mathematics

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Bi-Slant Submanifolds of Para Hermitian Manifolds

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Received: 26 June 2019; Accepted: 9 July 2019; Published: 11 July 2019

Abstract: In this paper, we introduce the notion of bi-slant submanifolds of a para Hermitian manifold. They naturally englobe CR, semi-slant, and hemi-slant submanifolds. We study their first properties and present a whole gallery of examples.

Keywords: semi-Riemannian manifold; para Hermitian manifold; para Kaehler manifold; para-complex; totally real; CR; slant; bi-slant; semi-slant; hemi-slant; anti-slant submanifolds

MSC: 53C15; 53C25; 53C40; 53C50

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1. Introduction

In [1], B.-Y.Chen introduced slant submanifolds of an almost Hermitian manifold, as those submanifolds for which the angle θ between JX and the tangent space is constant, for any tangent vector field X. They play an intermediate role between complex submanifolds ($\theta = 0$) and totally real ones ($\theta = \pi/2$). Since then, the study of slant submanifolds has produced an incredible amount of results and examples in two different ways: various ambient spaces and more general submanifolds.

On the one hand, J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández analyzed slant submanifolds of a Sasakian manifold in [2], and B. Sahin did so in almost product manifolds in [3]. The study of slant submanifolds in a semi-Riemannian manifold has been also initiated: B.-Y. Chen, O. Garay, and I. Mihai classified slant surfaces in Lorentzian complex space forms in [4,5]. K. Arslan, A. Carriazo, B.-Y. Chen, and C. Murathan defined slant submanifolds of a neutral Kaehler manifold in [6], while A. Carriazo and M. J. Pérez-García did so in neutral almost contact pseudo-metric manifolds in [7]. Moreover, M. A. Khan, K. Singh, and V. A. Khan introduced slant submanifolds in LP-contact manifolds in [8], and P. Alegre studied slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds in [9]. Finally, slant submanifolds of para Hermitian manifolds were defined in [10].

On the other hand, some generalizations of both slant and CR submanifolds have also been defined in different ambient spaces, such as semi-slant [11–13], hemi-slant [14,15], bi-slant [16], or generic submanifolds [17].

In this paper, we continue on this line, introducing semi-slant, hemi-slant, and bi-slant submanifolds of para Hermitian manifolds.

2. Preliminaries

Let $\widetilde{M}$ be a $2n$-dimensional manifold. If it is endowed with a structure (J, g), where J is a $(1,1)$ tensor and g is a semi-defined metric, satisfying:

$$J^2X = X, \quad g(JX, Y) + g(X, JY) = 0, \tag{1}$$

for any vector fields X, Y on $\widetilde{M}$, it is called a *para Hermitian manifold*. It is said to be *para Kaehler* if, in addition, $\widetilde{\nabla} J = 0$, where $\widetilde{\nabla}$ is the Levi–Civita connection of g.

Let now M be a semi-Riemannian submanifold of $(\widetilde{M}, J, g)$. The Gauss and Weingarten formulas are given by:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X N, \tag{3}$$

for any tangent vector fields X, Y and any normal vector field V, where h is the second fundamental form of M, A_V is the Weingarten endomorphism associated with V, and $\nabla^{\perp}$ is the normal connection.

The Gauss and Codazzi equations are given by:

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W)), \tag{4}$$

$$(\widetilde{R}(X, Y)Z)^{\perp} = (\widetilde{\nabla}_X h)(Y, Z) - (\widetilde{\nabla}_Y h)(X, Z), \tag{5}$$

for any vectors fields X, Y, Z, W tangent to M.

For every tangent vector field X, we write:

$$JX = PX + FX, \tag{6}$$

where PX is the tangential component of JX and FX is the normal one. For every normal vector field V,

$$JV = tV + fV,$$

where tV and fV are the tangential and normal components of JV, respectively.

For such a submanifold of a para Kaehler manifold, taking the tangent and normal part and using the Gauss and Weingarten formulas (2) and (3):

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y = A_{FY} X + th(X, Y), \tag{7}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y = -h(X, PY) + fh(X, Y), \tag{8}$$

for all tangent vector fields X, Y.

In [10], we introduced the notion of slant submanifolds of para Hermitian manifolds, taking into account that we cannot measure the angle for light-like vector fields:

Definition 1 ([10])**.** *A semi-Riemannian submanifold M of a para Hermitian manifold $(\widetilde{M}, J, g)$ is called* slant submanifold *if for every space-like or time-like tangent vector field X, the quotient $g(PX, PX)/g(JX, JX)$ is constant.*

Remark 1. *It is clear that, if M is a para-complex submanifold, then $P \equiv J$, and so, the above quotient is equal to one. On the other hand, if M is totally real, then $P \equiv 0$ and the quotient equals zero. Therefore, both para-complex and totally real submanifolds are particular cases of slant submanifolds. A neither para-complex nor totally real slant submanifold will be called a* proper slant.

Three cases can be distinguished, corresponding to three different types of proper slant submanifolds:

Definition 2 ([10])**.** *Let M be a proper slant semi-Riemannian submanifold of a para Hermitian manifold $(\widetilde{M}, J, g)$. We say that it is of:*

Type 1 *if for any space-like (time-like) vector field X, PX is time-like (space-like), and* $\dfrac{|PX|}{|JX|} > 1$,

Type 2 *if for any space-like (time-like) vector field X, PX is time-like (space-like), and* $\dfrac{|PX|}{|JX|} < 1$,

Type 3 *if for any space-like (time-like) vector field X, PX is space-like (time-like).*

These three types can be characterized as follows:

Theorem 1 ([10])**.** *Let M be a semi-Riemannian submanifold of a para Hermitian manifold $(\widetilde{M}, J, g)$. Then,*

(1) *M is a slant of Type 1 if and only if for any space-like (time-like) vector field X, PX is time-like (space-like), and there exists a constant $\lambda \in (1, +\infty)$ such that:*

$$P^2 = \lambda Id. \tag{9}$$

We write $\lambda = \cosh^2\theta$, with $\theta > 0$.

