Analysis of a stochastic 2D–Navier-Stokes model with infinite delay

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Abstract

Some results concerning a stochastic 2D Navier-Stokes system when the external forces contain hereditary characteristics are established. The existence and uniqueness of solutions in the case of unbounded (infinite) delay are first proved by using the classical technique of Galerkin approximations. The local stability analysis of constant solutions (equilibria) is also carried out by exploiting two approaches. Namely, the Lyapunov function method and by constructing appropriate Lyapunov functionals. The asymptotic stability and hence, the uniqueness of equilibrium solution are obtained by constructing Lyapunov functionals. Moreover, some sufficient conditions ensuring the polynomial stability of the equilibrium solution in a particular case of unbounded variable delay will be provided. Exponential stability for other special cases of infinite delay remains as an open problem.

Key words: Stochastic Navier-Stokes equation, equilibrium solution, polynomial stability, unbounded variable delay.
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1. Introduction

In this paper we will analyze the following stochastic 2D-Navier-Stokes equation with infinite delay

$$\frac{du}{dt} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g_1(t,u_t) + g_2(t,u_t)\frac{dW(t)}{dt}, \text{ in } (\tau,T)\times\mathcal{O}, \tag{1.1}$$

$$div\ u \equiv 0, \text{ in } (\tau,T)\times\mathcal{O}, \tag{1.2}$$

$$u = 0, \text{ on } (\tau,T)\times\partial\mathcal{O}, \tag{1.3}$$

$$u(\tau+\theta,x) = \phi(\theta,x),\ \theta\in(-\infty,0],\ x\in\mathcal{O}, \tag{1.4}$$

where $\mathcal{O}\subset\mathbb{R}^2$ is a bounded open set with regular boundary $\partial\mathcal{O}$, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, ϕ is the initial datum, f is a nondelayed external force field, and g_1, g_2 are external forces containing some hereditary characteristics (memory, unbounded variable or distributed delay, etc), and $W(t)$ is a Wiener process on a suitable probability space to be described below.

As it is well known, Navier-Stokes equations (and its variants) are considered suitable models to describe the motion of many important fluids, like water, oil, air, etc, and its long-time behavior is considered

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as an interesting and important problem in the theory of fluid dynamics, and has been receiving much attention over the last decades (see [2, 3, 4, 18, 19, 36] and references therein).

Due to the importance of considering some delay terms in the models (for instance, in control problems, or when some memory is present in the real phenomenon), Caraballo and his collaborators have been investigating on delay versions of Navier-Stokes systems and its variants in the case of bounded delay. For instance, in the papers [5, 6, 7, 8, 10, 11, 14, 15] it has been developed an extensive theory for Navier-Stokes equations (and some variants) with bounded delay, considering the phase space C_H of continuous functions in a bounded interval with values in the Hilbert space H. In the case of unbounded or infinite delay, Marín-Rubio and his collaborators investigated their asymptotic behavior in the papers [20, 21, 29, 30, 31], by considering the phase space $C_\gamma(H)$ defined as

$$C_\gamma(H) = \Big\{\phi \in C((-\infty, 0]; H) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } H\Big\}\ (\gamma > 0).$$

We would like to mention [22, 23, 24] as general references for the existence of solutions to finite dimensional systems with infinite delay.

However, it is sensible to admit that the real phenomenon can be better described if some stochasticity or randomness are included in the equations, since uncertainty and noise are present almost everywhere in the real world. Therefore, it is very interesting to investigate about stochastic and delay versions of the Navier-Stokes equations. It is worth mentioning that, as far as we know, the literature on this topic is not very extensive. For instance, in [17, 37] the authors established some sufficient conditions for the exponential stability to stochastic Navier-Stokes equations with bounded variable delay, assuming the existence of solutions, and Taniguchi proved, in [35], the existence and asymptotic behavior of energy solutions to Navier-Stokes equations driven by Levy processes and external forcing terms with finite delay.

Up to date, we do not know any published work on stochastic Navier-Stokes equations with infinite delay, neither with distributed delay nor with unbounded variable delay. It is also worth mentioning that many authors used the phase space $C_\gamma(H)$ when dealt with differential equations with infinite delay in the deterministic framework, and proved exponential stability and convergence. Nevertheless, the methods that are used to prove exponential stability and the sufficient conditions stated in the literature only work for differential equations with distributed delay, and cannot be applied to the case of unbounded variable delay case, such as for example, the stochastic pantograph equation. Fortunately, Liu et al. [28] solved this problem, in the deterministic framework, by choosing

$$BCL_{-\infty}(H) = \Big\{\phi \in C((-\infty, 0]; H) : \lim_{\theta\to-\infty} \phi(\theta) \text{ exists in } H\Big\},$$

as the phase space, although only stability was established, but not asymptotic stability in general. However, in some special unbounded variable delay cases, it was proved the polynomial stability of steady-state solutions. Motivated by these previous results from the deterministic field, we will analyze a stochastic 2D-Navier-Stokes model with infinite delay. More precisely, we will first prove the existence and uniqueness of solution to Eq. (1.1), then focus on the stability analysis with unbounded variable delay, and will establish some stability results. The asymptotic stability with polynomial decay will be proved in some particular and special situation, for instance, in the case of proportional delay.

The stability results of stochastic Navier-Stokes equation with unbounded variable delay are new. Besides, enlightened by [13], which studied the exponential behavior and stability of stochastic Navier-Stokes equations, in the present work, we will weaken the conditions on the nonlinear stochastic term $g_2(t, u_t)$, but still ensuring the local stability.

The contents of the paper are as follows. In Section 2, we recall some notations, abstract spaces and operators which will be used along this paper. We will also establish the assumptions to be imposed on the delay terms and introduce some auxiliary lemmas. The existence and uniqueness of solutions to Eq. (1.1) will be carried out in Section 3. The main technique to prove the existence of solutions will be the classical Galerkin approximation. However, to establish the uniqueness of solution the already known technique and the Gronwall Lemma are not enough in the stochastic case because we cannot bound the trilinear term directly as it is usually done in the deterministic case. Therefore, new nontrivial technical lemmas are proved in order to overcome this difficulty. Two methods to analyze the asymptotic behavior of solutions to the problem are exploited in Section 4. Finally, some comments are included in Section 5.

2. Preliminaries

Although the notation and results included in this section may seem somehow repetitive, since they can be found in several already published references, we prefer to recall them for the sake of completeness and to make easier the reading of the paper.

Let us first consider the following usual abstract spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(O))^2 : div\ u = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(O))^2$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(O))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_O u_j(x) v_j(x) dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(O))^2$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(O))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_O \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. We will use $\|\cdot\|_*$ for the norm in V', and $\langle \cdot,\cdot \rangle$ for the duality pairing between V and V'. Now we define $A : V \to V'$ by $\langle Au, v \rangle = ((u, v))$, and the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_O u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall\ u, v, w \in V.$$

Note that the trilinear form b satisfies the following inequalities which will be used later in proofs (see [31, p. 2015])

$$|b(u, v, u)| \le |u|_4^2 \|v\| \le 2^{-1/2} |u| \|u\| \|v\|, \quad \forall\ u, v \in V. \tag{2.1}$$

Let $(\Omega, \mathfrak{F}, P)$ be a probability space on which an increasing and right continuous family $\{\mathfrak{F}_t\}_{t\in[0,\infty)}$ of complete sub-$\sigma-$algebra of $\mathfrak{F}$ is defined. Let $\beta_n(t) (n = 1, 2, 3, \cdots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathfrak{F}, P)$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n'} \beta_n(t) e_n, \quad t \ge 0,$$

where λ'_n $(n = 1, 2, 3, \cdots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda'_n < +\infty$, and $\{e_n\}$ $(n = 1, 2, 3, \cdots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Let $Q \in \mathcal{L}(K, K)$ be the operator defined by $Qe_n = \lambda'_n e_n$. The above $K-$valued stochastic process $W(t)$ is called a $Q-$Wiener process. Given real numbers $a < b$, and a separable Hilbert space $\mathcal{H}$, we will denote by $I^2(a, b; \mathcal{H})$ the space of all processes $X \in L^2(\Omega \times (a, b), \mathfrak{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$ (where $\mathcal{B}((a, b))$ denotes the Borel $\sigma-$algebra on (a, b)) such that $X(t)$ is $\mathfrak{F}_t-$measurable a.e. $t \in (a, b)$. The space $I^2(a, b; \mathcal{H})$ is a closed subspace of $L^2(\Omega \times (a, b), \mathfrak{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$.

We will denote by $C(a, b; \mathcal{H})$ the Banach space of all continuous functions from $[a, b]$ into $\mathcal{H}$ equipped with sup norm. We will write $L^2(\Omega; C(a, b; \mathcal{H}))$ instead of $L^2(\Omega, \mathfrak{F}, dP; C(a, b; \mathcal{H}))$.

Let us also consider a real number $T > 0$. If we consider a function $x \in C(-\infty, T; \mathcal{H})$, for each $t \in [0, T]$ we will denote $x_t \in C(-\infty, 0; \mathcal{H})$ the function defined by $x_t(\theta) = x(t + \theta)$, $\forall \theta \in (-\infty, 0]$. Moreover, if $y \in L^2(-\infty, T; \mathcal{H})$, we will also denote $y_t \in L^2(-\infty, 0; \mathcal{H})$, for a.e. $t \in (0, T)$, by $y_t(\theta) = y(t + \theta)$ a.e. $\theta \in (-\infty, 0]$.

Let us consider now the phase space

$$BCL_{-\infty}(H) = \left\{ \phi \in C((-\infty, 0]; H) : \lim_{\theta \to -\infty} \phi(\theta) \text{ exists in } H \right\},$$

which is a Banach space with the norm

$$\|\phi\|_{BCL_{-\infty}(H)} = \sup_{\theta \in (-\infty, 0]} |\phi(\theta)|.$$

We now enumerate the assumptions on the delay terms g_1, g_2. For $g_i : [\tau, T] \times BCL_{-\infty}(H) \to (L^2(O))^2$, $i = 1, 2$, we may assume:

(g1) For any $\xi \in BCL_{-\infty}(H)$, the mappings $[\tau, T] \ni t \mapsto g_i(t, \xi) \in (L^2(O))^2$ are measurable.

(g2) $g_i(\cdot, 0) = 0$.

