NOWHERE HÖLDERIAN FUNCTIONS AND PRINGSHEIM SINGULAR FUNCTIONS IN THE DISC ALGEBRA

L. BERNAL-GONZÁLEZ, A. BONILLA, J. LÓPEZ-SALAZAR,
AND J.B. SEOANE-SEPÚLVEDA

Abstract. We prove the existence of dense linear subspaces, of infinitely generated subalgebras and of infinite dimensional Banach spaces in the disc algebra all of whose nonzero members are not α-hölderian at any point of the unit circle for any $\alpha > 0$. This completes the recently established result of topological genericity of this kind of functions, as well as the corresponding lineability statements about functions that are nowhere differentiable at the boundary. Topological and algebraic genericity is also studied for the family of boundary-smooth holomorphic functions that are Pringsheim singular at any point of the unit circle.

1. Introduction, Notation and Preliminaries

The existence of real functions that are continuous everywhere but differentiable nowhere in a given real interval I is well known at least from Weierstrass [50]. Using the Baire category theorem, Banach and Mazurkiewicz [7, 42] obtained that, in fact, the set of these functions is residual –that is, its complement is of first category– in the space of continuous functions on I endowed with the topology of uniform convergence on compacta. Giving a step forward, we have that the more stringent class of continuous nowhere hölderian functions on I is also residual [5]. In a different –but related– order of ideas, the family of smooth functions on I that are not analytic at any point of I is also known to be residual in the space of smooth functions on I under its natural topology (see [26, 43]). It is our aim in this paper to shed light on the structure of these families of special functions, not only in the topological sense, but also in the algebraic sense. We focus our attention on those periodic functions that can be extended holomorphically on the unit disc in the complex plane.

Before going on, let us fix some notation. As usual, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$ and $\mathfrak{c}$ will denote the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the field of rational numbers, the real line, the complex plane, the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and the

2010 *Mathematics Subject Classification.* Primary 30H50; Secondary 15A03, 26A27, 26E10, 46E10.

Key words and phrases. Nowhere hölderian function, Pringsheim singular function, disc algebra, lineability, spaceability, algebrability.

cardinality of the continuum, respectively. If I is a compact interval of $\mathbb{R}$, then $C(I)$ will represent the Banach space of all real- or complex-valued continuous functions defined on I, endowed with the maximum norm. The space $C^\infty(I)$ of smooth functions $I \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), that is, the family of all infinitely many differentiable functions on I, when endowed with the topology of uniform convergence of functions and all derivatives, becomes an F-space, that is, a complete metrizable topological vector space.

If $S \subset \mathbb{C}$, then $\overline{S}$ and ∂S will stand for the closure and the boundary of S in $\mathbb{C}$, respectively. If Ω is a domain, that is, a nonempty connected open subset of $\mathbb{C}$, then $H(\Omega)$ represents, as usual, the space of all holomorphic functions on Ω. By $A(\mathbb{D})$ we will denote the disc algebra, that is, the space of those continuous functions on $\overline{\mathbb{D}}$ that belong to $H(\mathbb{D})$. This space becomes a Banach space when endowed with the topology of uniform convergence on $\overline{\mathbb{D}}$. The symbol $A^\infty(\mathbb{D})$ will stand for the set of functions $f \in H(\mathbb{D})$ that are smooth on the boundary, that is, such that every derivative $f^{(n)}$ $(n \in \mathbb{N}_0)$ can be continuously extended to $\overline{\mathbb{D}}$. This family is, again, an F-space under the topology of uniform convergence of all derivatives on $\overline{\mathbb{D}}$.

If $S \subset \mathbb{C}$, $z_0 \in S$ and $\alpha \in (0,1]$, then a function $f : S \to \mathbb{C}$ is called α-hölderian at z_0 whenever there are a constant $C > 0$ and a neighborhood U of z_0 in S such that $|f(z) - f(z_0)| \le C\,|z - z_0|^\alpha$ for all $z \in U$. Note that 1-hölderian equals lipschitzian, that differentiable implies lipschitzian, and that α-hölderian for some α implies continuous, both reverses being not true. A function $f : S \to \mathbb{C}$ is called *nowhere hölderian on* S provided that, for any $z_0 \in S$ and any $\alpha \in (0,1]$, f is not α-hölderian at z_0.

