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Necessary and Sufficient Optimality Conditions for Vector Equilibrium Problems on Hadamard Manifolds

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Abstract: The aim of this paper is to show the existence and attainability of Karush–Kuhn–Tucker optimality conditions for weakly efficient Pareto points for vector equilibrium problems with the addition of constraints in the novel context of Hadamard manifolds, as opposed to the classical examples of Banach, normed or Hausdorff spaces. More specifically, classical necessary and sufficient conditions for weakly efficient Pareto points to the constrained vector optimization problem are presented. The results described in this article generalize results obtained by Gong (2008) and Wei and Gong (2010) and Feng and Qiu (2014) from Hausdorff topological vector spaces, real normed spaces, and real Banach spaces to Hadamard manifolds, respectively. This is done using a notion of Riemannian symmetric spaces of a noncompact type as special Hadamard manifolds.

Keywords: vector equilibrium problem; generalized convexity; hadamard manifolds; weakly efficient pareto points

1. Introduction

The pursuit of equilibrium is a ubiquitous horizon in practically all areas of human activity. For example, in economics, the dynamics of offer and demand are typically described as equilibrium problems. In the same way, physical or social phenomena such as the distribution of particles in a container, traffic flow or telecommunication networks can be accurately conceptualized in terms of equilibrium.

However, it was not until Fan [1] that equilibrium theory was applied in the context of Euclidean spaces. Mathematically, the simplest definition of a equilibrium problem consists in finding $x \in S$ such that

$$F(x, y) \geq 0, \forall y \in S$$

where $S \subseteq \mathbb{R}^p$ is a nonempty closed set and $F : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}$ is an equilibrium bifunction, i.e., $F(x, x) = 0$ for all $x \in S$.

Some of the main mathematical problems that can be phrased as equilibrium problems are:

- The weak minimum point of a multiobjective function $f = (f_1, \dots, f_p)$ over a closed set $S \subseteq \mathbb{R}^p$ is any $\bar{x} \in S$ such that for any $y \in S$, $\exists i$ such that $f_i(y) - f_i(\bar{x}) \geq 0$. Finding a weak minimum point can be reduced to solving an equilibrium problem by virtue of setting

$$F(x, y) = \max_{i=1, \dots, p} [f_i(y) - f_i(x)].$$

- The Stampacchia variational inequality problem demands finding $\bar{x} \in S$ such that

$$\langle G(\bar{x}), y - \bar{x} \rangle \geq 0, \forall y \in S$$

where $G : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $S \subseteq \mathbb{R}^p$ is a closed set. This problem is also an equilibrium problem where

$$F(x, y) = \langle G(x), y - x \rangle.$$

- Nash equilibrium problems in a non-cooperative game with p players where each player i has a set of possible strategies $K_i \subseteq \mathbb{R}^{n_i}$ aim to minimize a loss function $f_i : K \rightarrow \mathbb{R}$ with $K = K_1 \times \dots \times K_p$. Thus, a Nash equilibrium point is any $\bar{x} \in K$ such that no player can reduce its loss by unilaterally changing their strategy, i.e., any $\bar{x} \in K$ such that

$$f_i(\bar{x}) \leq f_i(\bar{x}(y_i))$$

holds for any $y_i \in K_i$ for any $i = 1, \dots, p$, with $\bar{x}(y_i)$ denoting the vector obtained from \bar{x} by replacing \bar{x}_i with y_i . Therefore, this problem amounts to solving an equilibrium problem with

$$F(x, y) = \sum_{i=1}^p [f_i(x(y_i)) - f_i(x)].$$

Despite their apparent diversity, all the above-mentioned problems can be framed as particular cases of the vector equilibrium problem and thus can all be encompassed in a single mathematical picture. Due to the power of this formulation, it is of great interest to obtain and study the Karush–Kuhn–Tucker (KKT) optimality conditions for the solution of such, more general problems.

Thanks to their capacity to provide such a fundamental insight, vector equilibrium problems are an active branch of non-linear analysis with plenty of publications being made up to this date. For example, in 2003, authors such as Iusem and Sosa [2] studied the relation between equilibrium problems and some auxiliary convex problems. In addition, over the past century, the field of physics departed from euclidean geometry as a space in which to allocate its theories, opting instead for more complex spaces also known as manifolds. A historical landmark that illustrates this example is Einstein's theory of gravity that revolves around the concept of space-time curvature on a Riemannian manifold. Other less known but equally fundamental applications in the fields of physics involve the appearance of symplectic manifolds in the treatment of Hamiltonian vector fields or Noether's theorem.