(g3) For $i = 1, 2$, there exist $L_{g_i} > 0$ such that, for any $t \in [\tau, T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g_i(t, \xi) - g_i(t, \eta)| \le L_{g_i} \|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Remark 2.1. *(i) As pointed out in [31], condition* $(g2)$ *is not a restriction. Indeed, if* $|g_i(\cdot, 0)| \in L^2(\tau, T)$*, we could redefine* $\hat{f}_i(t) = f_i(t) + g_i(t, 0)$ *and* $\hat{g}_i(t, \cdot) = g_i(t, \cdot) - g_i(t, 0)$*. In this way the problem is exactly the same,* $\hat{f}$ *and* $\hat{g}$ *satisfy the required assumptions.*
(ii) Conditions $(g2)$ *and* $(g3)$ *imply that*

$$|g_i(t, \xi)| \le L_{g_i} \|\xi\|_{BCL_{-\infty}(H)}, \quad i = 1, 2,$$

whence $|g_i(t, \xi)| \in L^\infty(\tau, T)$.

Here, we exhibit two examples of delay forcing terms which satisfy $(g1) - (g3)$. Later on, to illustrate the different methods for the stability analysis, we focus on the unbounded variable delay case. Readers are referred to [28] for detailed proofs ensuring that these two examples satisfy $(g1) - (g3)$.

Example 1. A forcing term with unbounded variable delay

Let $G : [\tau, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a measurable function satisfying $G(t, 0) = 0$ for all $t \in [\tau, T]$, and assume that there exists $M > 0$ such that

$$|G(t, u) - G(t, v)|_{\mathbb{R}^2} \leq M|u - v|_{\mathbb{R}^2}, \quad \forall u, v \in \mathbb{R}^2.$$

Consider a function $\rho(\cdot) : [0, +\infty) \to [0, +\infty)$, which is going to play the role of the variable delay. Assume that $\rho(\cdot)$ is measurable and define $g(t, \xi)(x) = G(t, \xi(-\rho(t))(x))$ for each $\xi \in BCL_{-\infty}(H)$, $x \in \mathcal{O}$ and $t \in [\tau, T]$. Notice that, in this case, the delayed term g in our problem becomes

$$g(t, \xi) = G(t, \xi(-\rho(t))).$$

Examples 2. A forcing term with infinite distributed delay

Let $G : [\tau, T] \times \mathbb{R}_- \times \mathbb{R}^2 \to \mathbb{R}^2$ be measurable function satisfying $G(t, s, 0) = 0$ for all $(t, s) \in [\tau, T] \times (-\infty, 0]$, and there exists a function $\alpha(s) \in L^1(-\infty, 0)$ such that

$$|G(t, s, u) - G(t, s, v)|_{\mathbb{R}^2} \leq \alpha(s)|u - v|_{\mathbb{R}^2}, \quad \forall u, v \in \mathbb{R}^2, \ \forall (t, s) \in [\tau, T] \times (-\infty, 0].$$

Define $g(t, \xi)(x) = \int_{-\infty}^0 G(t, s, \xi(s)(x))ds$ for each $\xi \in BCL_{-\infty}(H)$, $t \in [\tau, T]$, and $x \in \mathcal{O}$. Then the delayed term g in our problem becomes

$$g(t, \xi) = \int_{-\infty}^0 G(t, s, \xi(s))ds.$$

Next we state the definition of weak solution for problem (1.1).

Definition 2.2. *Let $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ be an initial process where $\mathfrak{F}_t = \mathfrak{F}_0$ for $t \leq 0$. A stochastic process $u(t), t \in (-\infty, T]$, is said to be a weak solution of Eq.*(1.1)*, if*

(1a) $u(t)$ *is* $\mathfrak{F}_t$*-adapted, for* $t \leq T$*,*

(1b) $u(\cdot) \in I^2(-\infty, T; V) \cap L^2(\Omega; C(-\infty, T; H))$*,*

(1c) The following equation holds as an identity in V'*, a.s.*

$$u(t) = \phi(0) - \nu \int_0^t Au(s)ds - \int_0^t B(u(s))ds + \int_0^t (f(s) + g_1(s, u_s))\, ds + \int_0^t g_2(s, u_s)dW(s), \quad t \in [0, T].$$

(1d) $u(t) = \phi(t), \quad t \in (-\infty, 0]$*, a.s.*

The following lemma (see Sritharan and Sundar [34]) will play a crucial role in the proof of uniqueness of solution.

Lemma 2.3. *There exists* $\lambda > 0$ *such that for any* $u, v \in V$*,*

$$-2(B(u) - B(v), u - v) - \nu(A(u - v), u - v) \leq \lambda|v|_4^4|u - v|^2.$$

Finally, λ_1 will denote the number defined by

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{||v||^2}{|v|^2} > 0.$$

3. Existence and uniqueness of solutions

In this section we establish existence and uniqueness of weak solutions for Eq. (1.1). We begin with the uniqueness.

Lemma 3.1. *Let* $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ *be an initial function with* $E[\sup_{-\infty<s\le 0} |\phi(s)|^4] < \infty$. *If* $(g1)-(g3)$ *are fulfilled, then there exists at most one weak solution to* (1.1).

Proof. Let $u(t)$ and $v(t)$ be two solutions to (1.1) with the same initial value $u(s) = v(s) = \phi(s),\ s \le 0$. Let $N > 1$ be any fixed integer and

$$\tau_N := \inf\left\{t \le T : \int_0^t |v(s)|_4^4 ds \ge N\right\}.$$

Without loss of generality we may assume that $E\int_0^T |v(s)|_4^4 ds < \infty$. Actually, this is a direct consequence of Lemma 3.4. Set

$$r(t) := \exp\left(-\lambda \int_0^t |v(s)|_4^4 ds\right),$$

where $\lambda > 0$ is the one in Lemma 2.3. Hence,

$$r(t \wedge \tau_N) \ge \exp(-\lambda N).$$

Applying Itô's formula to the function $r(t)|u(t) - v(t)|^2$, we have that

$$\begin{aligned} r(t)|u(t) - v(t)|^2 = &-\lambda \int_0^t r(s)|v(s)|_4^4|u(s) - v(s)|^2 ds \\ &+ 2\int_0^t r(s)\,(u(s) - v(s), -\nu A(u(s) - v(s)) - B(u(s)) + B(v(s)))\,ds \\ &+ 2\int_0^t r(s)\,(u(s) - v(s), g_1(s, u_s) - g_1(s, v_s))\,ds \\ &+ 2\int_0^t r(s)\,(u(s) - v(s), g_2(s, u_s) - g_2(s, v_s))\,dW(s) \\ &+ \int_0^t r(s)|g_2(s, u_s) - g_2(s, v_s)|^2 ds. \end{aligned}$$

Thanks to Lemma 2.3, taking the supremum (w.r.t. t) and then taking expectation.

$$\begin{aligned} &E\left[\sup_{0\le l\le t\wedge\tau_N} r(l)|u(l) - v(l)|^2\right] + \nu E\int_0^{t\wedge\tau_N} r(s)\|u(s) - v(s)\|^2 ds \\ &\le 2E\left[\sup_{0\le l\le t\wedge\tau_N} \left|\int_0^l r(s)(u(s) - v(s), g_1(s, u_s) - g_1(s, v_s))ds\right|\right] \\ &\quad + 2E\left[\sup_{0\le l\le t\wedge\tau_N} \left|\int_0^l r(s)(u(s) - v(s), g_2(s, u_s) - g_2(s, v_s))dW(s)\right|\right] \\ &\quad + E\left[\sup_{0\le l\le t\wedge\tau_N} \int_0^l r(s)|g_2(s, u_s) - g_2(s, v_s)|^2 ds\right]. \end{aligned}$$

The first term on the right-hand side of the above inequality can be bounded by

$$
\begin{aligned}
&2E\left[\sup_{0\le l\le t\wedge\tau_N}|\int_0^l r(s)(u(s)-v(s),g_1(s,u_s)-g_1(s,v_s))ds|\right]\\
&\le \frac{1}{4}E\left[\sup_{0\le l\le t\wedge\tau_N} r(l)|u(l)-v(l)|^2\right]+4L_{g_1}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.
\end{aligned}
$$

On the other hand, Burkholder-Davis-Gundy's inequality yields

$$
\begin{aligned}
&2E\left[\sup_{0\le l\le t\wedge\tau_N}|\int_0^l r(s)(u(s)-v(s),g_2(s,u_s)-g_2(s,v_s))dW(s)|\right]\\
&\le 8E\left\{\sup_{0\le l\le t\wedge\tau_N} r^{1/2}(l)|u(l)-v(l)|\cdot[\int_0^t r(s)|g_2(s,u_s)-g_2(s,v_s)|^2_{L_2^0}ds]^{1/2}\right\}\\
&\le \frac{1}{4}E[\sup_{0\le l\le t\wedge\tau_N} r(l)|u(l)-v(l)|^2]+64L_{g_2}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.
\end{aligned}
$$

And

$$
E\left[\sup_{0\le l\le t\wedge\tau_N}\int_0^l r(s)|g_2(s,u_s)-g_2(s,v_s)|^2ds\right]\le L_{g_2}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.
$$

From the previous inequalities we have

$$
\begin{aligned}
&E[\sup_{0\le s\le t}|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2]+2\nu E\int_0^t r(s\wedge\tau_N)\|u(s\wedge\tau_N)-v(s\wedge\tau_N)\|^2ds\\
&\le (8L_{g_1}^2+130L_{g_2}^2)e^{\lambda N}\int_0^t E\sup_{0\le\tau\le s}|u(\tau\wedge\tau_N)-v(\tau\wedge\tau_N)|^2ds.
\end{aligned}
$$

By the Gronwall Lemma,

$$
E[\sup_{0\le s\le t}|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2]=0.
$$

Thus, for any fixed $N>1$,

$$
u(t\wedge\tau_N)=v(t\wedge\tau_N),\quad a.e.,\ \omega\in\Omega.
$$

By Markov's inequality,

$$
P(\tau_N<T)=P\left(\int_0^t|v(s)|_4^4ds\ge N\right)\le\frac{E\int_0^t|v(s)|_4^4ds}{N},
$$

since $E\int_0^t|v(s)|_4^4ds<\infty$, we obtain that $\tau_N\to T$ as $N\to\infty$. Consequently, $u(t)=v(t)$, a.e., $\omega\in\Omega$, for all $t\le T$. The proof is completed.