If $I \subset \mathbb{R}$ is an interval and $x_0 \in I$, then a smooth function $f : I \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called *Pringsheim-singular at* x_0 provided that the radius of convergence $R(f, x_0)$ of the Taylor series associated to f at x_0 equals 0 or, equivalently, the sequence

$$\{|f^{(n)}(x_0)/n!|^{1/n} : n \in \mathbb{N}\}$$

is not bounded. Note that this is a notion strictly stronger than mere non-analyticity at x_0. By $\mathcal{PS}(I)$ we shall denote the set of all functions $f \in C^\infty(I)$ that are Pringsheim-singular at any point of I. Again, the family $\mathcal{PS}(I)$ turns out to be residual in $C^\infty(I)$ (see [12, 45, 48, 49]).

In a natural way, we say that a function $f : \mathbb{T} \to \mathbb{C}$ is *smooth* if the associated periodic function $t \in \mathbb{R} \mapsto f(e^{it}) \in \mathbb{C}$ belongs to $C^\infty(\mathbb{R})$, and that such an f is *Pringsheim-singular* at a point $z_0 = e^{it_0} \in \mathbb{T}$ whenever the function $t \mapsto f(e^{it})$ is Pringsheim-singular at t_0.

Some additional terminology, coming from the new theory of lineability, will be used. For an account of results on lineability, the reader is referred to the survey [20] and the monograph [3], or [2, 21, 24, 31–33] in order to see the many fields in which this theory has made an impact so far. Assume that

X is a vector space and α is a cardinal number. Then a subset $A \subset X$ is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$.
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.

If, in addition, X is a topological vector space, then the subset A is said to be:

- *spaceable* in X whenever there is a closed infinite-dimensional vector subspace M of X such that $M \setminus \{0\} \subset A$.
- *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$.
- *α-dense-lineable* in X whenever there is a dense vector subspace M of X with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.

And, provided that X is a vector space contained in some (linear) algebra, then A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$. Recall that if X is contained in a commutative algebra, then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \dots, f_N \in B$, we have $P(f_1, \dots, f_N) \neq 0$ and $P(f_1, \dots, f_N) \in A$.

In 2000, Hencl [37] was able to show that every separable infinite-dimensional Banach space is isometrically isomorphic to a space of continuous functions on $[0,1]$ that are, except for the null function, nowhere hölderian on $[0,1]$. In particular, the family of continuous nowhere hölderian functions on $[0,1]$ is spaceable in $C([0,1])$, which improves corresponding results by a number of mathematicians on continuous nowhere differentiable functions (see [30,35,39,46]). In 2007, Bayart and Quarta [11] established the algebrability of the mentioned family, with the additional property that the existent algebra is dense in $C([0,1])$. The strong $\mathfrak{c}$-algebrabilty of the same family was finally proved by Bartoszewicz *et al.* in [8]. Concerning Pringsheim-singular functions, the $\mathfrak{c}$-dense-lineability of $\mathcal{PS}([0,1])$ in $C^\infty([0,1])$ is shown in [17] (see [14,23] and [3, Section 1.4] for previous weaker results). The strong $\mathfrak{c}$-algebrability of the (bigger) sets of smooth nowhere analytic functions and of smooth functions that are Pringsheim-singular at a cofinite subset of $[0,1]$ were respectively proved in [8] and [17] (see also [10,29]).

