreshold amplitude $A=0.35$ and a fundamental frequency $\Omega=2\pi/T=0.0024$ (all quantities in dimensionless units).

As r increases, the maximum value of $R_{out}^{(0)}$ increases. Namely, the longer the potential remains asymmetric during each half cycle, the larger the maximum height in the SNR is. Therefore $R_{out}^{(0)}$ is quite sensitive to the duration of the pulses within each half cycle.

In Fig. 3 the behavior of $G^{(0)}$ with the noise strength D is depicted for the same values of r as in Fig. 2. For all the cases, there exists a range of noise values such that $G^{(0)}$ is larger than unity. The peak value of $G^{(0)}$ has a nonmonotonic behavior with r. At the lowest value of r considered, the SR gain has a rather large peak value. Then, as r is increased, the peak of the SR gain decreases, in agreement with the observations in Ref. [8,9]. As the duration of the pulses gets larger so that r is closer to 1, the tendency of the SR gain maximum reverses and a considerable increase in the maximum is observed.

The features above can be rationalized by noting that the SR gain depends on $R_{out}^{(0)}$ and on $R_{in}^{(0)}$. As pointed out in Ref.

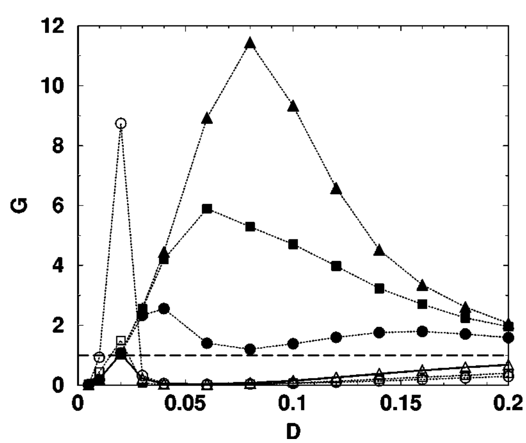

FIG. 3. The SR gain $G^{(0)}$ vs the noise strength D for the same parameter values as in Fig. 2. The solid line connects the G values for $r=0.7$ as a guide to depict the nonmonotonic behavior of $G^{(0)}$ with D, which is otherwise obscured by the filled symbols (all quantities in dimensionless units).

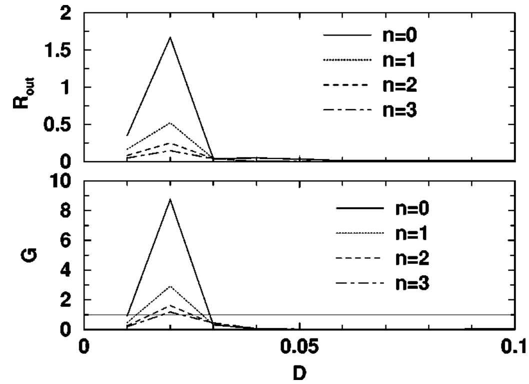

FIG. 4. SNR evaluated at the first few overtones, $R_{out}^{(n)}$ (upper panel) and $G^{(n)}$ (lower panel) vs D, for an input signal with $A=0.35$, $\Omega=0.0024$, $r=0.1$. The solid line in the lower panel is a guide to the eye to indicate the value 1 (all quantities in dimensionless units).

[9], for fixed values of the noise strength D and amplitude A, the input $R_{in}^{(0)}$ always increases with r [see Eq. (13)]. Also, for given values of A and r, $R_{in}^{(0)}$ decreases monotonically with D. On the other hand, the results depicted in Fig. 2 indicate that there are two main effects on the location of the maximum of $R_{out}^{(0)}$ as r increases. First, as noted before, the maximum height increases as r increases. Second, the maxima appear at increasingly larger values of D as the duration of the pulses increases. This second effect manifests itself clearly for pulses of sufficiently long duration, namely for r larger than ≈ 0.9, while it is almost unnoticeable for smaller values of r. For low values of r (let us say $r=0.1$), even though the peak of $R_{out}^{(0)}$ is the smallest one appearing in Fig. 2, the corresponding value of $R_{in}^{(0)}$ is so small (due to the smallness of r) that the SR gain reaches the large values depicted in Fig. 3. As r increases, the height of the $R_{out}^{(0)}$ maximum also increases, appearing at an approximately constant value of the noise strength. Thus the increase of $R_{in}^{(0)}$ with r counterbalances the increase of $R_{out}^{(0)}$ in such a way that the SR gain decreases. Finally, for long duration pulses, the shift to the right of the $R_{out}^{(0)}$ maximum and the large increase in its height are the cause of the increase in the maximum gain observed in Fig. 3.