Smooth Riemannian manifolds are spaces that contain curvature, as opposed to Euclidean spaces which are flat everywhere. This can be mathematically expressed as $ax + by \notin M, \forall x, y \in M, a, b \in \mathbb{R}$, where M is a Riemannian manifold. Nonetheless, Riemannian geometry constitutes a generalization of the Euclidean case. This can be easily understood by introducing the notion of tangent planes. For any point of a smooth curved space, say a 2-Sphere, it is always possible to define a flat tangent plane to that point; i.e., a Euclidean space. We can think of this in the same way we think of the Earth to be flat at local scales while overall being spherical. Indeed, all curved manifolds locally resemble Euclidean space, which is a vital property for our understanding of them. However, cartography can empirically tell us that flat projections of curved surfaces onto planes fails to faithfully represent the real dimensions of the objects that live on the original curved surface especially at large scales

where the locality condition starts weakening. Thus, metricity is no longer trivial and measurements of distances need to account for such curvature.

At this point, we can already see how Euclidean spaces are simply Riemannian manifolds for which the tangent plane to any of its points is identical to the plane itself. Thus, in Euclidean spaces, vectors living on the surface are equivalent to vectors living on its tangent space. It is this key feature of Euclidean geometry that allows for the simple definition of distance as the dot product. Thus, given a vector u , if allocated in an Euclidean space, its length is given by $|u|^2 = \langle u, u \rangle$. On the other hand, in non-flat spaces it is necessary to account for the distortion of the distances when projected to the tangent space. Riemannian manifolds are those equipped with a so called “metric tensor”; commonly denoted k_{ij} , that allows us to adequately define distances; i.e., $|u|^2 = k_{ij}u^i u^j$. (see Section 2 for more details).

This new definition of length has direct short comings in minimization and equilibrium. The Euclidean line element, the shortest connection between two points on a flat surface, is replaced on manifolds by a geodesic equation which plays the role of straight lines in non-flat spaces. This can be seen from the fact that geodesic curves are solutions to the Euler–Lagrange equations which minimize the functional of the Lagrangian given by the metric of such space, $\mathcal{L} = k_{ij}dx^i dx^j$, and as such describe the trajectories that minimize the action necessary to move from A to B. For example, the orbits of planets obey geodesics despite clearly not being straight in a Euclidean sense.

A Hadamard manifold is a simply connected complete Riemannian manifold of non-positive sectional curvature. The motivation of the study of Hadamard spaces is that they share some properties with Euclidean spaces. One of them is the separation theorem (see Ferreira and Oliveira [3]).

In addition, for any two points in M , there exists a minimal geodesic joining these two points. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of M is a global diffeomorphism. Moreover, the exp map is defined on the whole tangent space ([4]).

However, the minimization of functions on a Hadamard manifold is locally equivalent to the smoothly constrained optimization problem on a Euclidean space, due to the fact that every C^∞ Hadamard manifold can be isometrically embedded in an Euclidean space by virtue of John Nash’s embedding theorem. This is consistent with the intuition we previously laid out.

The study of optimization problems on Hadamard manifolds is a powerful tool. This is due to the fact that, generally, solving nonconvex constrained problems in \mathbb{R}^n with the Euclidean metric can be also framed as solving the unconstrained convex minimization problem in the Hadamard manifold feasible set with the affine metric (see [5]). In Colao et al. [5] the existence of solutions for equilibrium problems under some suitable conditions on Hadamard manifolds and their applications to Nash equilibrium for non-cooperative games was studied. In the same way, in Németh [6] the existence and uniqueness results for variational inequality problems on Hadamard manifolds were obtained.

Moreover, many optimization problems cannot be solved in linear spaces, for example, controlled thermonuclear fusion research (see [7]), signal processing, numerical analysis and computer vision (see [8,9]) require Hadamard manifold structures for their modeling. Also, geometrical structures hidden in data sets of machine learning problems are studied in terms of manifolds. In the field of medicine, Hadamard manifolds have been used in the analysis of magnetic resonances to quantify the growth of tumors and consequently deduce their state of progression, as shown by Fletcher et al. [10]. The geometry necessary to understand and perform these techniques is best understood through the use of manifolds and symmetric structures. For example, the set of symmetric positive definite matrices used in magnetic resonance imaging to study Alzheimer’s disease [11] is one case in which this translation to manifolds is necessary. In addition, other problems in computer vision, signal processing or learning algorithms employ geodesic curves when addressing optimization problems. Finally, in economics, the search of Nash–Stampacchia equilibria points using Hadamard manifolds has been used by Kristály [12].