(2) *M is a slant of Type 2 if and only if for any space-like (time-like) vector field X, PX is time-like (space-like), and there exists a constant $\lambda \in (0, 1)$ such that:*

$$P^2 = \lambda Id. \tag{10}$$

We write $\lambda = \cos^2\theta$, with $0 < \theta < 2\pi$.

(3) *M is a slant of Type 3 if and only if for any space-like (time-like) vector field X, PX is space-like (time-like), and there exists a constant $\lambda \in (-\infty, 0)$ such that:*

$$P^2 = \lambda Id. \tag{11}$$

We write $\lambda = -\sinh^2\theta$, with $\theta > 0$.

In every case, we call θ the slant angle*.*

Remark 2. *It was proven in [10] that Conditions (9), (10), and (11) also hold for every light-like vector field, as every light-like vector field can be decomposed as a sum of one space-like and one time-like vector field. Furthermore, every slant submanifold of Type 1 or 2 must be a neutral semi-Riemannian manifold.*

Para-complex and totally real submanifolds can also be characterized by P^2. In [10], we did not consider that case, but it will be useful in the present study.

Theorem 2. *Let M be a semi-Riemannian submanifold of a para Hermitian manifold $(\widetilde{M}, J, g)$. Then,*

1) *M is a para-complex submanifold if and only if $P^2 = Id$.*
2) *M is a totally real submanifold if and only if $P^2 = 0$.*

Proof. If M is para-complex, $P^2 = J^2 = Id$ directly. Conversely, if $P^2 = Id$, from:

$$g(JX, JX) = g(PX, PX) + g(FX, FX),$$

we have:

$$-g(X, J^2X) = -g(X, P^2X) + g(FX, FX),$$

then

$$-g(X, X) = -g(X, X) + g(FX, FX),$$

and hence, $g(FX, FX) = 0$, which implies $F = 0$.

The second statement can be proven in a similar way. □

3. Slant Distributions

In [11], N. Papaghiuc introduced *slant distributions* in a Kaehler manifold. Given an almost Hermitian manifold, $(\widetilde{N}, J, g)$, and a differentiable distribution D, it is called a slant distribution if for any nonzero vector $X \in D_x$, $x \in \widetilde{N}$, the angle between JX and the vector space D_x is constant, that is it is independent of the point x. If $P_D X$ is the projection of JX over D, they can be characterized as $P_D^2 = \lambda I$. This, together with the definition of slant submanifolds of a para Hermitian manifold, aims us to give the following:

Definition 3. *A differentiable distribution D on a para Hermitian manifold $(\widetilde{M}, J, g)$ is called a* slant distribution *if for every non-light-like $X \in D$, the quotient $g(P_D X, P_D X)/g(JX, JX)$ is constant.*

A distribution is called *invariant* if it is a slant with slant angle zero, that is if $g(P_D X, P_D X)/g(JX, JX) = 1$ for all non-light-like $X \in D$. It is called *anti-invariant* if $P_D X = 0$ for all $X \in D$. In other cases, it is called a *proper slant distribution*.

With this definition, every one-dimensional distribution defines an anti-invariant distribution in $\widetilde{M}$, so we are just going to take under study non-trivial slant distributions, that is with dimensions greater than one. Just like for slant submanifolds, we can consider three cases depending on the casual character of the implied vector fields.

Obviously, a submanifold M is a slant submanifold if and only if TM is a slant distribution.

Definition 4. *Let D be a proper slant distribution of a para Hermitian manifold $(\tilde{M}, J, g)$. We say that it is of:*

Type 1 *if for every space-like (time-like) vector field X, $P_D X$ is time-like (space-like), and* $\dfrac{|P_D X|}{|JX|} > 1$,

Type 2 *if for every space-like (time-like) vector field X, if $P_D X$ is time-like (space-like), and* $\dfrac{|P_D X|}{|JX|} < 1$,

Type 3 *if for every space-like (time-like) vector field X, $P_D X$ is space-like (time-like).*

These slant distributions can be characterized as slant submanifolds as in Theorem 1 [10].

Theorem 3. *Let D be a distribution of a para Hermitian metric manifold $\widetilde{M}$. Then,*

(1) *D is a slant distribution of Type 1 if and only for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like), and there exits a constant $\lambda \in (1, +\infty)$ such that:*

$$P_D^2 = \lambda I \tag{12}$$

Moreover, in such a case, $\lambda = \cosh^2 \theta$.

(2) *D is a slant distribution of Type 2 if and only for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like), and there exits a constant $\lambda \in (0, 1)$ such that:*

$$P_D^2 = \lambda I \tag{13}$$

Moreover, in such a case, $\lambda = \cos^2 \theta$.

(3) *D is a slant distribution of Type 3 if and only for any space-like (time-like) vector field X, $P_D X$ is space-like (time-like), and there exits a constant $\lambda \in (0, +\infty)$ such that:*

$$P_D^2 = \lambda I \tag{14}$$

Moreover, in such a case, $\lambda = \sinh^2 \theta$.

In each case, we call θ the slant angle.