Remark 3.2. *In the previous proof, we used Markov's inequality, and that is why we need the fourth moment of solutions be finite, see [32] for more details about Markov's inequality.*

□

Let $0 < T \leq \infty$ and thus $T = \infty$ means $[0, T] = [0, \infty)$. Let $\{w_j\}_{j=1}^{\infty} \subset D(A)$ be a complete orthonormal basis in $(L^2(O))^2$. Now we use the Galerkin approximation method to prove the existence of weak solutions to Eq. (1.1). Set

$$u_n(t) = \sum_{j=1}^{n} \alpha_{nj}(t) w_j,$$

where $\alpha_{nj}(t)$ are determined by the following ordinary differential stochastic systems:

$$\begin{aligned}(u_n(t), w_j) = (u_{0n}, w_j) &+ \int_0^t (-\nu A u_n(s) + P_n B(u_n(s)) + P_n f(s), w_j) ds \\ &+ \int_0^t (P_n g_1(s, u_{ns}), w_j) ds + \int_0^t (P_n g_2(s, u_{ns}) dW(s), w_j), \quad j = 1, 2, \cdots, n,\end{aligned}$$

with an initial value $u_n(t) = P_n \phi(t)$, $t \in (-\infty, 0]$, where $u_{0n} = u_n(0) = P_n \phi(0) = \sum_{j=1}^{n} (u_0, w_j) w_j$, and $u_0 = \phi(0)$.

Consider the next stochastic equation

$$\begin{aligned}&u_n(t) = u_{0n} + \int_0^t (-\nu A u_n(s) + B(u_n(s)) + P_n f(s)) ds + \int_0^t P_n g_1(s, u_{ns}) ds + \int_0^t P_n g_2(s, u_{ns}) dW(s). \\ &u_n(t) = P_n \phi(t), t \in (-\infty, 0],\end{aligned}$$

where $u_{0n} = u_n(0) = P_n \phi(0)$.

Lemma 3.3. *Let $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ be an initial function with $E[\sup_{-\infty<s\leq 0} |\phi(s)|^4] < \infty$, with $u_0 = \phi(0)$. Assume that $(g_1) - (g_3)$ are satisfied. Then, if $f \in I^2(0, T; H)$, there exists a constant $c_0 > 0$ such that*

$$E[\sup_{0\leq s\leq t} |u_n(s)|^2] + \int_0^t E\|u_n(s)\|^2 ds \leq c_0, \quad \textit{uniformly in } n \geq 1.$$

Proof. By Itô's formula for $|u_n(t)|^2$,

$$\begin{aligned}|u_n(t)|^2 = |P_n u_0|^2 &+ 2\int_0^t (-\nu A u_n(s) - B(u_n(s)), u_n(s)) ds + 2\int_0^t (f(s) + g_1(s, u_{ns}), u_n(s)) ds \\ &+ 2\int_0^t (g_2(s, u_{ns}), u_n(s)) dW(s) + \int_0^t |g_2(s, u_{ns})|^2 ds.\end{aligned} \tag{3.1}$$

Note that $(u_n(t), -B(u_n(t))) = 0$. Taking supremum w.r.t t in (3.1) and expectation, we obtain

$$\begin{aligned}&E[\sup_{0\leq s\leq t} |u_n(s)|^2] + 2\nu \int_0^t E\|u_n(s)\|^2 ds \\ &\leq E|u_0|^2 + 2E[\sup_{0\leq\tau\leq t} \int_0^\tau |u_n(s)||f(s)| ds] + 2E[\sup_{0\leq\tau\leq t} \int_0^\tau |u_n(s)||g_1(s, u_{ns})| ds] \\ &\quad + 2E[\sup_{0\leq\tau\leq t} |\int_0^\tau (u_n(s), g_2(s, u_{ns})) dW(s)|] + E[\sup_{0\leq\tau\leq t} \int_0^\tau |g_2(s, u_{ns})|^2 ds] \\ &= E|u_0|^2 + J_1 + J_2 + J_3 + J_4.\end{aligned}$$

$$J_1 = 2E[\sup_{0\leq\tau\leq t} \int_0^\tau |u_n(s)||f(s)| ds] \leq \int_0^t E[\sup_{0\leq\tau\leq s} |u_n(\tau)|^2] ds + E\int_0^t |f(s)|^2 ds.$$

$$J_2 = 2E[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)||g_1(s,u_{ns})|ds] \le \frac{1}{4}E[\sup_{0\le s\le t}|u_n(s)|^2]$$
$$+ 4L_{g_1}^2\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^2]ds + 4L_{g_1}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4].$$

By Burkholder-Davis-Gundy's inequality,

$$\begin{aligned} J_3 &= 2E[\sup_{0\le\tau\le t}|\int_0^\tau (u_n(s), g_2(s,u_{ns}))dW(s)|] \\ &\le 8E[(\int_0^t |u_n(s)|^2|g_2(s,u_{ns})|^2 ds)^{1/2}] \\ &\le \frac{1}{4}E[\sup_{0\le s\le t}|u_n(s)|^2] + 64L_{g_2}^2\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^2]ds + 64L_{g_2}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4]. \end{aligned}$$

$$J_4 = E[\sup_{0\le\tau\le t}|\int_0^\tau |g_2(s,u_{ns})|^2 ds] \le L_{g_2}^2\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^2]ds + L_{g_2}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4].$$

$$\begin{aligned} &\frac{1}{2}E[\sup_{0\le s\le t}|u_n(s)|^2] + 2\nu\int_0^t E\|u_n(s)\|^2 ds \\ &\le E|u_0|^2 + E\int_0^t |f(s)|^2 ds + (4L_{g_1}^2 + 65L_{g_2}^2)E[\sup_{-\infty<s\le 0}|\phi(s)|^4] + (1 + 4L_{g_1}^2 + 65L_{g_2}^2)\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^2]ds. \end{aligned}$$

Then the conclusion follows directly from the Gronwall Lemma.

□

Lemma 3.4. *Let $\phi \in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$ be an initial function with $E[\sup_{-\infty<s\le 0}|\phi(s)|^4] < \infty$, and $u_0 = \phi(0)$. Assume that $(g1)-(g3)$ are satisfied. If $f \in I^4(0,T;H)$, then there exists $\delta > 0$, which is independent of n and will be specified later, such that $E\int_0^t |u_n(s)|_4^4 ds < \delta$.*

Proof. Applying the Itô formula to $|u_n(t)|^4$,

$$\begin{aligned} |u_n(t)|^4 &= |P_n u_0|^4 + 4\int_0^t |u_n(s)|^2(u_n(s), -\nu Au_n(s) - B(u_n(s)))ds + 4\int_0^t |u_n(s)|^2(f(s) + g_1(s,u_{ns}), u_n(s))ds \\ &\quad + 4\int_0^t |u_n(s)|^2(u_n(s), g_2(s,u_{ns}))dW(s) + 6\int_0^t |u_n(s)|^2|g_2(s,u_{ns})|_{L_2^0}^2 ds. \end{aligned}$$

Taking supremum and expectation,

$$\begin{aligned} &E\left[\sup_{0\le\tau\le t}|u_n(\tau)|^4\right] + 4\nu E\left[\int_0^t |u_n(s)|^2\|u_n(s)\|^2 ds\right] \\ &\le E|u_0|^4 + 2E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2(|f|^2 + |u_n(s)|^2)ds\right] + 2E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2(L_{g_1}^2|u_{ns}|^2 + |u_n(s)|^2)ds\right] \\ &+ 4E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2(u_n(s), g_2(s,u_{ns}))dW(s)\right] + 6L_{g_2}^2 E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2|u_{ns}|^2 ds\right] \\ &= E|u_0|^4 + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now we estimate $I_i,\ i = 1,2,3,4$, one by one.

$$I_1 = 2E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2(|f|^2 + |u_n(s)|^2)ds\right] \le 3\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^4]ds + E\int_0^t |f(s)|^4 ds.$$

$$I_2 = 2E\left[\sup_{0\le\tau\le t}\int_0^\tau |u_n(s)|^2(L_{g_1}^2|u_{ns}|^2 + |u_n(s)|^2)ds\right] \le 2(1+L_{g_1}^2)\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^4]ds + L_{g_1}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4].$$

Using Burkholder-Davis-Gundy's inequality,

$$I_3 \le \frac{1}{2}E[\sup_{0\le\tau\le t}|u_n(\tau)|^4] + 256L_{g_2}^2\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^4]ds + 256L_{g_2}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4].$$

$$I_4 \le 6L_{g_2}^2\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^4]ds + 6L_{g_2}^2 E[\sup_{-\infty<s\le 0}|\phi(s)|^4].$$

Consequently,

$$\frac{1}{2}E\left[\sup_{0\le\tau\le t}|u_n(\tau)|^4\right] + 4\nu E\left[\int_0^t |u_n(s)|^2\|u(s)\|^2 ds\right] \le c_f + c_g\int_0^t E[\sup_{0\le\tau\le s}|u_n(\tau)|^4]ds.$$

By the Gronwall Lemma, there exists a $C_0 > 0$ such that

$$E\left[\sup_{0\le\tau\le t}|u_n(\tau)|^4\right] + 8\nu E\left[\int_0^t |u_n(s)|^2\|u(s)\|^2 ds\right] \le C_0.$$

From inequality (2.1), we obtain that

$$|u_n(s)|_4 \le 2^{-\frac{1}{4}}|u_n(s)|^{1/2}\|u_n(s)\|^{1/2}.$$

Thus, there exists a constant $\delta = \frac{C_0}{16\nu}$ such that

$$E\int_0^t |u_n(s)|_4^4 ds \le \frac{1}{2}E\int_0^t |u_n(s)|^2\|u_n(s)\|^2 ds < \delta.$$

The proof is completed. □

Under more suitable assumption, we can prove the existence and uniqueness of solutions of our problem. There is a positive constant λ (the same as the one in Lemma 2.3) such that for all $u, v \in L^2(-\infty, T; V)$ and for all $t \in [0, T]$, it holds

$$\begin{aligned}
&\int_0^t |g_2(s,u_s) - g_2(s,v_s)|^2 ds - \nu\int_0^t \|u(s)-v(s)\|^2 ds \\
&\le \lambda\int_0^t |v(s)|_4^4|u(s)-v(s)|^2 ds + 2\int_0^t (B(u)-B(v), u(s)-v(s))ds \\
&\quad + \nu\int_0^t (Au(s)-Av(s), u(s)-v(s))ds - 2\int_0^t (g_1(s,u_s)-g_1(s,v_s), u(s)-v(s))ds.
\end{aligned} \tag{3.2}$$

We have the next theorem:

Theorem 3.5. *Let $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ be an initial function such that $E[\sup_{-\infty<s\le 0}|\phi(s)|^4] < \infty$, and $u_0 = \phi(0)$. Assume that $E\int_0^t |f(s)|^4 ds < \infty$ and that $(g1) - (g3)$ and (3.2) are fulfilled. Then, there exists a unique solution u to the problem*

$$\begin{aligned}
&u(t) = \phi(0) - \nu\int_0^t Au(s)ds - \int_0^t B(u(s))ds + \int_0^t (f(s) + g_1(s,u_s))\,ds + \int_0^t g_2(s,u_s)dW(s), \\
&u(t) = \phi(t), \quad t \in (-\infty, 0],
\end{aligned} \tag{3.3}$$

where the equation holds as an identity in V' almost surely for every $t \in [0, T]$.