It is known that a convex environment has good properties for the search of optimal points. In Ferreira [13], the author gives necessary and sufficient conditions for convex functions on Hadamard manifolds. A significant generalization of the convex functions are the invex functions, introduced by Hanson [14], where the x - y vector is replaced by any function $\eta(x, y)$. The main result of invex functions states that a scalar function is invex if and only if every critical point is a global minimum solution. This property is essential to obtain optimal points through algorithms, due to the coincidence of critical points and solutions being always assured. In Barani and Pourayeyali [15] and Hosseini and Pourayeyali [16], the relation between invexity and monotonicity using the mean value theorem is studied. Ruiz-Garzón et al. [17] showed that invexity can be characterized in the context of Riemannian manifolds for both scalar and vector cases, in a similar way to Euclidean spaces. Recently, in Ahmad et al. [18] the authors introduced the log-preinvex and log-invex functions on Riemannian manifolds and the mean value theorem on Cartan-Hadamard manifolds.

In the same way, several authors have studied vector equilibrium problems. Ansari and Flores-Bazán [19] were capable of providing a theorem of existence of solutions to vector quasi-equilibrium problems. Furthermore, a characterization for a weakly efficient Pareto point for the vector equilibrium problems with constraints under convexity conditions on real Hausdorff topological vector spaces were presented by Gong [20]. In the following years, scalarization results for the solutions to the vector equilibrium problems were also given by Gong [21]. Later, optimality conditions for weakly efficient Pareto points to vector equilibrium problems with constraints in real normed spaces were investigated by Wei and Gong [22]. Also, sufficient conditions of weakly efficient Pareto points on real Banach spaces for vector equilibrium and vector optimization problems with constraints under generalized invexity were obtained by Feng and Qiu [23].

Motivated by Gong's works mentioned above, our objective will focus on extending the KKT necessary and sufficient conditions for constrained vector equilibrium problems obtained in topological or normed spaces to other environments like the Hadamard manifolds, not present in the literature up to date of publication. Hence, we propose a generalization that extends the linear space definition to Hadamard manifolds, by virtue of substituting line segments by geodesic arcs. We will see that the KKT classic conditions for constrained vector optimization are a particular case of the ones obtained for constrained vector equilibrium problem.

The organization of the paper is as follows: In Section 2, we discuss notation, differentials and invex function concepts on Hadamard manifolds. Section 3 is devoted to proving the main results obtained in this paper, and studying the necessary and sufficient optimality conditions for weakly efficient points of the constrained vector equilibrium problem. Section 4 dwells on how the previous results can be reduced to classical KKT conditions for constrained vector optimization problems, first obtained by William Karush [24] and rediscovered by Harold Kuhn and Albert Tucker [25]. Finally, an example is presented as well as the final conclusions.

2. Preliminaries

Let M be a C^∞ -manifold modeled on a Hilbert space H endowed with a Riemannian metric g_x on a tangent space $T_x M$. We denote by $T_x M$ the tangent space of M at x , by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of M , by \tilde{TM} an open neighborhood of the submanifold M of TM . The corresponding norm is denoted by $\|\cdot\|_x$ and the length of a piecewise C^1 curve $\alpha : [a, b] \rightarrow M$ is defined by

$$L(\alpha) = \int_a^b \|\alpha'(t)\|_{\alpha(t)} dt.$$

We define d as the distance which induces the original topology on M such that

$$d(x, y) = \inf\{L(\alpha) \mid \alpha \text{ is a piecewise } C^1 \text{ curve joining } x \text{ and } y \forall x, y \in M\}.$$

If d is the distance induced by the Riemannian metric k_{ij} then any Riemannian manifold (M, k_{ij}) can be converted into a metric space (M, d) . The derivatives of the curves at a point x on the manifold lies in a vector space $T_x M$. Whatever path α joining x and y in M such that $L(\alpha) = d(x, y)$ is a geodesic.

Let $\exp : \bar{T}M \rightarrow M$ be the Riemannian exponential map defined as $\exp_x(V) = \alpha_V(1)$ for every $V \in \bar{T}M$, where α_V is the geodesic starting at x with velocity V (i.e., $\alpha(0) = x, \alpha'(0) = V$).