All results in the preceding paragraphs still hold, with essentially the same proofs, if we restrict ourselves to *periodic* functions that are continuous nowhere hölderian or smooth Pringsheim-singular. In fact, respective explicit examples of these kinds of functions are known (see, respectively, [38] and [14, Remark 2.2.1]):

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2^n \pi x), \qquad g(x) = \sum_{n=1}^{\infty} b_n^{1-n} \sin(b_n x),$$

where $b_n := 2\big(2 + c_n + [c_{n-1} + \sum_{j=1}^{n-1} b_j^{n+1-j}]\big)$, $c_n := (n+1)!(n+1)^{n+1}$ and the term inside the square brackets in the expression of b_n is defined as 0 if $n = 1$. Consequently, we can in fact consider functions that are, respectively, continuous or smooth on $\mathbb{T}$.

In this order of ideas, recall that not every continuous function f on $\mathbb{T}$ can be extended to a function $F \in A(\mathbb{D})$ (this takes place if and only if $\int_0^{2\pi} f(e^{i\theta})\, e^{in\theta}\, d\theta = 0$ for all $n \in \mathbb{N}_0$). Therefore, the extensions of the previous residuality and lineability results from $C(\mathbb{T})$ to $A(\mathbb{D})$ is not automatic. Recently, Eskenazis [27] has established the residuality of the family of those $F \in A(\mathbb{D})$ such that $F|_{\mathbb{T}}$ is nowhere differentiable on $\mathbb{T}$. Even more, Eskenazis and Makridis [28] have proved that the more stringent class of functions $F \in A(\mathbb{D})$ such that $F|_{\mathbb{T}}$ is nowhere hölderian on $\mathbb{T}$ is residual in $A(\mathbb{D})$. In fact, it is shown that, generically, both $\Re f|_{\mathbb{T}}$ and $\Im f|_{\mathbb{T}}$ are nowhere hölderian on $\mathbb{T}$. (See also [40], where for certain domains Ω and closed sets $J \subset \partial\Omega$, the residuality of the set of those $F \in H(\Omega)$ such that F is continuous on $\overline{\Omega}$ and $F|_J$ is nowhere lipschitzian on J is studied). Turning to an algebraic point of view, the lineability –in its various degrees– of the class of functions F in the disc algebra whose restriction to $\mathbb{T}$ is nowhere differentiable/lipschitzian has been recently analyzed in [18].

In this paper, we go deeper into the subject and undertake the study of lineability of the family of those functions in the disc algebra whose restriction to the unit circle is nowhere hölderian, and of the family of those holomorphic functions on the unit disc that are smooth on the boundary but also Pringsheim singular (P-singular, from now on) at every point of this boundary. This will be performed in Sections 3 and 4, respectively. Section 2 will be devoted to provide a number of auxiliary statements.

2. Some auxiliary results

The next lemma tells us essentially that both properties of non-hölderian and P-singular are preserved under composition with holomorphic functions. The first part is a complex version of Theorem 4.7 in [8], and can be verified along the same lines. Nevertheless, we furnish a (slightly modified) proof for the sake of completeness. The second part is in the same spirit as [8, Theorem 4.10], but the proof of this must be refined in order to give the conclusion of our lemma.

Lemma 2.1. *Let us suppose that $f : \mathbb{T} \to \mathbb{C}$ is a function, $z_0 \in \mathbb{T}$, Ω is a domain in $\mathbb{C}$ such that $f(\mathbb{T}) \subset \Omega$, $\varphi \in H(\Omega)$ is not constant and $g : \mathbb{T} \to \mathbb{C}$ is defined as $g = \varphi \circ f$. We have:*

(a) *If f is not α-hölderian at z_0 for any $\alpha \in (0,1]$, then g is not α-hölderian at z_0 for any $\alpha \in (0,1]$.*
(b) *If f is smooth on $\mathbb{T}$ and P-singular at z_0, then g is also P-singular at the point z_0.*

Proof. (a) Fix $\alpha > 0$ and $k \in \mathbb{N}$. On the one hand, due to the assumption, there exists a sequence $(z_m)_{m=1}^{\infty} \subset \mathbb{T} \setminus \{z_0\}$ such that $z_m \to z_0$ and

$$\lim_{m\to\infty} \left| \frac{f(z_m) - f(z_0)}{(z_m - z_0)^{\frac{\alpha}{k}}} \right| = +\infty.$$

Without loss of generality we can assume $f(z_m) \neq f(z_0)$ for all $m \in \mathbb{N}$. The continuity of f at z_0 entails $f(z_m) \to f(z_0)$ as $m \to \infty$.