Proof. By lemmas 3.3 and 3.4, there exists a subsequence $u_n(t)$ (relabeled the same) which converges weakly to $u(t) \in L^2(\Omega; L^\infty(0,T;H)) \cap L^2(\Omega \times [0,T]; V) \cap L^4(\Omega \times [0,T]; (L^4(O))^2)$. Moreover,

$$-\nu A u_n - B(u_n) \rightharpoonup \chi \text{ weakly in } L^2(\Omega \times [0,T]; V'),$$

$$g_1(t, u_{nt}) \rightharpoonup \zeta \text{ weakly in } L^2(\Omega \times [0,T]; H),$$

$$g_2(t, u_{nt}) \rightharpoonup \sigma \text{ weakly in } L^2(\Omega \times [0,T]; H).$$

We use the absolutely continuous function φ_k on $[0,T]$ with $\varphi_k' \in L^2(0,T)$ and $\varphi_k(T) = 0$ defined as follows

$$\varphi_k(s) = \begin{cases} 1 & 0 \le s \le t - \frac{1}{2k}, \\ \frac{1}{2} + k(t-s) & t - \frac{1}{2k} < s \le t + \frac{1}{2k}, \\ 0 & t + \frac{1}{2k} < s \le T. \end{cases}$$

Applying Itô's formula to $(u_n(s), \xi)\varphi_k(s)$, $\xi \in (H_0^1(O))^2$, we deduce

$$\begin{aligned} 0 = (u_{n0}, \xi)\varphi_k(0) - k \int_{t-\frac{1}{2k}}^{t+\frac{1}{2k}} (u_n(s), \xi) ds + \int_0^T (-\nu A u_n(s) - B(u_n(s)), \xi)\varphi_k(s) ds \\ + \int_0^T (g_1(s, u_{ns}), \xi)\varphi_k(s) ds + \int_0^T (g_2(s, u_{ns}), \xi)\varphi_k(s) dW(s) + \int_0^T (f(s), \xi)\varphi_k(s) ds. \end{aligned}$$

Let $k \to \infty$ in the previous inequality, then

$$\begin{aligned} (u_n(t), \xi) = (u_{n0}, \xi) + \int_0^T (-\nu A u_n(s) - B(u_n(s)), \xi) ds \\ + \int_0^T (g_1(s, u_{ns}), \xi) ds + \int_0^T (g_2(s, u_{ns}), \xi) dW(s) + \int_0^T (f(s), \xi) ds. \end{aligned}$$

Taking limits when $n \to \infty$,

$$u(t) = u_0 + \int_0^T (\chi(s) + f(s) + \zeta) ds + \int_0^T \sigma dW(s).$$

Define $\rho(t) = \int_0^t |z(s)|_4^4 ds$, $z \in L^4(\Omega \times [0,T]; (L^4(O))^2)$ and $z(s) = \phi(s)$, $s \le 0$. Using the Itô formula with $\exp(-\lambda\rho(t))|u(t)|^2$ and $\exp(-\lambda\rho(t))|u_n(t)|^2$, respectively,

$$\begin{aligned} E e^{-\lambda\rho(t)}|u(t)|^2 = E|u_0|^2 - E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4 |u(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi(s) + f(s) + \zeta, u(s)) ds \\ + E\int_0^t e^{-\lambda\rho(s)}|\sigma|^2 ds, \end{aligned}$$

and

$$\begin{aligned} E e^{-\lambda\rho(t)}|u_n(t)|^2 = E|u_{n0}|^2 - E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4 |u_n(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu A u_n(s) - B(u_n(s)) + f(s), u_n(s)) ds \\ + 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s, u_{ns}), u_n(s)) ds + E\int_0^t e^{-\lambda\rho(s)}|g_2(s, u_{ns})|^2 ds. \end{aligned}$$

Define α_n, β_n and γ_n as follows

$$\begin{aligned}\alpha_n &= -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u_n(s)-z(s)|^2ds + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s)-B(u_n(s)), u_n(s)-z(s))ds\\ &\quad - 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u_n(s)-z(s))ds + 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,u_{ns})-g_1(s,z_s), u_n(s)-z(s))ds\\ &\quad + E\int_0^t e^{-\lambda\rho(s)}|g_2(s,u_{ns})-g_2(s,z_s)|^2ds.\end{aligned}$$

$$\begin{aligned}\beta_n &= -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u_n(s)|^2ds + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s)-B(u_n(s)), u_n(s))ds\\ &\quad + 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,u_{ns}), u_n(s))ds + E\int_0^t e^{-\lambda\rho(s)}|g_2(s,u_{ns})|^2ds.\end{aligned}$$

$$\begin{aligned}\gamma_n &= -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4(|z(s)|^2-2(u_n(s),z(s)))ds\\ &\quad + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s)-B(u_n(s)), -z(s))ds - 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z(s)) + g_1(s,z_s), u_n(s)-z(s))ds\\ &\quad + 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,u_{ns}), -z(s))ds + E\int_0^t e^{-\lambda\rho(s)}(g_2(s,z_s)-2g_2(s,u_{ns}), g_2(s,z_s))ds.\end{aligned}$$

Obviously,

$$\alpha_n = \beta_n + \gamma_n.$$

By Lemma 2.3 and (3.2), we have $\alpha_n \le 0$.

$$\begin{aligned}0 &\ge \liminf_{n\to\infty} \alpha_n\\ &\ge -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)-z(s)|^2ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi, u(s)-z(s))ds\\ &\quad - 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u(s)-z(s))ds + 2E\int_0^t e^{-\lambda\rho(s)}(\zeta - g_1(s,z_s), u(s)-z(s))ds\\ &\quad + E\int_0^t e^{-\lambda\rho(s)}|\sigma - g_2(s,z_s)|^2ds.\end{aligned}$$

Take $z(t) = u(t)$ in the previous inequality, it follows that $\sigma = g_2(t,u_t)$, $t \in [0,T]$, where we use the fact that $e^{-\lambda\rho(t)}$ is bounded for $t \in [0,T]$. On the other hand, notice that

$$\beta_n = Ee^{-\lambda\rho(t)}|u_n(t)|^2 - E|u_n(0)|^2 - 2E\int_0^t e^{-\lambda\rho(s)}(f(s), u_n(s))ds.$$

$$\begin{aligned}\liminf_{n\to\infty}\beta_n &\ge Ee^{-\lambda\rho(t)}|u(t)|^2 - E|u(0)|^2 - 2E\int_0^t e^{-\lambda\rho(s)}(f(s), u(s))ds\\ &= -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)|^2ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta, u(s))ds\\ &\quad + E\int_0^t e^{-\lambda\rho(s)}|g_2(s,u_s)|^2ds.\end{aligned}$$

$$\liminf_{n\to\infty}\gamma_n \ge -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4(|z(s)|^2-2(u(s),z(s)))ds+2E\int_0^t e^{-\lambda\rho(s)}(\chi,-z(s))ds$$
$$-2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az-B(z),u(s)-z(s))ds-2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,z_s),u(s)-z(s))ds$$
$$+2E\int_0^t e^{-\lambda\rho(s)}(\zeta,-z(s))ds+E\int_0^t e^{-\lambda\rho(s)}(g_2(s,z_s)-2g_2(s,u_s),g_2(s,z_s))ds.$$

$$0\ge\liminf_{n\to\infty}\alpha_n=\liminf_{n\to\infty}\beta_n+\liminf_{n\to\infty}\gamma_n$$
$$\ge -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)-z(s)|^2ds+2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta,u(s)-z(s))ds$$
$$-2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az-B(z),u(s)-z(s))ds-2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,z_s),u(s)-z(s))ds$$
$$+E\int_0^t e^{-\lambda\rho(s)}|g_2(s,z_s)-g_2(s,u_s)|^2ds.$$

Thus

$$0\le E\int_0^t e^{-\lambda\rho(s)}|g_2(s,z_s)-g_2(s,u_s)|^2ds$$
$$\le 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az-B(z)+g_1(s,z_s),u(s)-z(s))ds-2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta,u(s)-z(s))ds$$
$$+\lambda E\int_0^t e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)-z(s)|^2ds.$$

For any fixed $w\in L^2(\Omega\times[0,T];V)\cap L^4(\Omega\times[0,T];(L^4(O))^2)$, set $z(t)=u(t)-\theta w(t)$, then

$$0\le 2E\int_0^t e^{-\lambda\rho(s)}(-\nu A(u-\theta w)-B(u-\theta w)+g_1(s,u_s-\theta w),w)ds$$
$$-2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta,w)ds+\theta\lambda E\int_0^t e^{-\lambda\rho(s)}|z(s)|_4^4|w|^2ds.$$

Let $\theta\to 0$, for any $w\in L^2(\Omega\times[0,T];V)\cap L^4(\Omega\times[0,T];(L^4(O))^2)$, we obtain

$$E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta+\nu Au(s)+B(u(s))-g_1(s,u_s),w)ds\le 0.$$

Since $L^2(\Omega\times[0,T];V)\cap L^4(\Omega\times[0,T];(L^4(O))^2)$ is dense in $L^2(\Omega\times[0,T];H)$,

$$e^{-\lambda\rho(t)}(\chi+\zeta+\nu Au(t)+B(u(t))-g_1(t,u_t))=0,\quad a.e.\ t\in[0,T],\quad \omega\in\Omega.$$

Hence,

$$u(t)=u_0-\int_0^t(\nu Au(s)+B(u(s)))ds+\int_0^t f(s)ds+\int_0^t g_1(s,u_s)ds+\int_0^t g_2(s,u_s)dW(s),\quad a.e.\ \omega\in\Omega.$$

Therefore, there exists a unique weak solution to (1.1) on $[0,T]$. This completes the proof. □

Corollary 3.6. *Assume that* $(g1)-(g3)$ *hold and* $E\int_0^t|f(s)|^4ds<\infty$. *If*

$$\nu\lambda_1>2L_{g_1}+L_{g_2}^2,$$

then, for every $\phi\in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$ *such that* $E[\sup_{-\infty<s\le 0}|\phi(s)|^4]<\infty$ *and* $\phi(0)=u_0$, *there exists a unique solution* u *to Eq.*(3.3) *with initial value* ϕ.