Assume now that η is a map $\eta : M \times M \rightarrow TM$ defined on the product manifold such that

$$\eta(x, y) \in T_y M, \forall x, y \in M.$$

Definition 1. [26] A subset S_1 of M is considered totally convex if S_1 contains every geodesic $\alpha_{x,y}$ of M whose endpoints x and y belong to S_1 .

On a Hadamard manifold M , we can define the function η as $\eta(x, y) = \alpha'_{x,y}(0)$ for all $x, y \in M$. This function plays the same role of $x - y \in \mathbb{R}^n$. Here $\alpha_{x,y}$ is the unique minimal geodesic joining y to x as follows

$$\alpha_{x,y} = \exp_y(\lambda \exp_y^{-1} x) \quad \forall \lambda \in [0, 1].$$

Example 1. Let $M = \mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$ endowed with the Riemannian metric defined by $g(y) = y^{-2}$ be a Hadamard manifold. Hyperbolic spaces and geodesic spaces, more precisely, a Busemann non-positive curvature (NPC) space are examples of Hadarmard manifolds.

We will need an adequate concept of the differential:

Definition 2. [27] A mapping $f_i : M \rightarrow \mathbb{R}$ is said to be a differential map along the geodesic $\alpha_{x,y}$ at $y \in M$ if and only if the limit

$$f'_i(y) = \lim_{\lambda \rightarrow 0} \frac{f_i(\exp_y(\lambda \eta(x, y))) - f_i(y)}{\lambda \|\eta(x, y)\|}$$

exists.

The gradient of a real-valued C^∞ function $f = (f_1, \dots, f_p) : S_1 \subseteq M \rightarrow \mathbb{R}^n$ on M in x , denoted by $\text{grad} f_x = (f'_1(x), f'_2(x), \dots, f'_n(x))$, is the unique vector in $T_x M$ such that $df_x(X) = \langle \text{grad} f_x, X \rangle$ for all X in $T_x M$ is the differential of f at \bar{x} of X .

Remark 1. The differential of f at \bar{x} of X is similar to the definition of directional derivative in the Euclidean space.

Let $S_1 \subset M$ be a nonempty open totally convex subset and let $F : S_1 \times S_1 \rightarrow \mathbb{R}^p, g : S_1 \rightarrow \mathbb{R}^p$ be mappings.

Definition 3. We define the constraint set $S = \{x \in S_1 : g(x) \in -\mathbb{R}_+^p\}$ and consider the vector equilibrium problem with constraints (VEPC): find $x \in S$ such that

$$F(x, y) \notin -\mathbb{R}_+^p \setminus \{0\}, \forall y \in S$$

where \mathbb{R}_+^p is the non-negative orthant of \mathbb{R}^p .

We recall the classical concept:

Definition 4. A vector $x \in S$ satisfying $F(x, y) \notin -\text{int } \mathbb{R}_+^p, \forall y \in S$ is called a weakly efficient Pareto point to the VEPC.

Notation 1. We denote as $H_x(y) = F(x, y), \forall y \in S_1$, given $x \in S$, where $H : S_1 \rightarrow \mathbb{R}^p$ is a mapping.

Inspired by the concept of convexity on a linear space, the notion of invexity function concept on Hadamard manifolds has become a successful tool in vector optimization. This generalized definition was notably provided by Hanson in [14].

Definition 5. Let S_1 be a nonempty open totally convex subset of a Hadamard manifold M . A differentiable $h : S_1 \rightarrow \mathbb{R}^p$ function is said to be a \mathbb{R}_+^p -invex at $\bar{x} \in S_1$ respect to $\eta : M \times M \rightarrow TM$ if there exist $\eta(x, \bar{x}) \in T_{\bar{x}}M$ such that

$$h(x) - h(\bar{x}) - dh_{\bar{x}}(\eta(x, \bar{x})) \in \mathbb{R}_+^p.$$

Using the previously stated definitions, we can obtain the sufficient conditions for optimality by virtue of the assumption of invexity of the functions of the problem.