Proof. If D is a slant distribution of Type 1, for any space-like tangent vector field $X \in D$, $P_D X$ and JX are also time-like, where we have used (1). They satisfy $|P_D X|/|JX| > 1$. Therefore, there exists $\theta > 0$ such that:

$$\cosh\theta = \frac{|P_D X|}{|JX|} = \frac{\sqrt{-g(P_D X, P_D X)}}{\sqrt{-g(JX, JX)}}. \tag{15}$$

Considering $P_D X$ instead of X, we obtain:

$$\cosh\theta = \frac{|P_D^2 X|}{|JP_D X|} = \frac{|P_D^2 X|}{|P_D X|}. \tag{16}$$

Now,

$$g(P_D^2 X, X) = g(JP_D X, X) = -g(P_D X, JX) = -g(P_D X, P_D X) = |P_D X|^2. \tag{17}$$

Therefore, using (15), (16), and (17):

$$g(P_D^2 X, X) = |P_D X|^2 = |P_D^2 X||JX| = |P_D^2 X||X|.$$

Since both X and $P_D^2 X$ are space-like, it follows that they are collinear, that is $P_D^2 X = \lambda X$. Finally, from (15), we deduce that $\lambda = \cosh^2\theta$.

Everything works in a similar way for any time-like tangent vector field $Y \in D$, but now, $P_D Y$ and JY are space-like, and so, instead of (15), we should write:

$$\cosh\theta = \frac{|P_D Y|}{|JY|} = \frac{\sqrt{g(P_D Y, P_D Y)}}{\sqrt{g(JY, JY)}}.$$

Since $P_D^2 X = \lambda X$, for any space-like or time-like $X \in D$, it also holds for light-like vector fields, and so, we have that $P_D^2 = \lambda Id_D$.

The converse is just a simple computation.

For the second case, let D be a slant distribution of Type 2, for any space-like or time-like vector field $X \in D$, $|P_D X|/|JX| < 1$, and so, there exists $\theta > 0$ such that:

$$\cos\theta = \frac{|P_D X|}{|JX|} = \frac{\sqrt{-g(P_D X, P_D X)}}{\sqrt{-g(JX, JX)}}.$$

Proceeding as before, we prove that $g(P_D^2 X, X) = |P_D^2 X||X|$, and as both X and $P_D^2 X$ are space-like vector fields, it follows that they are collinear, that is $P_D^2 X = \lambda X$. The converse is just a direct computation.

Finally, if D is a slant distribution of Type 3, for any space-like vector field $X \in D$, $P_D X$ is also space-like, and there exists $\theta > 0$ such that:

$$\sinh\theta = \frac{|P_D X|}{|JX|} = \frac{\sqrt{g(P_D X, P_D X)}}{\sqrt{-g(JX, JX)}}.$$

Once more, we can prove that $g(P_D^2 X, X) = |P_D^2 X||X|$ and $P_D^2 X = \lambda X$. Again, the converse is a direct computation. □

Remember that an *holomorphic distribution* satisfies $JD = D$, so every holomorphic distribution is a slant distribution with angle zero, but the converse is not true. It is called a *totally real distribution* if $JD \subseteq T^{\perp}M$; therefore, every totally real distribution is anti-invariant but the converse does not always hold. For holomorphic and totally real distributions, the following necessary conditions are easy to prove:

Theorem 4. *Let D be a distribution of a submanifold of a para Hermitian metric manifold $\widetilde{M}$.*

(1) *If D is a holomorphic distribution, then $|P_D X| = |JX|$, for all $X \in D$.*
(2) *If D is a totally real distribution, then $|P_D X| = 0$, for all $X \in D$.*

However, the converse results do not hold if D is not TM; in such a case $TM = D \oplus \nu$, and for a unit vector field X:

$$JX = P_D X + P_\nu X + FX.$$

Therefore from:

$$g(JX, JX) = g(P_D X, P_D X) + g(P_\nu X, P_\nu X) + g(FX, FX),$$

and $|P_D X| = |JX|$, in the case that $P_D X$ is also space-like, it is only deduced that:

$$g(P_\nu X, P_\nu X) + g(FX, FX) = -2,$$

or, in the case it is time-like,

$$g(P_\nu X, P_\nu X) + g(FX, FX) = 0.$$

Therefore, in general $FX \neq 0$, and D is not invariant.

Similarly, it can be shown that the converse of the second statement does not always hold.

Theorem 5. *Let M be a semi-Riemannian submanifold of a para Hermitian metric manifold $\widetilde{M}$.*

1) *The maximal holomorphic distribution is characterized as $D = \{X/FX = 0\}$.*
2) *The maximal totally real distribution is characterized as $D^\perp = \{X/PX = 0\}$.*

Proof. For the first statement, if a distribution D is holomorphic, obviously $F\rceil_D = 0$. For the converse, consider $D = \{X/FX = 0\}$. We should prove that it is a holomorphic distribution. Let $X \in D$, $JX = TX$ be tangent to M, and:

$$g(FJX, V) = g(J^2 X, V) = g(X, V) = 0,$$

for all $V \in T^\perp M$. Therefore, $FJX = 0$. That implies $JX \in D$ for all $X \in D$, so D is holomorphic.

The second statement is trivial. □

4. Bi-Slant, Semi-Slant and Hemi-Slant Submanifolds

In [11], *semi-slant submanifolds* of an almost Hermitian manifold were introduced as those submanifolds whose tangent space could be decomposed as a direct sum of two distributions, one totally real and the other a slant distribution. In [16], *anti-slant submanifolds* were introduced as those whose tangent space is decomposed as a direct sum of an anti-invariant and a slant distribution; they were called *hemi-slant submanifolds* in [14]. Finally, in [12], the authors defined *bi-slant submanifolds* with both distributions slant ones.