Proof. It is not difficulty to verify that assumption $\nu\lambda_1>2L_{g_1}+L_{g_2}^2$ implies (3.2). Therefore, the proof is finished thanks to Theorem 3.5. □

4. Asymptotic behavior of solutions

In this section, we analyze the long time behavior of solutions in a neighborhood of an (steady-state) equilibrium solution to (1.1). It is worth mentioning that the concept usually used as "equilibrium" in the context of stochastic differential equations is the so called "stationary solution", i.e., a stochastic process (solution) whose distribution is time-independent and generally is not a constant stochastic process. For instance, in reference [12], it is analyzed the existence and uniqueness of exponentially stable stationary solution that are stochastic solutions, where the finite-dimensional distributions are invariant under a suitable action of $\mathbb{R}^+$ on Ω. In some cases dealing with the theory of random dynamical systems, as it is the case in [12], when the random attractor becomes a single point, then it is a stationary solution in the previous sense. However it is our first objective in this paper, to analyze first the deterministic stationary solutions which are nothing but equilibria or steady-state solutions. We will call them equilibrium points or solutions. In future works we will analyze the existence and stability properties of stationary solutions not necessarily constant, as in the paper [12].

First, we state a general result ensuring the existence and uniqueness of equilibrium solutions. Then we show two methods that can be used to study the stability properties: the Lyapunov function method as well as the another one based on the construction of Lyapunov functionals. Both cases will be related to the model considered in Example 1, namely, the unbounded variable delay case. We also would like to point out that, although we will provide some sufficient condition ensuring the asymptotic stability of equilibrium solutions, to prove that this stability is indeed exponential remains as an open problem in general in the unbounded variable delay case. Nevertheless we will be able to prove polynomial asymptotic stability in some particular cases which have some relevance in applications.

4.1. Existence and uniqueness of equilibrium solutions

To investigate the existence and properties of equilibrium solutions to (4.1), we need to impose some extra assumptions. Namely, we assume that f is independent of time, i.e., $f(t) \equiv f \in V'$, and $g_i,\ i = 1, 2,$ defined now for all positive times, also is somehow *autonomous*. Indeed, if we assume directly $g_i,\ i = 1, 2$ to be autonomous, then the delay should have a distributed or fixed (constant) form, but an infinite variable delay would not be possible to be considered within our functional setting. Therefore the explicit presence of t in the operator cannot be removed if we wish to keep the variable delay case within our set-up. Namely, we introduce a new assumption for $g_i,\ i = 1, 2$. Denote by j the trivial immersion $j : H \to BCL_{-\infty}(H)$ given by $j(u) = \hat{u}$ with $\hat{u}(\theta) = u$ for all $\theta \le 0$. We require now that g_i fulfills

($g4$) $g_i(s, \xi) = g_i(t, \xi)$ $i = 1, 2$ for any $s, t \in \mathbb{R}_+$ and $\xi \in j(H)$.

If $(g2)-(g4)$ holds, we trivially have that $\tilde{g}_i : H \to (L^2(O))^2$ defined as $\tilde{g}_i(u) = g_i(0, j(u))$, i.e., $\tilde{g}_i = g_i|_{\mathbb{R}_+ \times j(H)}$, is of course autonomous, Lipschitz (with the same Lipschitz constant L_{g_i}) and $\tilde{g}_i(0) = 0$, $i = 1, 2$.

For convenience, we consider our model in an abstract formulation as

$$\frac{du}{dt} + \nu Au + B(u) = f + g_1(t, u_t) + g_2(t, u_t)\frac{dW}{dt}, \tag{4.1}$$

A equilibrium solution u_∞ to (4.1) must satisfy, almost surely,

$$\nu Au_\infty + B(u_\infty) = f + g_1(t, u_\infty) + g_2(t, u_\infty)\frac{dW}{dt},$$

By using $(g4)$ and then integral above equation on $[0, t]$, we have

$$(\nu A u_\infty + B(u_\infty) - f - \tilde{g_1}(u_\infty))\, t = \int_0^t \tilde{g_2}(u_\infty)\, dW(s), \quad \forall t > 0, \tag{4.2}$$

and, according to [13, Remark 3.1], this means that u_∞ must be a equilibrium solution of the deterministic equation as $t \to +\infty$, in other words

$$\nu A u_\infty + B(u_\infty) = f + \tilde{g_1}(u_\infty), \; P - almost\ surely, \tag{4.3}$$

which is an equality in V' and is a deterministic case of equation (4.2).

Thus, to discuss the stability of weak solutions to stochastic (4.1), we first need to consider the existence of equilibrium solutions to equation (4.3).

To carry out our analysis, we assume that the forcing terms g_1, g_2 are defined by

$$g_i(t, \hat{u}) = G_i(u), \; i = 1, 2, \text{for all } u \in H,$$

where $G_i : \mathbb{R}^2 \to \mathbb{R}^2, \; i = 1, 2$ are functions satisfying

$$G_1(0) = 0, \tag{4.4}$$

and there exists $M_i > 0, \; i = 1, 2$ for which

$$|G_i(u) - G_i(v)|_{\mathbb{R}^2} \le M_i |u - v|_{\mathbb{R}^2}, \; \forall u, v \in \mathbb{R}^2, \; i = 1, 2. \tag{4.5}$$

Recall that by $G_i(u)$ we denote the element in H defined by $G_i(u)(x) = G_i(u(x))$ for all $x \in \mathcal{O}$. Then equation (4.2) and equation (4.3) can be rewritten respectively as

$$\nu A u_\infty + B(u_\infty) - f - G_1(u_\infty) = G_2(u_\infty) \frac{W(t)}{t}, \quad \forall t > 0, \tag{4.6}$$

and

$$\nu A u_\infty + B(u_\infty) = f + G_1(u_\infty). \tag{4.7}$$

Remark 4.1. *As it is pointed out in [13, Remark 3.1], any equilibrium solution, for instance u_∞, to* (4.6) *is also a equilibrium solution to equation* (4.7) *, but it is possible that equation* (4.7) *possesses more than one equilibrium solution, for example u_1, and $u_1 \neq u_\infty$. However, if we assume that equation* (4.7) *has a unique equilibrium solution u_1, then $u_1 = u_\infty$, and in this case it must hold $G_2(u_\infty) = 0$ since* (4.6) *must be fulfilled for all $t > 0$.*

Now we present a sufficient condition which ensures the existence and uniqueness of a equilibrium solution to equation (4.7).

Theorem 4.2. *Suppose that* (4.4)-(4.5) *hold true and $\nu > \lambda_1^{-1} M_1$. Then,*

(a) for all $f \in V'$ there exists at least one equilibrium solution to (4.7)*;*
(b) if $f \in (L^2(\mathcal{O}))^2$, the equilibrium solutions belong to $D(A)$;
(c) if $(\nu - \lambda_1^{-1} M_1)^2 > (2\lambda_1)^{-\frac{1}{2}} \|f\|_*$, then the equilibrium solution to (4.7) *is unique.*

Proof. The proof can be carried out by a similar method to that of [13, Lemma 3.1, p.720] and this is why we omit the proof. Readers are also referred to [11, Theorem 4.1, p.1087] for more details. □

From now on, we always suppose that the stochastic equation (4.1) has a time-independent solution u_∞, which satisfies equation (4.6). Actually, this can happen when we take g_2 in such a way that g_2 vanishes at an equilibrium point, for instance, $g_2(t, u) = G_2(u - u_\infty)$. See [13, Remark 4.3] for more details.

4.2. Local stability: A direct approach

In this subsection, we prove the local stability of equilibrium solution by a straightforward way. Suppose that $\rho \in C^1([0,+\infty))$, $\rho(t) \geq 0$ for all $t \geq 0$ and $\rho_* = \sup_{t\geq 0} \rho'(t) < 1$.