3. Main Results

Next, we will obtain a characterization for the weakly efficient points of VEPC through the application of necessary and sufficient optimality conditions. We start with the necessary conditions:

Theorem 1. [Necessary KKT-conditions] Let S_1 be a nonempty open totally convex subset of a Hadamard manifold M and let $F : S_1 \times S_1 \rightarrow \mathbb{R}^p$, $g : S_1 \rightarrow \mathbb{R}^p$, $\eta : M \times M \rightarrow TM$ be mappings. Let $F(\bar{x}, \bar{x}) = H_{\bar{x}}(\bar{x}) = 0$. Assume that H and g are differentiable at $\bar{x} \in S$. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -int \mathbb{R}_+^p$. If \bar{x} is a weakly efficient Pareto point to the VEPC, then there exists $v \in \mathbb{R}_+^p \setminus \{0\}$, $u \in \mathbb{R}_+^p$ such that

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \forall x \in S_1 \quad (1)$$

$$ug(\bar{x}) = 0. \quad (2)$$

Proof. Let there be $\bar{x} \in S$ as a weakly efficient Pareto point to the VEPC. We denote by

$$W = \{(y, z) \in \mathbb{R}^p \times \mathbb{R}^p : \text{there exists } x \in S_1, \text{ such that } y - dH_{\bar{x}}(\eta(x, \bar{x})) \in int \mathbb{R}_+^p, \\ z - [g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \in int \mathbb{R}_+^p\}.$$

It may be noted that W is a nonempty open totally convex set. This proof can be divided into five steps:

Step 1. We have to prove that $(0, 0) \notin W$. By reduction ad absurdum, if $(0, 0) \in W \Rightarrow \exists x_0 \in S_1$, such that

$$dH_{\bar{x}}(\eta(x_0, \bar{x})) \in -int \mathbb{R}_+^p, \quad g(\bar{x}) + dg_{\bar{x}}(\eta(x_0, \bar{x})) \in -int \mathbb{R}_+^p. \quad (3)$$

From the differentiability we obtain that

$$dH_{\bar{x}}(\eta(x_0, \bar{x})) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [H_{\bar{x}}(exp_{\bar{x}}(\lambda \eta(x_0, \bar{x}))) - H_{\bar{x}}(\bar{x})] \in -int \mathbb{R}_+^p \quad (4)$$

$$g(\bar{x}) + dg_{\bar{x}}(\eta(x_0, \bar{x})) = g(\bar{x}) + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [g(exp_{\bar{x}}(\lambda \eta(x_0, \bar{x}))) - g(\bar{x})] \in -int \mathbb{R}_+^p. \quad (5)$$

As $-int \mathbb{R}_+^p$ is an open set, then $\exists \lambda_0, 0 < \lambda_0 < 1$ such that

$$\frac{1}{\lambda_0} [H_{\bar{x}}(exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))) - H_{\bar{x}}(\bar{x})] \in -int \mathbb{R}_+^p \quad (6)$$

$$g(\bar{x}) + \frac{1}{\lambda_0} [g(exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))) - g(\bar{x})] \in -int \mathbb{R}_+^p. \quad (7)$$

By hypothesis, from $g(\bar{x}) \in -\mathbb{R}_+^p$, $F(\bar{x}, \bar{x}) = H_{\bar{x}}(\bar{x}) = 0$, and $\frac{1}{\lambda_0} > 1$, then

$$H_{\bar{x}}[\exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))] \in -\text{int } \mathbb{R}_+^p \quad \text{and} \quad g(\exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))) \in -\text{int } \mathbb{R}_+^p. \tag{8}$$

As S_1 is a totally convex set we have that

$$\exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x})) \in S_1, \quad F(\bar{x}, \exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))) \in -\text{int } \mathbb{R}_+^p \tag{9}$$

and

$$g(\exp_{\bar{x}}(\lambda_0 \eta(x_0, \bar{x}))) \in -\text{int } \mathbb{R}_+^p \tag{10}$$

stands in contradiction with $\bar{x} \in S$ as a weakly efficient Pareto point to the VEPC, consequently $(0, 0) \notin W$.

Step 2. We will prove that there exists a multiplier $v \in \mathbb{R}_+^p$. As W is an open set and the separation theorem holds (see Theorem 2.13 and Remark 2.14 in [28]) or [3]), there exists $(v, u) \neq (0, 0) \in \mathbb{R}^p \times \mathbb{R}^p$ such that

$$vy + uz > 0, \quad \forall (y, z) \in W. \tag{11}$$

Let $(y, z) \in W$ be a point then $\exists x \in S_1$ such that

$$y - dH_{\bar{x}}(\eta(x, \bar{x})) \in \text{int } \mathbb{R}_+^p, \quad z - [g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \in \text{int } \mathbb{R}_+^p. \tag{12}$$

For any $r \in \text{int } \mathbb{R}_+^p$, $s \in \text{int } \mathbb{R}_+^p$, $t', t'' > 0$, we have $(y + t'r, z) \in W$ and $(y, z + t''s) \in W$.