Definition 5. *A semi-Riemannian submanifold M of a para Hermitian manifold $(\widetilde{M}, J, g)$ is called a* bi-slant submanifold *if the tangent space admits a decomposition $TM = D_1 \oplus D_2$ with both D_1 and D_2 slant distributions.*

It is called a semi-slant submanifold *if $TM = D_1 \oplus D_2$ with D_1 a holomorphic distribution and D_2 a proper slant distribution. In such a case, we will write $D_1 = D_T$.*

It is called a hemi-slant submanifold *if $TM = D_1 \oplus D_2$ with D_1 a totally real distribution and D_2 a proper slant distribution. In such a case, we will write $D_1 = D_\perp$.*

Remark 3. *As we have said before, being holomorphic (totally real) is a stronger condition than being a slant with slant angle zero ($\pi/2$).*

We write π_i, the projections over D_i and $P_i = \pi_i \circ P$, $i = 1, 2$.

Let us consider two different para Kaehler structures over $\mathbb{R}^4$:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and:

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Using the examples of slant submanifolds of $\mathbb{R}^4$ given in [10] and making products, we can obtain examples of bi-slant submanifolds in $\mathbb{R}^8$. To present different examples with all the combinations of slant distributions, we consider the following para Kaehler structures over $\mathbb{R}^8$:

$$J_2 = \begin{pmatrix} J & \Theta \\ \Theta & J \end{pmatrix}, \qquad g_2 = \begin{pmatrix} g & \Theta \\ \Theta & g \end{pmatrix},$$

$$J_3 = \begin{pmatrix} J_1 & \Theta \\ \Theta & J \end{pmatrix}, \qquad g_3 = \begin{pmatrix} g_1 & \Theta \\ \Theta & g \end{pmatrix},$$

$$J_4 = \begin{pmatrix} J_1 & \Theta \\ \Theta & J_1 \end{pmatrix}, \qquad g_4 = \begin{pmatrix} g_1 & \Theta \\ \Theta & g_1 \end{pmatrix},$$

where Θ is the corresponding null matrix.

Example 1. *For any $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 \neq 1$, and $c^2 + d^2 \neq 1$,*

$$x(u_1, v_1, u_2, v_2) = (au_1, v_1, bu_1, u_1, cu_2, v_2, du_2, u_2)$$

defines a bi-slant submanifold in $(\mathbb{R}^8, J_2, g_2)$, with slant distributions $D_1 = Span\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}\right\}$ and $D_2 = Span\left\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_2}\right\}$. We can see the different types in the Table 1:

Table 1. Types for Example 1.

	D_1	D_2	
Type 1	$a^2 + b^2 > 1, \quad b^2 < 1$	$c^2 + d^2 > 1, \quad c^2 < 1$	$(\mathbb{R}^8, J_2, g_2)$
Type 2	$a^2 + b^2 > 1, \quad b^2 > 1$	$c^2 + d^2 > 1, \quad c^2 > 1$	$P_1^2 = \frac{a^2}{-1 + a^2 + b^2} Id_1$
time-like Type 3	$a^2 + b^2 < 1$	$c^2 + d^2 < 1$	$P_2^2 = \frac{c^2}{-1 + c^2 + d^2} Id_2$

Remark 4. *The decomposition of TM into two slant distributions is not unique, for example, if we choose $\tilde{D}_1 = Span\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_2}\right\}$ and $\tilde{D}_2 = Span\left\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_1}\right\}$ in the previous example, both distributions are anti-invariant, that is $P(\tilde{D}_1) = \tilde{D}_2$ and $P(\tilde{D}_2) = \tilde{D}_1$; therefore, $P_1 = 0$ and $P_2 = 0$. However, they are not totally real distributions.*

Example 2. *Taking $a = 0$ in the previous example, we obtain a semi-slant submanifold, and taking $b = 1$, we obtain a hemi-slant submanifold.*

Example 3. *For any a, b, c, d with $a^2 - b^2 \neq 1$, $c^2 - d^2 \neq 1$:*

$$x(u_1, v_1, u_2, v_2) = (u_1, av_1, bv_1, v_1, u_2, cv_2, dv_2, v_2),$$

defines a bi-slant submanifold, with slant distributions $D_1 = Span\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}\right\}$ and $D_2 = Span\left\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_2}\right\}$. We can see the different types in the Table 2:

Table 2. Types for Example 3.

	D_1	D_2	
Type 1	$b^2 - a^2 < 1, \quad b^2 > 1$	$d^2 - c^2 < 1, \quad d^2 > 1$	$(\mathbb{R}^8, J_2, g_2)$
Type 2	$b^2 - a^2 < 1, \quad b^2 < 1$	$d^2 - c^2 < 1, \quad d^2 < 1$	$P_1^2 = \frac{a^2}{1 + a^2 - b^2} Id_1$
space-like Type 3	$b^2 - a^2 > 1$	$d^2 - c^2 > 1$	$P_2^2 = \frac{c^2}{1 + c^2 - d^2} Id_2$
Type 1	$b^2 - a^2 > 1, \quad a^2 > 1$	$d^2 - c^2 < 1, \quad d^2 > 1$	$(\mathbb{R}^8, J_3, g_3)$
Type 2	$b^2 - a^2 > 1, \quad a^2 < 1$	$d^2 - c^2 < 1, \quad d^2 < 1$	$P_1^2 = \frac{a^2}{1 + a^2 - b^2} Id_1$
space-like Type 3	$b^2 - a^2 < 1$	$d^2 - c^2 > 1$	$P_2^2 = \frac{c^2}{1 + c^2 - d^2} Id_2$
Type 1	$b^2 - a^2 > 1, \quad a^2 > 1$	$d^2 - c^2 > 1, \quad c^2 > 1$	$(\mathbb{R}^8, J_4, g_4)$
Type 2	$b^2 - a^2 > 1, \quad a^2 < 1$	$d^2 - c^2 > 1, \quad c^2 < 1$	$P_1^2 = \frac{a^2}{1 + a^2 - b^2} Id_1$
space-like Type 3	$b^2 - a^2 < 1$	$d^2 - c^2 < 1$	$P_2^2 = \frac{c^2}{1 + c^2 - d^2} Id_2$