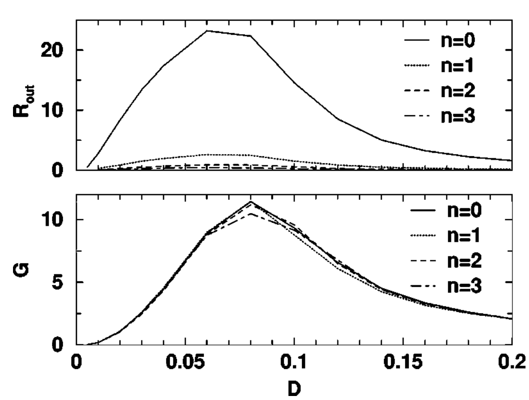

FIG. 5. SNR evaluated at the first few overtones, $R_{out}^{(n)}$ (upper panel) and $G^{(n)}$ (lower panel) vs D, for an input signal with $A=0.35$, $\Omega=0.0024$, $r=1$ (all quantities in dimensionless units).

Next, we evaluate the SNR and the gain at higher harmonics. In Fig. 4 we present the results for $R_{out}^{(n)}$ (upper panel) and $G^{(n)}$ (lower panel) for the first few harmonics and $r=0.1$. The SNR and the gain evaluated at the different overtones show peaks of decreasing heights around $D=0.01$. The gain associated to the first three harmonics is larger than unity for some region of D values, indicating a nonlinear behavior. Notice that for $D=0.02$, the average output is quite distorted with respect to the input shape [see Fig. 1 (upper panel)]. On the other hand, for $D=0.1$, when the distortion of the output signal is minor, $R_{out}^{(n)}$ and $G^{(n)}$ are very small.

In the next figure, Fig. 5, we evaluate the same quantities as in Fig. 4, but for an input signal with $r=1$. The nonmonotonic behavior of the different $R_{out}^{(n)}$ with D is clear. The shape of the output implies that $Q_u^{(n)}$ [see Eq. (11)] scales as $1/(2n+1)^2$. $Q_l^{(n)}$ is quite independent of n. The values should then decrease as $1/(2n+1)^2$ as observed. On the other hand, the behavior of $G^{(n)}$ (lower panel) with D is quite similar for all the different harmonics considered, as the behavior of $R_{out}^{(n)}$ with D is compensated by a similar behavior of $R_{in}^{(n)}$.

In conclusion, our study indicates that strong nonlinear SR effects show up in noisy bistable systems driven by subthreshold multifrequency periodic signals. This is manifested in the SNR and the gain associated to several overtones of the signal, not only for small values of r, but also for rectangular inputs ($r\approx 1$). Large SR gains are observed independently of the degree of distortion of the input.

We acknowledge the support of the Dirección General de Enseñanza Superior of Spain (BFM2002-03822) and the Junta de Andalucía.

[1] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. Lett. **91**, 210601 (2003).
[2] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. E **68**, 061104 (2003).
[3] J. Casado-Pascual, C. Denk, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. E **67**, 036109 (2003).
[4] M. I. Dykman, D. G. Luchinsky, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, Nuovo Cimento D **17**, 660 (1995).
[5] M. DeWeese and W. Bialek, Nuovo Cimento D **17**, 733 (1995).
[6] R. N. Mantegna, B. Spagnolo, and M. Trapanese, Phys. Rev. E **63**, 011101 (2000).
[7] K. Loerincz, Z. Gingl, and L. B. Kiss, Phys. Lett. A **224**, 63 (1996).
[8] Z. Gingl, P. Makra, and R. Vajtai, Fluct. Noise Lett. **1**, L181 (2001).
[9] P. Makra, Z. Gingl, and L. B. Kish, Fluct. Noise Lett. **2**, L147 (2002).
[10] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[11] P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989); Phys. Rev. A **44**, 8032 (1991).
[12] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[13] F. Chapeau-Blondeau, Phys. Lett. A **232**, 41 (1997).
[14] P. Hänggi, M. Inchiosa, D. Fogliatti, and A. R. Bulsara Phys. Rev. E **62**, 6155 (2000).
[15] M. Morillo and J. Gómez-Ordóñez, Phys. Rev. E **51**, 999 (1995).
[16] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Europhys. Lett. **58**, 342 (2002).