From Equation (11) we have that

$$v(y + t'r) + u(z) > 0, \quad \forall r \in \text{int } \mathbb{R}_+^p, t' > 0. \tag{13}$$

Then

$$vr > \frac{-uz - vy}{t'}. \tag{14}$$

Letting $t' \rightarrow \infty$ we get $vr \geq 0, \forall r \in \text{int } \mathbb{R}_+^p$ and therefore $vr \geq 0$ for all $r \in \mathbb{R}_+^p$, that is $v \in \mathbb{R}_+^p$. In the same way, we can show that $u \in \mathbb{R}_+^p$.

Step 3. We will prove that $v \neq 0$, thus is, $v \in \mathbb{R}_+^p \setminus \{0\}$. By reduction ad absurdum, if $v = 0$, from Equation (11) we get

$$uz > 0, \quad \forall (y, z) \in W. \tag{15}$$

According to the hypothesis, $\exists x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -\text{int } \mathbb{R}_+^p$; then, we obtain

$$(dH_{\bar{x}}(\eta(x_1, \bar{x})) + r, g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) + s) \in W, \quad \forall r \in \text{int } \mathbb{R}_+^p \quad \forall s \in \text{int } \mathbb{R}_+^p. \tag{16}$$

Therefore, from Equation (11) we have that

$$u[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) + s] > 0, \quad \forall s \in \text{int } \mathbb{R}_+^p \tag{17}$$

$$us > -u[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x}))]. \tag{18}$$

As $[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x}))] \in -\text{int } \mathbb{R}_+^p$, and if $s = 0$, we get $u \cdot 0 = 0 > 0$, which implies a contradiction, thus $v \neq 0$.

Step 4. We will prove the first KKT condition.

Since

$$(dH_{\bar{x}}(\eta(x, \bar{x})) + r, g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x})) + s) \in W, \quad x \in S_1, r \in \text{int } \mathbb{R}_+^p, s \in \text{int } \mathbb{R}_+^p. \tag{19}$$

From Equation (11) we get

$$v[dH_{\bar{x}}(\eta(x, \bar{x})) + r] + u[g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x})) + s] > 0, \forall x \in S_1, r \in \text{int } \mathbb{R}_+^p, s \in \text{int } \mathbb{R}_+^p. \tag{20}$$

Letting $r \rightarrow 0, s \rightarrow 0$, we obtain

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + u[g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \geq 0, \quad \forall x \in S_1. \tag{21}$$

Step 5. We will prove the second KKT condition. As

$$(dH_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'r, g(\bar{x}) + dg_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t's) \in W, \forall r \in \text{int } \mathbb{R}_+^p, s \in \text{int } \mathbb{R}_+^p, t' > 0. \tag{22}$$

From Equation (11) we have that

$$v[dH_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'r] + u[g(\bar{x}) + dg_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t's] = t'vr + ug(\bar{x}) + t'us > 0. \tag{23}$$

Letting $t' \rightarrow 0$, we obtain $ug(\bar{x}) \geq 0$. Noting that $g(\bar{x}) \in -\mathbb{R}_+^p$ and $u \in \mathbb{R}_+^p$, we have that $ug(\bar{x}) \leq 0$, in consequence

$$ug(\bar{x}) = 0 \tag{24}$$

and therefore $\exists v \in \mathbb{R}_+^p \setminus \{0\}, u \in \mathbb{R}_+^p$ such that KKT conditions

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \forall x \in S_1 \tag{25}$$

$$ug(\bar{x}) = 0 \tag{26}$$

hold. \square

Let us see now the reciprocal of the previous theorem. To obtain it we first need conditions of invexity.

Theorem 2. [Sufficient KKT-conditions] Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $F : S_1 \times S_1 \rightarrow \mathbb{R}^p, g : S_1 \rightarrow \mathbb{R}^p$ be mappings. Let $F(\bar{x}, \bar{x}) = H(\bar{x}) = 0$. Assume that H and g are differentiable at $\bar{x} \in S_1$. H and g are \mathbb{R}_+^p -invex at \bar{x} respect to η on S_1 . If there exist $v \in \mathbb{R}_+^p \setminus \{0\}$ and $u \in \mathbb{R}_+^p$ such that

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \forall x \in S_1 \tag{27}$$

$$ug(\bar{x}) = 0 \tag{28}$$

then \bar{x} is a weakly efficient Pareto point to the VEPC.