Now, we are interested in those bi-slant submanifolds of an almost para Hermitian manifold that are Lorentzian. Let us remember that the only odd-dimensional slant distributions are the totally real ones and that Type 1 and 2 are neutral distributions. Taking this into account, the only possible cases are the following:

(i) M_1^{2s+1} with $TM = D_1 \oplus D_2$, where D_1 is a one-dimensional, time-like, anti-invariant distribution and D_2 is a space-like, Type 3 slant distribution.
(ii) M_1^{2s+2} with $TM = D_1 \oplus D_2$, where D_1 is a two-dimensional, neutral, slant distribution of Type 1 or 2 and D_2 is a space-like, Type 3 slant distribution.

With Examples 1 and 3, we can obtain examples for Case (ii). It only remains to construct a Case (i) example.

Example 4. *Consider in $\mathbb{R}^6$ the almost para Hermitian structure given by:*

$$J_5 = \begin{pmatrix} J & \Theta \\ \Theta & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}, \quad g_5 = \begin{pmatrix} g & \Theta \\ \Theta & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix},$$

with Θ the corresponding null matrix.

For any $k > 1$,

$$x(u, v, w) = (u, k\cosh v, v, k\sinh v, w, 0)$$

defines a bi-slant submanifold in $(\mathbb{R}^6, J_5, g_5)$ *with* $D_1 = Span\left\{\frac{\partial}{\partial w}\right\}$ *a totally real distribution and* $D_2 = Span\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}$ *a Type 3 slant distribution with* $P_2^2 = \frac{1}{k^2-1} Id_{|D_2}$.

We can present a bi-slant submanifold, with the same angle for both slant distributions, that is not a slant submanifold.

Example 5. *The submanifold of* $(\mathbb{R}^8, J_2, g_2)$ *defined by:*

$$x(u_1, v_1, u_2, v_2) = (u_1, v_1 + u_2, u_1, u_1, u_2, v_2, \sqrt{3}u_2, u_2 - v_1),$$

is a bi-slant submanifold. The slant distributions are $D_1 = Span\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}\right\}$ *and* $D_2 = Span\left\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_2}\right\}$, *with* $P_1^2 = \frac{1}{2} Id_1$ *and* $P_2^2 = \frac{1}{2} Id_2$. *It is not a slant submanifold.*

5. Semi-Slant Submanifolds of a Para Kaehler Manifold

It is always interesting to study the integrability of the involved distributions.

Proof. Let M be a semi-slant submanifold of a para Hermitian manifold. Both the holomorphic and the slant distributions are P invariant. □

Proof. Let be $TM = D_T \oplus D_2$ the decomposition with D_T holomorphic and D_2 the slant distribution. Of course D_T is invariant as $JD_T = D_T$ implies $PD_T = D_T$. Now, consider $X \in D_2$,

$$JX = P_1X + P_2X + FX.$$

Given $Y \in D_T$, $g(P_1X, Y) = g(JX, Y) = -g(X, JY) = 0$, as D_T is invariant. Moreover, for all $Z \in D_2$, $g(P_1X, Z) = 0$. Therefore $P_1X = 0$ and $PX = P_2X$, so $PD_2 \subseteq D_2$. □

Theorem 6. *Let* M *be a semi-slant submanifold of a para Kaehler manifold. The holomorphic distribution is integrable if and only if* $h(X, JY) = h(JX, Y)$ *for all* $X, Y \in D_T$.

Proof. For $X, Y \in D_T$, $PX = JX$, $FX = 0$, $PY = JY$, and $FY = 0$. From (8), it follows that $F[X, Y] = h(X, PY) - h(Y, PX)$. Then, $[X, Y] \in D_T$, that is D_T is integrable, if and only if $h(X, JY) = h(JX, Y)$. □

Theorem 7. *Let* M *be a semi-slant submanifold of a para Kaehler manifold. The slant distribution is integrable if and only if:*

$$\pi_1(\nabla_X PY - \nabla_Y PX) = \pi_1(A_{FY}X - A_{FX}Y), \tag{18}$$

for all $X, Y \in D_2$, *where* π_1 *is the projection over the invariant distribution* D_T.

Proof. From (7), $P_1\nabla_X Y = \pi_1(\nabla_X PY - th(X, Y) - A_{FY}X)$. Then:

$$P_1[X, Y] = \pi_1(\nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X).$$

Then, (18) is equivalent to $P_1[X, Y] = 0$. As $P_1[X, Y] = \pi_1 P[X, Y] = 0$, it holds if and only if $P[X, Y] \in D_2$. Finally, from Theorem 5, D_2 is P invariant, so we obtain $[X, Y] \in D_2$. □

Now, we study the conditions for the involved distributions being totally geodesic.