Theorem 4.3. *Assume that the forcing term $g_i(t,u_t)$ are given by $g_i(t,u_t) = G_i(u(t-\rho(t)))$, $i = 1,2$, which satisfy (4.4)-(4.5). Assume that there exists $c_1 > 0$, depending only on $\mathcal{O}$, such that if $f \in (L^2(\mathcal{O}))^2$ and $\nu > \lambda_1^{-1}M_1 + (2\lambda_1)^{-\frac{1}{4}}\|f\|_*^{\frac{1}{2}}$ satisfies in addition*

$$2\nu > \frac{(2-\rho_*)M_1 + M_2^2}{\lambda_1(1-\rho_*)} + \frac{c_1|f|}{\lambda_1(\nu - \lambda_1^{-1}M_1)}. \tag{4.8}$$

Then there exists a unique equilibrium solution $u_\infty \in D(A)$ of (4.7), and for all $\phi \in I^2(-\infty,0;V) \cap L^2(\Omega; BCL_{-\infty}(H))$, the corresponding solution u of (1.1) with $f(t) \equiv f$ satisfies

$$E|u(t) - u_\infty|^2 \leq E|u_0 - u_\infty|^2 + \frac{M_1 + M_2^2}{(1-\rho_*)}\int_{-\infty}^0 E[|\phi(s) - u_\infty|^2]ds. \tag{4.9}$$

Proof. Let $f \in (L^2(\mathcal{O}))^2$ be fixed. Consider u the solution of (1.1) for $f(t) \equiv f$. By Theorem 4.2 (c), we obtain that equation (4.7) has a unique equilibrium solution $u_\infty \in D(A)$. Apply now Itô's formula to $|u(t) - u_\infty|^2$,

$$\begin{aligned}|u(t) - u_\infty|^2 = |u_0 - u_\infty|^2 &+ 2\int_0^t (-\nu A(u-u_\infty) - B(u) + B(u_\infty), u - u_\infty)ds \\ &+ 2\int_0^t (G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds \\ &+ 2\int_0^t (G_2(u(s-\rho(s))) - G_2(u_\infty), u - u_\infty)dW(s) \\ &+ \int_0^t |G_2(u(s-\rho(s))) - G_2(u_\infty)|_{L_2^0}^2 ds,\end{aligned} \tag{4.10}$$

and take then expectation,

$$\begin{aligned}E|u(t) - u_\infty|^2 = E|u_0 - u_\infty|^2 &- 2\nu\int_0^t E[\|u - u_\infty\|^2]ds - 2\int_0^t E(B(u) - B(u_\infty), u - u_\infty)ds \\ &+ 2\int_0^t E(G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds \\ &+ \int_0^t E|G_2(u(s-\rho(s))) - G_2(u_\infty)|_{L_2^0}^2 ds.\end{aligned} \tag{4.11}$$

Observe that

$$2(B(u) - B(u_\infty), u - u_\infty) = 2b(u - u_\infty, u_\infty, u - u_\infty) \leq \frac{c_1}{\sqrt{\lambda_1}}\|u - u_\infty\|^2\|u_\infty\|,$$

and since,

$$\nu A u_\infty + B(u_\infty) = f + G_1(u_\infty),$$

$$\|u_\infty\| \leq \frac{|f|}{\sqrt{\lambda_1}(\nu - \lambda_1^{-1}M_1)}.$$

On the other hand,

$$2\int_0^t E(G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds$$
$$\leq \frac{(2-\rho_*)M_1}{\lambda_1(1-\rho_*)}\int_0^t E[\|u-u_\infty\|^2]ds + \frac{M_1}{(1-\rho_*)}\int_{-\infty}^0 E[\sup_{-\infty<s\leq 0}|\phi(s)-u_\infty|^2]ds.$$

The last term on the right-hand side of (4.11) is bounded by

$$\int_0^t E|G_2(u(s-\rho(s))) - G_2(u_\infty)|^2_{L_2^0}ds \leq \frac{M_2^2}{\lambda_1(1-\rho_*)}\int_0^t E[\|u-u_\infty\|^2]ds + \frac{M_2^2}{(1-\rho_*)}\int_0^t E[\sup_{-\infty<s\leq 0}|\phi(s)-u_\infty|^2]ds.$$

Hence,

$$E|u(t)-u_\infty|^2 \leq E|u_0-u_\infty|^2 + \left(-2\nu + \frac{c_1|f|}{\lambda_1(\nu-\lambda_1^{-1}M_1)} + \frac{(2-\rho_*)M_1}{\lambda_1(1-\rho_*)} + \frac{M_2^2}{\lambda_1(1-\rho_*)}\right)\cdot\int_0^t E[\|u-u_\infty\|^2]ds$$
$$+ \frac{M_1+M_2^2}{(1-\rho_*)}\int_{-\infty}^0 E[\sup_{-\infty<s\leq 0}|\phi(s)-u_\infty|^2]ds.$$

Therefore, by (4.8) we have

$$E|u(t)-u_\infty|^2 \leq E|u_0-u_\infty|^2 + \frac{M_1+M_2^2}{(1-\rho_*)}E[|\phi(s)-u_\infty|^2_{L^2(-\infty,0;H)}].$$

The proof is complete. □

Remark 4.4. *In order to obtain that the weak solution to Eq.*(4.1) *converges exponentially to u_∞ and thus u_∞ is exponentially stable in the mean square by this technique, we need that $\rho(t)$ be bounded. See [28] for details.*

4.3. Asymptotic stability via Lyapunov method

In this subsection, we aim to show first the asymptotic stability of trivial solution by constructing suitable Lyapunov functionals of the following class of nonlinear stochastic partial differential equations, and later we will apply these abstract results to our Navier-Stokes model. See [16, 33] for more details.

Let us consider the following problem

$$\begin{aligned} du(t) &= (A(t,u(t)) + f(t,u_t))dt + g(t,u_t)dW(t), \quad t\in[0,T],\\ u(t) &= \phi(t), \quad t\in(-\infty,0], \end{aligned} \tag{4.12}$$

where $A(t,\cdot): V \to V'$ with $\langle A(t,u),u\rangle \leq 0$, for all $v\in V$, $f(t,\cdot): BCL_{-\infty}(H) \to H$ and $g(t,\cdot): BCL_{-\infty}(H) \to \mathcal{L}(K,H)$ satisfy the following Lipschitz conditions: there exist L_f, L_g such that for all $t\geq 0$ and all $\xi,\eta\in BCL_{-\infty}(H)$,

$$\begin{aligned} |f(t,\xi) - f(t,\eta)| &\leq L_f\|\xi-\eta\|_{BCL_{-\infty}(H)},\\ |g(t,\xi) - g(t,\eta)| &\leq L_g\|\xi-\eta\|_{BCL_{-\infty}(H)}. \end{aligned} \tag{4.13}$$

The existence and uniqueness of solution to (4.12) can be proved by a similar process as we did in Section 3. For a fixed $T>0$, given an initial value $\phi\in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$, a solution to (4.12) is a process $u(\cdot)\in I^2(-\infty,T;V)\cap L^2(\Omega;C(-\infty,T;H))$ such that

$$\begin{aligned} u(t) &= \phi(0) + \int_0^t A(s,u(s))ds + \int_0^t f(s,u_s)ds + \int_0^t g(s,u_s)dW(s), \quad t\in[0,T],\ P-a.s.\\ u(t) &= \phi(t), \quad t\in(-\infty,0], \end{aligned} \tag{4.14}$$

where the first equality is defined in V'.

From now on, we are interested in the long-time behavior of the solutions to (4.12). To this end, we need the Itô formula for the solutions of (4.14). We define an associate operator L which is usually called the "generator" of equation (4.14). To deal with the stochastic differential of the process $\sigma(t) = v(t, u(t))$, where $u(t)$ is a solution of equation (4.12), and the function $v(t, u) : [0, \infty) \times V \to \mathbb{R}_+$ has continuous partial derivatives

$$v'_t(t, u) = \frac{\partial v(t, u)}{\partial t}, \quad v'_u(t, u) = \frac{\partial v(t, u)}{\partial u}, \quad v''_{uu}(t, u) = \frac{\partial^2 v(t, u)}{\partial u^2},$$

the Itô formula for $\sigma(t)$ reads

$$d\sigma(t) = Lv(t, u(t))dt + \langle v'_u(t, u(t)), g(t, u_t)dW(t)\rangle,$$

where the generator L is defined in the following way

$$Lv(t, u(t)) = v'_t(t, u(t)) + \langle v'_u(t, u(t)), A(t, u_t) + f(t, u_t)\rangle + \frac{1}{2}Tr[v''_{uu}(t, u(t))g(t, u_t)Qg^*(t, u_t)].$$

The generator L can be applied also for some functionals $V(t, \phi) : [0, \infty) \times L^2(\Omega; BCL_{-\infty}(H)) \to \mathbb{R}_+$. Indeed, assume that a functional $V(t, \phi)$ can be represented in the form $V(t, \phi) = W(t, \phi(0), (\phi(\theta))_{\theta<0})$. When we particularize for fixed $\bar{\phi} = \bar{u}_t$, where $\bar{u}(\cdot)$ is a solution to (4.12), then we can define a function $v : [0, \infty) \times H \to \mathbb{R}_+$ as $v(t, u) = W(t, u, (\bar{u}_t(\theta))_{\theta<0})$, the terms appearing in the generator L for this function can be calculated as

$$v'_t(t, u) = \frac{\partial}{\partial t}\left[W(t, u, (\bar{u}_t(\theta))_{\theta<0})\right], \quad v'_u(t, u) = \frac{\partial}{\partial u}\left[W(t, u, (\bar{u}_t(\theta))_{\theta<0})\right], \quad v''_{uu}(t, u) = \frac{\partial^2}{\partial u^2}\left[W(t, u, (\bar{u}_t(\theta))_{\theta<0})\right].$$

Denoting by D be the set of functionals $V(t, \phi)$ which can be written as described above (i.e. $V(t, \phi) = W(t, \phi(0), (\phi(\theta))_{\theta<0})$), and which have a continuous derivative with respect to the variable t and two continuous derivatives with respect to the variable u, then the generator for the solution $\bar{u}(\cdot)$ of (4.12), according to the functional $V(t, \phi)$ above, reads as

$$LV(t, \bar{u}_t) = v'_t(t, \bar{u}(t)) + \langle v'_u(t, \bar{u}(t)), A(t, \bar{u}(t)) + f(t, \bar{u}_t)\rangle + \frac{1}{2}Tr[v''_{uu}(t, \bar{u}(t))g(t, \bar{u}_t)Qg^*(t, \bar{u}_t)].$$

For functionals from D, the Itô formula implies

$$E[V(t, u_t) - V(s, u_s)] = \int_s^t ELV(r, u_r)dr, \quad t \geq s. \tag{4.15}$$

Next proposition is a generalization of Theorem 2.1 in [33, p. 34] to an infinite dimensional framework. More precisely, Theorem 2.1 in [33] was stated and proved for stochastic ordinary differential equations with finite delays while we will prove it for stochastic partial differential equations with unbounded delays.