Proof. On the assumption that H and g are \mathbb{R}_+^p -invex at \bar{x} respect to η on S_1 then

$$dH_{\bar{x}}(\eta(x, \bar{x})) \in H_{\bar{x}}(x) - H_{\bar{x}}(\bar{x}) - \mathbb{R}_+^p = H_{\bar{x}}(x) - \mathbb{R}_+^p, \forall x \in S_1 \tag{29}$$

$$dg_{\bar{x}}(\eta(x, \bar{x})) \in g(x) - g(\bar{x}) - \mathbb{R}_+^p, \forall x \in S_1. \tag{30}$$

From $v \in \mathbb{R}_+^p \setminus \{0\}, u \in \mathbb{R}_+^p$ and (27) we obtain that

$$vH_{\bar{x}}(x) + u(g(x) - g(\bar{x})) = vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \quad \forall x \in S_1. \tag{31}$$

From hypothesis (28), we get on the one hand that:

$$vH_{\bar{x}}(x) + ug(x) \geq 0, \quad \forall x \in S_1. \tag{32}$$

On the other hand, we will show that \bar{x} is a weakly efficient Pareto point to the VEPC. If not, consequently by definition $\exists y_0 \in S$ such that

$$F(\bar{x}, y_0) \in -\text{int } \mathbb{R}_+^p. \quad (33)$$

From $v \in \mathbb{R}_+^p \setminus \{0\} \Rightarrow vF(\bar{x}, y_0) < 0$.

Since $y_0 \in S$, we have $g(y_0) \in -\mathbb{R}_+^p$, so $ug(y_0) \leq 0$ because of $u \in \mathbb{R}_+^p$ and then

$$vF(\bar{x}, y_0) + ug(y_0) < 0 \quad (34)$$

stands in contradiction with (32) and therefore \bar{x} is a weakly efficient Pareto point to the VEPC. \square

Remark 2. Theorem 3.1 in [20] on real Hausdorff topological vector spaces and Theorem 3.2 and Theorem 3.4 in [22] on real normed spaces are particular cases of Theorems 1 and 2 obtained in this paper on Hadamard manifolds. The same is true for Theorems 3.1 and 3.3 in [23] on real Banach spaces.

To sum up, we obtain the KKT optimality conditions for weakly efficient Pareto points to the vector equilibrium problems with constraints. These results are not only necessary but also sufficient.

4. Application

As a particular case of the results obtained in the previous section, we will obtain the optimality conditions of KKT for constrained vector optimization problems.

Let us consider the constrained multiobjective programming (CVOP) defined as:

$$\begin{aligned} \text{(CVOP)} \quad & \min f(x) \\ & \text{subject to:} \\ & g(x) \leq 0 \\ & x \in X \subseteq M \end{aligned}$$

where $f = (f_1, \dots, f_p) : X \subseteq M \rightarrow \mathbb{R}^p$, $g = (g_1, \dots, g_m) : X \subseteq M \rightarrow \mathbb{R}^m$ are differentiable multiobjective functions on the open set $X \subseteq M$ and let M be a Hadamard manifold.

As a consequence of the previous theorems and considering CVOP as a particular case of VEPC we have the KKT classical conditions.

Corollary 1. Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $f, g : S_1 \rightarrow \mathbb{R}^p$ be mappings. Assume that f and g are differentiable at $\bar{x} \in S$. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -\text{int } \mathbb{R}_+^p$. If \bar{x} is a weakly efficient Pareto point to the CVOP, then there exist $v \in \mathbb{R}_+^p \setminus \{0\}$, $u \in \mathbb{R}_+^p$ such that

$$vdf_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \forall x \in S_1 \quad (35)$$

$$ug(\bar{x}) = 0. \quad (36)$$

Corollary 2. Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $f, g : S_1 \rightarrow \mathbb{R}^p$ be mappings. Assume that f and g are differentiable at $\bar{x} \in S$. Assume that f and g are differentiable at $\bar{x} \in S$ and f and g are \mathbb{R}_+^p -invex respect at \bar{x} to η on S_1 . If there exist $v \in \mathbb{R}_+^p \setminus \{0\}$, $u \in \mathbb{R}_+^p$ such that

$$vdf_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \forall x \in S_1 \quad (37)$$

$$ug(\bar{x}) = 0 \quad (38)$$

then \bar{x} is a weakly efficient Pareto point to the CVOP.