Proof. Let M be a semi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. If the holomorphic distribution D_T is totally geodesic, then $(\nabla_X P)Y = 0$, and $\nabla_X Y \in D_T$ for any $X, Y \in D_T$. □

Proof. For a para Kaehler manifold taking $X, Y \in D_T$, (7)–(8), leads to:

$$\nabla_X PY - P\nabla_X Y - th(X,Y) = 0, \tag{19}$$

$$-F\nabla_X Y + h(X,PY) - fh(X,Y) = 0. \tag{20}$$

If D_T is totally geodesic, $(\nabla_X P)Y = 0$ and $F\nabla_X Y = 0$, which imply the result. □

Proof. Let M be a semi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. The slant distribution D_2 is totally geodesic if and only if $(\nabla_X F)Y = 0$, and $(\nabla_X P)Y = A_{FY}X$ for any $X, Y \in D_2$. □

Proof. If D_2 is a totally geodesic distribution, from (7) and (8), taking $X, Y \in D_2$:

$$\nabla_X PY - A_{FY}X - P\nabla_X Y = 0, \tag{21}$$

$$\nabla^{\perp}_X FY - F\nabla_X Y = 0. \tag{22}$$

which implies the given conditions. On the converse, if $(\nabla_X P)Y = A_{FY}X$, then $th(X,Y) = 0$, which implies $Jh(X,Y) = fh(X,Y)$. From (8) and $\nabla F = 0$, it holds that $h(X,PY) = nh(X,Y)$. Then, for $PY \in D_2$:

$$\lambda h(X,Y) = h(X,P^2Y) = f^2h(X,Y) = J^2h(X,Y) = h(X,Y),$$

and as D_2 is a proper slant distribution, $\lambda \neq 1$, it must be $h(X,Y) = 0$ for all $X, Y \in D_2$. □

Given two orthogonal distributions D_1 and D_2 over a submanifold, it is called a $D_1 - D_2$-*mixed totally geodesic* if $h(X,Y) = 0$ for all $X \in D_1$, $Y \in D_2$.

Proof. Let M be a semi-slant submanifold of a para Hermitian manifold $\widetilde{M}$. M is a mixed totally geodesic if and only if $A_N X \in D_i$ for any $X \in D_i$, $N \in T^{\perp}M$, $i = 1,2$. □

Proof. If M is a $D_T - D_2$ mixed totally geodesic, for any $X \in D_T$, $Y \in D_2$,

$$g(A_N X, Y) = g(h(X,Y), N) = 0,$$

which implies $A_N X \in D_T$. The same proof is valid for $X \in D_2$ and for the converse. □

Proof. Let M be a semi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. If $\nabla F = 0$, then either M is $D_T - D_2$-mixed totally geodesic or $h(X,Y)$ is a eigenvector of f^2 associated with the eigenvalue of one, for all $X \in D_T$, $Y \in D_2$. □

Proof. Let be $X \in D_T$, $Y \in D_2$, if $\nabla F = 0$, from (8), $fh(X,Y) = h(X,PY)$.

As D_T is holomorphic, that is J-invariant, D_2 is P-invariant. Therefore,

$$f^2h(X,Y) = fh(X,PY) = h(X,P^2Y) = h(X,P_2^2Y) = \lambda h(X,Y),$$

with $\lambda = \cosh^2\theta$ ($\cos^2\theta, \sinh^2\theta$, respectively). However, also:

$$f^2h(Y,X) = fh(Y,PX) = h(Y,P^2X) = h(Y,X).$$

From both equations, either $h(X,Y) = 0$ or it is an eigenvalue of f^2 associated with $\lambda = 1$. □

Proof. Let M be a mixed totally geodesic semi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. If D_T is integrable, then $PA_N X = A_N PX$, for all $X \in D_T$ and $N \in T^{\perp}M$. □

Proof. From Theorem 6, $h(X, JY) = h(Y, JX)$ for all $X, Y \in D_T$,

$$g(JA_N X, Y) = -g(A_N X, PY) = -g(N, h(X, PY)) = -g(N, h(Y, PX)) = -g(A_N PY, Y).$$

Given $Z \in D_2$,

$$g(JA_N X, Z) = -g(A_N X, PZ) = -g(N, h(X, PZ)) = 0,$$

because M is mixed totally geodesic. From both equations, $PA_N X = A_N PX$, which finishes the proof. □

Finally, the mixed totally geodesic characterization can be summarized with:

Theorem 8. *Let M be a proper semi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. M is a $D_T - D_2$-mixed totally geodesic if and only if $(\nabla_X P)Y = A_{FY}X$ and $(\nabla_X F)Y = 0$, for all X, Y in different distributions.*

Proof. On the one hand, if M is a $D_T - D_2$-mixed totally geodesic, let X, Y belong to different distributions. From (7) and (8), both conditions are deduced.

On the other hand, from (7) and $(\nabla_X P)Y = A_{FY}X$, it is deduced $th(X, Y) = 0$. From (8) and $(\nabla_X F)Y = 0$, it is deduced:

$$h(X, PY) = fh(X, Y), \tag{23}$$

for all X, Y in different distributions.

Therefore, for $X \in D_T$ and $Y \in D_2$:

$$f^2 h(X, Y) = h(X, P^2 Y) = \lambda h(X, Y)$$

and also:

$$f^2 h(Y, X) = h(Y, P^2 X) = h(Y, X).$$

As M is a proper semi-slant submanifold, $\lambda \neq 1$, and $h(X, Y) = 0$, so M is a mixed totally geodesic. □

6. Hemi-Slant Submanifolds of a Para Kaehler Manifold

We will also study the integrability of the involved distributions for a hemi-slant submanifold.