Proposition 4.5. *Let $V(t, \phi) : [0, \infty) \times L^2(\Omega; BCL_{-\infty}(H)) \to \mathbb{R}_+$ be a continuous functional such that for any solution $u(t)$ of problem (4.12) with $p \geq 2$, the following inequalities hold:*

$$\begin{aligned}
&EV(t, u_t) \geq \gamma_1 E|u(t)|^p, \quad \forall t \geq 0, \\
&EV(0, \phi) \leq \gamma_2 \|\phi\|_1^p, \\
&E[V(t, u_t) - V(0, \phi)] \leq -\gamma_3 \int_0^t E|u(s)|^p ds, \quad t \geq 0,
\end{aligned}$$

where $\|\phi\|_1^p := \sup_{\theta\le 0} E|\phi(\theta)|^p$*. Then the trivial solution of* (4.12) *is asymptotically p-stable (i.e. asymptotically stable in the pth-moment).*

Proof. For simplicity, we only prove the case when $p = 2$, although the proof for $p \neq 2$ can be obtained in a similar way. By the assumption, we know that for any $\phi \in L^2(\Omega; BCL_{-\infty}(H))$,

$$\gamma_1 E|u(t)|^2 \le EV(t, u_t) \le EV(0, \phi) \le \gamma_2 \|\phi\|_1^2 = \gamma_2 \sup_{\theta\le 0} E|\phi(\theta)|^2, \tag{4.16}$$

which implies that the trivial solution is stable.

Notice that, by (4.16), we have

$$\sup_{t\ge 0} E|u(t)|^2 \le \frac{\gamma_2}{\gamma_1}\|\phi\|_1^2. \tag{4.17}$$

On the other hand, it follows from the condition of this proposition, we find

$$\int_0^\infty E|u(s)|^2 ds \le \frac{1}{\gamma_3} EV(0, \phi) \le \frac{\gamma_2}{\gamma_3}\|\phi\|_1^2 < \infty, \tag{4.18}$$

Estimating the generator L to $U(t, u_t) = |u(t)|^2$ and using (4.13), we obtain

$$\begin{aligned}
ELU(t, u_t) = EL|u(t)|^2 &= 2E(u(t), A(t, u(t)) + f(t, u_t)) + E|g_2(t, u_t)|^2 \\
&\le 2E(u(t), f(t, u_t)) + E|g_2(t, u_t)|^2 \\
&\le E|u(t)|^2 + (L_f^2 + L_g^2)E|u_t|^2_{BCL_\infty(H)} \\
&= E|u(t)|^2 + (L_f^2 + L_g^2)E\sup_{\theta\le 0}|u(t+\theta)|^2 \\
&\le E|u(t)|^2 + (L_f^2 + L_g^2)E\sup_{\theta\le -t}|u(t+\theta)|^2 + (L_f^2 + L_g^2)E\sup_{\theta> -t}|u(t+\theta)|^2 \\
&\le E|u(t)|^2 + (L_f^2 + L_g^2)\|\phi\|_1^2 + (L_f^2 + L_g^2)\sup_{t\ge 0} E|u(t)|^2 \\
&\le \gamma_4,
\end{aligned} \tag{4.19}$$

where γ_4 is a positive constant. Since from (4.15),

$$E[U(t, u_t) - U(s, u_s)] = \int_s^t ELU(r, u_r)dr,$$

then by (4.19), we have for any $t_2 \ge t_1 \ge 0$,

$$|E|u(t_2)|^2 - E|u(t_1)|^2| \le \gamma_4(t_2 - t_1),$$

which means that the function $E|u(t)|^2$ is Lipschitz, and on account of (4.17)-(4.18), we obtain that $\lim_{t\to+\infty} E|u(t)|^2 = 0$. Therefore, the proof is finished. □

Theorem 4.6. *Assume that the forcing terms* $g_i(t, u_t)$ *are given by* $g_i(t, u_t) = G_i(u(t - \rho(t))), i = 1, 2,$ *which satisfy* (4.4)-(4.5)*. Let* $f = 0$ *and*

$$2\nu > \frac{(2 - \rho_*)M_1 + M_2^2}{\lambda_1(1 - \rho_*)},$$

then there exists a unique equilibrium solution $u_\infty = 0$ *to the problem* (4.7)*, and any weak solution* $u(t)$ *to* (4.1) *converges to zero in mean square. Then zero is asymptotically mean-square stable.*

Proof. We prove this theorem by constructing a Lyapunov functional following the general method of construction described in [9, 16, 27]. Set

$$V(t,\phi) = |\phi(0)|^2 + \frac{M_1 + M_2^2}{1-\rho_*}\int_{-\rho(t)}^{0} |\phi(s)|^2 ds.$$

Then we replace ϕ by u_t, and obtain

$$\begin{aligned} V(t,u_t) &= |\phi(0)|^2 + \frac{M_1 + M_2^2}{1-\rho_*}\int_{-\rho(t)}^{0} |\phi(s)|^2 ds \\ &= |u(t)|^2 + \frac{M_1 + M_2^2}{1-\rho_*}\int_{t-\rho(t)}^{t} |u(s)|^2 ds. \end{aligned}$$

Therefore,

$$\begin{aligned} LV(t,u_t) \le\ & 2(-\nu Au(t) - B(u(t)) + G_1(u(t-\rho(t))), u(t)) + |G_2(u(t-\rho(t)))|^2 \\ & + \frac{M_1 + M_2^2}{1-\rho_*}|u(t)|^2 - (M_1 + M_2^2)|u(t-\rho(t))|^2 \\ \le\ & -2\nu\|u(t)\|^2 + 2(G_1(u(t-\rho(t))), u(t)) + |G_2(u(t-\rho(t)))|^2 \\ & + \frac{M_1 + M_2^2}{1-\rho_*}|u(t)|^2 - (M_1 + M_2^2)|u(t-\rho(t))|^2 \\ \le\ & (-2\nu\lambda_1 + M_1)|u(t)|^2 + (M_1 + M_2^2)|u(t-\rho(t))|^2 + \frac{M_1 + M_2^2}{1-\rho_*}|u(t)|^2 - (M_1 + M_2^2)|u(t-\rho(t))|^2. \end{aligned}$$

Thanks to the above inequalities and the fact that $2\nu > \frac{(2-\rho_*)M_1 + M_2^2}{\lambda_1(1-\rho_*)}$, we obtain that there exists a positive constant γ, such that the Lyapunov functional $V(t,u_t)$ fulfills

$$LV(t,u_t) \le (-2\nu\lambda_1 + \frac{(2-\rho_*)M_1 + M_2^2}{(1-\rho_*)})|u(t)|^2 \le -\gamma|u(t)|^2 \le 0.$$

Therefore, the functional $V(t,u_t) = |u(t)|^2 + \frac{M_1+M_2^2}{1-\rho_*}\int_{t-\rho(t)}^{t}|u(s)|^2 ds$ satisfies the conditions in Proposition 4.5, thus the trivial solution of (4.7) is asymptotically mean-square stable, which also means that the equilibrium solution to (4.7) is unique. □

Remark 4.7. *As we can see from this proposition, by constructing appropriate Lyapunov functionals, we obtain the asymptotic stability of the trivial solution to* (4.7)*, which moreover ensures the uniqueness of this (steady-state) equilibrium solution. In this sense, the result we obtain by constructing Lyapunov functionals improves the former local stability of equilibrium solution by a direct approach. Notice that we have developed the result in the case that the unique equilibrium solution is* $u_\infty = 0$. *The general case can be considered by the classical shift of the equilibrium solution to the origin.*

4.4. Polynomial asymptotic stability for a special case

Up to now, two different methods have been used to study the stability of the solution to (1.1). However, instead of exponential stability, only local stability and asymptotical stability of equilibrium solution have been obtained. Actually, even for a very simple differential equation with unbounded variable delay, for example, the stochastic pantograph equation, in which the delay is given by $\rho(t) = (1-\lambda)t$ with $0 < \lambda < 1$, the exponential stability cannot be achieved. However, in this case, the polynomial decay to the equilibrium solution can be ensured. In fact, the polynomial stability is the best result we could obtain for this case, see [1, 25, 26] for details. In [1] authors studies the asymptotic growth and decay properties of solutions of the stochastic pantograph equation with multiplicative noise. Under appropriate conditions the solutions of stochastic pantograph equation decay to equilibrium solution with a polynomial rate in p-th mean and in the almost sure sense, which means that the equilibrium solution to the stochastic pantograph equation is polynomially stable . For more relevant references, readers are referred to references mentioned in [1] equilibrium solution

At light of the results in [1], in this subsection, we prove the polynomial stability of equilibrium solution to (1.1) with proportional delay. To do this, we need to introduce the following stochastic pantograph equation and some technical lemmas that are needed later.

An example of the deterministic pantograph equation reads

$$x'(t) = \bar{a}x(t) + \bar{b}x(\lambda t), \quad \forall t \geq 0, \ x(0) = x_0, \ \lambda \in (0, 1), \tag{4.20}$$

which has been studied in [1, 25, 26]. Recall that, for a continuous real-valued function h of a real variable, the Dini derivative D^+h is defined as

$$D^+h = \limsup_{\delta \downarrow 0} \frac{h(t+\delta) - h(t)}{\delta}.$$

The following lemma will be useful later.

Lemma 4.8. *(See [1, Lemma 3.4]) Let* $\bar{a} \in \mathbb{R}$, $\bar{b} > 0$, $\lambda \in (0, 1)$. *Assume* x *satisfies*

$$x'(t) = \bar{a}x(t) + \bar{b}x(\lambda t), \quad t \geq 0, \tag{4.21}$$

where $x(0) > 0$ *and suppose* $t \mapsto p(t)$ *is a continuous non-negative function defined on* $\mathbb{R}^+$ *satisfying*

$$D^+ p(t) \leq \bar{a}p(t) + \bar{b}p(\lambda t), \quad t \geq 0 \tag{4.22}$$

with $0 < p(0) < x(0)$. *Then* $p(t) \leq x(t)$ *for all* $t \geq 0$.

Lemma 4.9. *(See [1, Lemma 3.5]) Let* x *be the solution of* (4.20). *If* $\bar{a} < 0$, $\bar{b} \in \mathbb{R}$ *there exits* $C_1 = C_1(\bar{a}, \bar{b}, \lambda) > 0$ *such that*

$$\limsup_{t \to +\infty} \frac{|x(t)|}{t^\mu} = C_1|x(0)|$$

where $\mu \in \mathbb{R}$ *obeys*

$$0 = \bar{a} + |\bar{b}|\lambda^\mu. \tag{4.23}$$

Then, for some $C = C(\bar{a}, \bar{b}, \lambda) > 0$, *we have*

$$|x(t)| \leq C|x(0)|(1 + t)^\mu, \quad t \geq 0. \tag{4.24}$$

Notice that if $\mu < 0$, then (4.24) implies polynomial stability of the zero solution of (4.20). Now we use this idea to prove the polynomial stability of equilibrium solution of (1.1).