Proof. The proofs are similar to those already shown without further considering CVOP as particular cases of VEPC just by taking $F(x, y) = \max_{i=1, \dots, p} [f_i(y) - f_i(x)], \forall x, y \in M$. \square

Remark 3. Theorem 4.4 in [20] on real Hausdorff topological vector spaces and Corollary 3.3 in [23] on real Banach spaces are particular cases of Corollaries 1 and 2 obtained in this paper on Hadamard manifolds. Moreover, these results also coincide with Corollary 3.8 given by Ruiz-Garzón et al. [17].

We illustrate the previous results with another example:

Example 2. Let us consider the set $\Omega = \{p = (p_1, p_2) \in \mathbb{R}^2 : p_2 > 0\}$. Let K be a 2x2 matrix defined by $K(p) = (k_{ij}(p))$ with

$$k_{11}(p) = k_{22}(p) = \frac{1}{p_2^2}, \quad k_{12}(p) = k_{21}(p) = 0.$$

Endowing Ω with the Riemannian metric $\langle\langle u, v \rangle\rangle = \langle K(p)u, v \rangle$, we obtain a complete Riemannian manifold \mathbb{H}^2 , namely, the upper half-plane model of a hyperbolic space and $\text{grad } f(p) = K(p)^{-1} \nabla f(p)$.

Consider the CVOP:

$$\begin{aligned} \text{(CVOP)} \quad \text{Min } f(p) &= (f_1, f_2)(p) = (p_1, \ln p_2) \\ \text{subject to:} \\ g_1(p) &= 2p_1 - 2 \geq 0 \\ g_2(p) &= p_2 - 1 \geq 0 \end{aligned}$$

Given $\bar{p} = (1, 1)$ using the Riemannian metric k and f, g is \mathbb{R}_+^2 -invex at \bar{p} respect to $\eta(p, \bar{p}) = 2p - \bar{p}$ and there exists $q = \eta(p, \bar{p}) = (0, 1)$ we have that

$$\begin{aligned} df_{1(\bar{p})}(q) &= \left\langle \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = (p_2^2, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ df_{2(\bar{p})}(q) &= \left\langle \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ p_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = (0, p_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_2. \end{aligned}$$

The

$$\begin{aligned} dg_{1(\bar{p})}(q) &= \left\langle \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = (2p_2^2, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ dg_{2(\bar{p})}(q) &= \left\langle \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = (0, p_2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_2^2. \end{aligned}$$

We have

$$\begin{aligned} df_{\bar{p}}(q) &= (df_{1(\bar{p})}(q), df_{2(\bar{p})}(q)) = (0, p_2) \\ dg_{\bar{p}}(q) &= (dg_{1(\bar{p})}(q), dg_{2(\bar{p})}(q)) = (0, p_2^2) \end{aligned}$$

and therefore there exists $v = u = (1, 0)$ such that

$$\begin{aligned} vdf_{\bar{x}}(q) + udg_{\bar{x}}(q) &= 0 \\ ug(\bar{x}) &= 0 \end{aligned}$$

and then \bar{p} is a weakly efficient Pareto point to the CVOP.

5. Conclusions

In conclusion, we have shown the existence of KKT optimality conditions for weakly efficient Pareto points to the equilibrium vector problems with constraints on Hadamard manifolds, in particular, to constrained vector optimization problems. The main requirement we present for such characterization is the substitution of the segments by geodesics due to the introduction of non-euclidean spaces. This has proven to entail:

- The need for an extension of the concept of convex set to that of totally convex.
- The use of an adequate definition of differential functions in similar terms to those of directional derivatives in Euclidean space using an exponential Riemannian map.
- Generalizing the invexity definition by extending its classical definition given by Hanson [14] in order to obtain sufficient optimality conditions.

Thus, our study provides evidence of the logical continuity of the KKT formulation when extended to other contexts different from Banach spaces or norms, given in the literature by Gong [20] and Wei and Gong [22] and Feng and Qiu [23].

The strength of our method lays on the fact that it allows us to transform non-convex problems in Euclidean spaces to convex problems on Hadamard spaces in which the known properties of convexity can be safely applied. On the other hand, the weakness is that the method requires the manifold to have a nonpositive sectional curvature which limits the cases in which it can be employed. In addition, the dimensions of the Euclidean space tend to be larger than the manifold dimensions, making this approach sometimes not convenient.

The principal contribution of this paper is to obtain the classical KKT optimality conditions for vector equilibrium problems on Hadamard spaces, an unexplored field up to this date.

Finally, it would be interesting to continue studies in this line of research by virtue of considering other types of solutions to the vector equilibrium problem with constraints to which similar generalization have yet not been proposed.

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