Proof. Let M be a hemi-slant submanifold of a para Hermitian manifold. The slant distribution is P invariant. □

Proof. Let be $TM = D_\perp \oplus D_2$, the decomposition with $D_\perp$ totally real, and D_2 the slant distribution. Consider $X \in D_2$,

$$JX = P_1 X + P_2 X + FX.$$

Given $Y \in D_\perp$, $g(PX, Y) = g(JX, Y) = -g(X, JY) = 0$, as $D_\perp$ is totally real, therefore $PD_2 \subseteq D_2$. As $P_2^2 = \lambda Id$, given $X \in D_2$, $X = P\left(\frac{1}{\lambda}X\right)$, then $X \in PD_2$, and it is proven that $PD_2 = D_2$. □

Lemma 1. *Let M be a hemi-slant submanifold of a para Kaehler manifold. The totally real distribution is integrable if and only if $A_{FX}Y = A_{FY}X$ for all $X, Y \in D_\perp$.*

Proof. For $X, Y \in D_\perp$, $PX = 0$, $JX = FX$, $PY = 0$, and $JY = FY$. From (7), it follows that $P[X, Y] = A_{FX}Y - A_{FY}X$. Then, $[X, Y] \in D_\perp$, that is $D_\perp$ is integrable, if and only if $A_{FX}Y = A_{FY}X$. □

The following result is known for hemi-slant submanifolds of Kaehler manifolds [14]. We obtain the equivalent one for hemi-slant submanifolds of para Kaehler manifolds:

Theorem 9. *Let M be a hemi-slant submanifold of a para Kaehler manifold. The totally real distribution is always integrable.*

Proof. From the previous lemma, it is enough to prove $g(A_{FX}Y, Z) = g(A_{FY}X, Z)$, for $X, Y \in D_\perp$ and Z tangent. Then,

$$g(A_{FX}Y, Z) = g(h(Y, Z), FX) = g(-th(Y, Z), X) =$$

using (7):

$$= g(P\nabla_Z Y + A_{FY}Z, X) = g(A_{FY}Z, X) = g(A_{FY}X, Z),$$

which finishes the proof. □

Now, we study the integrability of the slant distribution.

Theorem 10. *Let M be a hemi-slant submanifold of a para Kaehler manifold. The slant distribution is integrable if and only if:*

$$\pi_1(\nabla_X PY - \nabla_Y PX) = \pi_1(A_{FY}X - A_{FX}Y), \tag{24}$$

for all $X, Y \in D_2$, where π_1 is the projection over the totally real distribution $D_\perp$.

The proof is analogous to the one of Theorem 7.

Lemma 2. *Let M be a hemi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. The totally real distribution $D_\perp$ is totally geodesic if and only if $(\nabla_X F)Y = 0$, and $P\nabla_X Y = -A_{FY}X$ for any $X, Y \in D_\perp$.*

Proof. From (7) and (8) for $X, Y \in D_\perp$:

$$-P\nabla_X Y - A_{FY}X - th(X, Y) = 0, \tag{25}$$

$$\nabla^\perp_X FY - F\nabla_X Y - fh(X, Y) = 0, \tag{26}$$

which imply the given conditions. □

The same proof of Lemma 5 is valid for the slant distribution of a hemi-slant distribution.

Lemma 3. *Let M be a hemi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. The slant distribution D_2 is totally geodesic if and only if $(\nabla_X F)Y = 0$, and $P\nabla_X Y = -A_{FY}X$ for any $X, Y \in D_2$.*

Remember that the classical De Rham–Wu Theorem [18,19], says that two orthogonal, complementary, and geodesic foliations (called a direct product structure) in a complete and simply connected semi-Riemannian manifold give rise to a global decomposition as a direct product of two leaves. Therefore, from the previous lemmas, it is directly deduced:

Theorem 11. *Let M be a complete and simply-connected hemi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. Then, M is locally the product of the integral submanifolds of the slant distributions if and only if $(\nabla_X F)Y = 0$, and $P\nabla_X Y = -A_{FY}X$ for both any $X, Y \in D_\perp$ or $X, Y \in D_2$.*

Finally, we can also study when a hemi-slant submanifold is mixed totally geodesic. We get a result similar to Proposition 8, but now the proof is much easier.

Proof. Let M be a hemi-slant submanifold of a para Kaehler manifold $\widetilde{M}$. M is a $D_\perp - D_2$-mixed totally geodesic if and only if $(\nabla_X P)Y = A_{FY}X$ and $(\nabla_X F)Y = 0$, for all X, Y in different distributions. □

Proof. Again, if M is a $D_\perp - D_2$-mixed totally geodesic and X, Y belong to different distributions, from (7) and (8), both conditions are deduced.

Now, if we suppose both conditions, from (7) and (8), it is deduced that $th(x, Y) = 0$ and $h(X, PY) = fh(X, Y)$. Therefore, taking $X \in D_2$ and $Y \in D_\perp$, we get $th(X, Y) = 0$ and $fh(X, Y) = 0$. Therefore, $h(X, Y) = 0$ and M is a mixed totally geodesic. □

7. CR-Submanifolds of a Para Kaehler Manifold

CR-submanifolds have been intensively studied in many environments [20]. Moreover, there are also some works about CR submanifolds of para Kaehler manifolds [21]. A semi-Riemannian submanifold M of an almost para Hermitian manifold is called a *CR-submanifold* if the tangent bundle admits a decomposition $TM = D \oplus D^{\perp}$ with D a holomorphic distribution, that is $JD = D$, and $D^{\perp}$ a totally real one, that is $JD \subseteq T^{\perp}M$.