Theorem 4.10. *Consider* (1.1) *with* $f = 0$, $g_1(t, u_t) = L_{g_1}u(qt)$, $g_2(t, u_t) = L_{g_2}u(qt)$ *with* $0 < q < 1$ *and* $\lambda_1\nu > 2|L_{g_1}| + L_{g_2}^2$, *then there exists a unique trivial solution* $u = 0$ *of* (1.1), *and all the solutions of* (1.1) *converge to zero polynomially, namely, there exist* $C > 0$ *and* $\mu < 0$ *such that*

$$E|u(t)|^2 < CE|u(0)|^2(1 + t)^{\mu}, \quad \text{for all } t \geq 0 \tag{4.25}$$

where μ *satisfies* $|L_{g_1}| - 2\nu\lambda_1 + (|L_{g_1}| + L_{g_2}^2)q^{\mu} = 0$.

Proof. Applying Itô's formula to $|u(t)|^2$, using a similar scheme as in Theorem 4.1 in [1], we obtain

$$E[|u(t + h)|^2] - E[|u(t)|^2] \leq (-2\nu\lambda_1 + |L_{g_1}|)E\int_t^{t+h} |u(s)|^2 ds + (|L_{g_1}| + L_{g_2}^2)E\int_t^{t+h} |u(qs)|^2 ds.$$

Denote by $w(t) = E|u(t)|^2$,

$$w'(t) \leq (-2\lambda_1\nu + |L_{g_1}|)w(t) + (|L_{g_1}| + L_{g_2}^2)w(qt). \tag{4.26}$$

By Lemma 4.8-4.9, there exist $C = C(L_{g_1}, L_{g_2}, \lambda_1, \nu) > 0$ and $\mu \in \mathbb{R}$ such that

$$w(t) \leq Cw(0)(1 + t)^{\mu}, \tag{4.27}$$

Since $-2\lambda_1\nu + 2|L_{g_1}| + L_{g_2}^2 < 0$, it holds that $\mu < 0$, and

$$E|u(t)|^2 \leq CE|u(0)|^2(1 + t)^{\mu}.$$

Then the polynomial decay of solutions follows directly. □

Remark 4.11. *(i) In this special case,* $g_i(t, u_t) = L_{g_i}u(qt)$, $i = 1, 2$ *with* $0 < q < 1$ *and* $f \equiv 0$. *As long as we have* $\lambda_1\nu > 2|L_{g_1}| + L_{g_2}^2$, *then we can prove that the solution converges polynomially to zero. In this sense, this result improves the stability results we established previously.*

(ii) In fact, our result can be extended to a more general case, namely, if the delay term $g(t, \phi)$ *is defined as* $g(t, \phi) = G(\phi(-(1 - \lambda)t))$, *with* G *satisfying a Lipschitz condition with Lipschitz constant* L_g.

5. Comments

In this work, we analyzed the existence and uniqueness of solutions of a 2D stochastic Navier-Stokes equation with infinite delay. We proved stability and asymptotic stability of the equilibrium solutions but were not able to obtain any result on the exponential stability, but on asymptotic stability and with polynomial decay in some particular case. These results are in concordance with what was pointed in [1], precisely, that convergence to equilibria need not be at an exponential rate for equations with unbounded delay. Actually, for unbounded variable delay, even in the simplest case, i.e., the stochastic pantograph equation, only polynomial stability can be obtained because the solutions behave in such polynomial way. Consequently, it is still an open problem to obtain sufficient conditions for the exponential stability of solutions for equations with other types of unbounded variable delay. We plan to investigate this direction in future.

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References

[1] J. A. D. Appleby and E. Buckwar. Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation. *Electron. J. Qual. Theory Differ. Equ.*, pages Paper No. 2, 32, 2016.

[2] A. Babin, A. Mahalov, and B. Nicolaenko. Global regularity of 3D rotating Navier-Stokes equations for resonant domains. *Appl. Math. Lett.*, 13(4):51–57, 2000.

[3] A. V. Babin. Attractors of Navier-Stokes equations. In *Handbook of mathematical fluid dynamics, Vol. II*, pages 169–222. North-Holland, Amsterdam, 2003.

[4] A. Babin, A. Mahalov, and B. Nicolaenko. Global regularity of 3D rotating Navier-Stokes equations for resonant domains. *Indiana Univ. Math. J.*, 48(3):1133–1176, 1999.

[5] T. Caraballo, A. M. Márquez-Durán, and J. Real. Asymptotic behaviour of the three-dimensional α-Navier-Stokes model with delays. *J. Math. Anal. Appl.*, 340(1):410–423, 2008.

[6] T. Caraballo, A. M. Márquez-Durán, and J. Real. Asymptotic behaviour of the three-dimensional α-Navier-Stokes model with locally Lipschitz delay forcing terms. *Nonlinear Anal.*, 71(12):e271–e282, 2009.

[7] T. Caraballo and J. Real. Attractors for 2D-Navier-Stokes models with delays. *J. Differential Equations*, 205(2):271–297, 2004.

[8] T. Caraballo and J. Real. Asymptotic behaviour and attractors of 2D-Navier-Stokes models with delays. In *EQUADIFF 2003*, pages 827–832. World Sci. Publ., Hackensack, NJ, 2005.

[9] T. Caraballo, J. Real, and L. Shaikhet. Method of Lyapunov functionals construction in stability of delay evolution equations. *J. Math. Anal. Appl.*, 334(2):1130–1145, 2007.

[10] T. Caraballo and X. Han. Stability of stationary solutions to 2D-Navier-Stokes models with delays. *Dyn. Partial Differ. Equ.*, 11(4):345–359, 2014.

[11] T. Caraballo and X. Han. A survey on Navier-Stokes models with delays: existence, uniqueness and asymptotic behavior of solutions. *Discrete Contin. Dyn. Syst. Ser. S*, 8(6):1079–1101, 2015.

[12] T. Caraballo, P. Kloeden, and B. Schmalfuß. Exponentially stable stationary solutions for stochastic evolution equations and their perturbation. *Appl. Math. Optim.*, 50(3):4183–207, 2004.

[13] T. Caraballo, J. A. Langa, and T. Taniguchi. The exponential behaviour and stabilizability of stochastic 2D-Navier-Stokes equations. *J. Differential Equations*, 179(2):714–737, 2002.

[14] T. Caraballo and J. Real. Navier-Stokes equations with delays. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457(2014):2441–2453, 2001.

[15] T. Caraballo and J. Real. Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459(2040):3181–3194, 2003.

[16] T. Caraballo and L. Shaikhet. Stability of delay evolution equations with stochastic perturbations. *Commun. Pure Appl. Anal.*, 13(5):2095–2113, 2014.

[17] H. Chen. Asymptotic behavior of stochastic two-dimensional Navier-Stokes equations with delays. *Proc. Indian Acad. Sci. Math. Sci.*, 122(2):283–295, 2012.

[18] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[19] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[20] J. García-Luengo, P. Marín-Rubio, and J. Real. Pullback attractors for 2D Navier-Stokes equations with delays and their regularity. *Adv. Nonlinear Stud.*, 13(2):331–357, 2013.

[21] J. García-Luengo, P. Marín-Rubio, and J. Real. Regularity of pullback attractors and attraction in H^1 in arbitrarily large finite intervals for 2D Navier-Stokes equations with infinite delay. *Discrete Contin. Dyn. Syst.*, 34(1):181–201, 2014.

[22] J. K. Hale. Retarded equations with infinite delays. In *Functional differential equations and approximation of fixed points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978)*, volume 730 of *Lecture Notes in Math.*, pages 157–193. Springer, Berlin, 1979.

[23] J. K. Hale and J. Kato. Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.*, 21(1):11–41, 1978.

[24] Y. Hino, S. Murakami, and T. Naito. *Functional-differential equations with infinite delay*, volume 1473 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[25] T. Kato. Asymptotic behavior of solutions of the functional differential equation $y^{'}(x) = ay(\lambda x)+by(x)$. pages 197–217, 1972.

[26] T. Kato and J. B. McLeod. The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. Amer. Math. Soc.*, 77:891–937, 1971.

[27] V. Kolmanovskii and L. Shaikhet. Construction of Lyapunov functionals for stochastic hereditary systems: a survey of some recent results. *Math. Comput. Modelling*, 36(6):691–716, 2002. Lyapunov's methods in stability and control.

[28] L. Liu, T. Caraballo, and P. Marín Rubio. Asymptotic behavior of 2d-Navier-Stokes equations with infinite delay. *J. Diff. Equ.(Accepted)*.

[29] P. Marín-Rubio, A. M. Márquez-Durán, and J. Real. Three dimensional system of globally modified Navier-Stokes equations with infinite delays. *Discrete Contin. Dyn. Syst. Ser. B*, 14(2):655–673, 2010.

[30] P. Marín-Rubio, A. M. Márquez-Durán, and J. Real. Pullback attractors for globally modified Navier-Stokes equations with infinite delays. *Discrete Contin. Dyn. Syst.*, 31(3):779–796, 2011.

[31] P. Marín-Rubio, J. Real, and J. Valero. Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case. *Nonlinear Anal.*, 74(5):2012–2030, 2011.

[32] C. C. Read. *Markov Inequality*, volume 7 of *Encyclopedia of Statistical Sciences*. John Wiley Sons, 2006.

[33] L. Shaikhet. *Lyapunov functionals and stability of stochastic functional differential equations*. Springer, Cham, 2013.

[34] S. S. Sritharan and P. Sundar. Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise. *Stochastic Process. Appl.*, 116(11):1636–1659, 2006.

[35] T. Taniguchi. The existence and asymptotic behaviour of energy solutions to stochastic 2D functional Navier-Stokes equations driven by Levy processes. *J. Math. Anal. Appl.*, 385(2):634–654, 2012.

[36] R. Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[37] M. J. Wei and T. Zhang. Exponential stability for stochastic 2D-Navier-Stokes equations with time delay. *Appl. Math. J. Chinese Univ. Ser. A*, 24(4):493–500, 2009.