Now, we make a study similar to the one made for generalized complex space forms in [22].

Examples of CR-submanifolds can be obtained from Example 1. Taking $a = 1, d = 0$, $D_1 = \text{Span}\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}\right\}$ is a totally real distribution, and $D_2 = \text{Span}\left\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_2}\right\}$ is a holomorphic distribution. Moreover:

(1) D_1 is Type 1 if $b^2 < 1$
(2) D_1 is Type 2 if $b^2 > 1$,
(3) D_2 is Type 2 if $c^2 > 1$,
(4) D_2 is Type 3 if $c^2 < 1$.

Therefore, we obtain examples of CR-submanifolds of Types 1-2, 1-3, 2-2, and 2-3. Taking $a = 0, d = 1$ we can obtain the Types 2-1, 2-2, 3-1, and again 3-2 examples.

For a para Kaehler manifold with constant holomorphic curvature for every non-light-like vector field, that is $\widetilde{R}(X, JX, JX, X) = c$, the curvature tensor is given by:

$$\widetilde{R}(X,Y)Z = \frac{c}{4}\{g(X,Z)Y - g(Y,Z)X + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}; \tag{27}$$

such a manifold is called a *para complex space form*.

Theorem 12. *Let M be a slant submanifold of a para Kaehler space form $\widetilde{M}(c)$. Then, M is a proper CR submanifold if and only if the maximal holomorphic subspace $D_p = T_pM \bigcap JT_pM$, $p \in M$, defines a non-trivial differentiable distribution on M such as:*

$$\widetilde{R}(D, D, D^{\perp}, D^{\perp}) = 0,$$

where $D^{\perp}$ denotes the orthogonal complementary of D on TM.

Proof. If M is a CR submanifold, from (27):

$$\widetilde{R}(X,Y)Z = 2g(X,JY)JZ,$$

for all $X, Y \in D$ and $Z \in D^{\perp}$, and this is normal to M; therefore, the equality holds.

On the other hand, let $D_p = T_pM \bigcap JT_pM$ be, and suppose $\widetilde{R}(D, D, D^{\perp}, D^{\perp}) = 0$. Again, from (27),

$$\widetilde{R}(X, JX, Z, W) = \frac{c}{2}g(X,X)g(JZ,W),$$

for every $X \in D$, $Z, W \in D^{\perp}$. Taking $X \neq 0$, a non-light-like vector, it follows that $g(JZ, W) = 0$. Then, JZ is orthonormal to $D^{\perp}$, and it is normal. Therefore, $D^{\perp}$ is anti-invariant, and M is a CR submanifold. □

There is a well-known result for CR submanifolds of a complex space form $\widetilde{M}(c)$ [22] establishing that if the invariant distribution is integrable, then the holomorphic sectional curvature determined by a unit vector field, $X \in D$, is upper bounded by the global holomorphic sectional curvature. That is, for every unit vector field X:

$$H(X) = R(X, JX, JX, X) \leq c.$$

The situation in the semi-Riemannian case, for a para complex space form, is completely different. From (27) and (4), for every non-light-like tangent unit vector field X, it holds that:

$$R(X, JX, JX, X) = c + g(h(X, X), h(JX, JX)) - g(h(X, JX), h(X, JX)).$$

Now, if D is integrable, from Theorem 6, $h(JX, JX) = h(X, J^2X) = h(X, X)$, and then:

$$H(X) = c + \|h(X, X)\|^2 - \|h(X, JX)\|^2.$$

A submanifold is called *totally umbilical* if there exists a normal vector field L such as $h(X, Y) = g(X, Y)L$ for all tangent vector fields X, Y. *Totally geodesic* submanifolds are particular cases with $L = 0$.

Theorem 13. *There do not exist proper CR totally umbilical submanifolds of a para complex space form $\widetilde{M}(c)$ with $c \neq 0$.*

Proof. From (27), it follows that:

$$(\widetilde{R}(X, Y)Z)^{\perp} = \frac{c}{4}\{g(X, JZ)FY - g(Y, JZ)FX + 2g(X, JY)FZ\},$$

for all X, Y, Z tangent vectors fields. Supposing M is a proper CR submanifold, we can choose two non-light-like vector fields $X \in D$ and $Z \in D^{\perp}$; for them:

$$(\widetilde{R}(X, JX)Z)^{\perp} = \frac{c}{2}g(X, X)FZ.$$

However, for a totally umbilical submanifold, Codazzi's equation (5) gives:

$$(\widetilde{R}(X, Y)Z)^{\perp} = \nabla^{\perp}_X g(Y, Z)L - g(\nabla_X Y, Z)L - g(Y, \nabla_X Z)L - \nabla^{\perp}_Y g(X, Z)L + g(\nabla_Y X, Z)L + g(X, \nabla_Y Z)L = 0.$$

Comparing both equations, if $c \neq 0$, it follows that $FZ = 0$, which is a contradiction. □

Moreover, the same proof is valid for asserting:

Corollary 1. *There do not exist proper semi-slant totally umbilical submanifolds of a para complex space form $\widetilde{M}(c)$ with $c \neq 0$.*

Author Contributions: The authors contributed equally to this work.

Funding: The work was supported by the MINECO-FEDER grant MTM2014-52197-P., the PAIDI group FQM-327 (Junta de Andalucía, Spain). The second author is also supported by the Instituto de Matemáticas de la Universidad de Sevilla (IMUS).

Acknowledgments: The authors would like to thank Benjamin Olea for his useful comments on some aspects of this paper and the referees for their suggestions, which have improved it.

Conflicts of Interest: The authors declare no conflict of interest.

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