On the other hand, since φ is nonconstant and holomorphic, there is $k \in \mathbb{N}$ with $\varphi^{(k)}(f(z_0)) \neq 0$. Assume that k is the least natural number with this property. Then

$$\lim_{m\to\infty} \left| \frac{\varphi(f(z_m)) - \varphi(f(z_0))}{(f(z_m) - f(z_0))^k} \right| = \left| \frac{\varphi^{(k)}(f(z_0))}{k!} \right| > 0.$$

Consequently, we get

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{g(z_m) - g(z_0)}{(z_m - z_0)^{\alpha}} \right| &= \lim_{m\to\infty} \left| \frac{\varphi(f(z_m)) - \varphi(f(z_0))}{(f(z_m) - f(z_0))^k} \right| \cdot \left| \frac{f(z_m) - f(z_0)}{(z_m - z_0)^{\frac{\alpha}{k}}} \right|^k \\ &= +\infty. \end{aligned}$$

This means that g is not α-hölderian at z_0.

(b) Here we proceed by improving part of the argument used in the proof of [17, Theorem 4.6]. Firstly, it is evident that g is smooth on $\mathbb{T}$. Next, we invoke the theory of formal power series, which can be found, for instance, in [22, Chap. 1]. Let $z_0 = e^{it_0}$, $f_0(t) := f(e^{it})$ and $g_0(t) := g(e^{it})$, so that $g_0 = \varphi \circ f_0$. Observe that

$$g_0 = \psi_2 \circ \widetilde{\varphi} \circ \widetilde{f_0} \circ \psi_1,$$

where $\psi_1(t) := t - t_0$, $\psi_2(z) := z + g(z_0)$, $\widetilde{f_0}(t) := f_0(t + t_0)$ and $\widetilde{\varphi}(z) := \varphi(z) - g(z_0)$. Since the fact of being P-singular is invariant under translations (ψ_1 and ψ_2, in this case), we may assume that $g_0 = \varphi \circ f_0$ with f_0 P-singular at $t_0 = 0$ and $g_0(0) = \varphi(f_0(0)) = 0$.

Since φ is not constant, there must be a first $N \in \mathbb{N}$ such that $\varphi^{(N)}(f_0(0)) \neq 0$. By [47, Theorem 10.32], there exists a neighborhood $U \subset \Omega$ of $f_0(0)$ as well as a one-to-one function $h \in H(U)$ such that $\varphi(z) = h(z)^N$ for all $z \in U$. Therefore $g_0 = (h \circ f_0)^N$ on some open real interval J containing 0. Let $G := h \circ f_0$, which belongs to $C^{\infty}(J)$. Hence $g_0 = G^N$ and $f_0 = h^{-1} \circ G$ on J, where h^{-1} is well defined and analytic on a neighborhood of $G(0) = 0$. If $f_0^*(z) := \sum_{n=0}^{\infty} \frac{f_0^{(n)}(0)}{n!} z^n$ and $G^*(z) := \sum_{n=1}^{\infty} \frac{G^{(n)}(0)}{n!} z^n$ are, respectively, the formal Taylor series of f_0 and G at the origin, then we get from the analyticity of h that G^* equals the formal Taylor series associated to $h \circ f_0^*$

at the origin. If $R(G,0) = R(G^*,0)$ were not 0 then, since $R(h^{-1},0) > 0$, we would have $R(f_0,0) = R(f_0^*,0) > 0$ (see [22, Chap. 1]), so contradicting that f_0 is P-singular at 0. Therefore $R(G,0) = 0$ and G is P-singular at 0.

Finally, since $g_0 = G^N$ on J, we get $R(g_0,0) = 0$ (and we are done). Indeed, suppose that $R(g_0,0) = R(g_0^*,0) > 0$, where g_0^* represents the formal Taylor series $g_0^*(z) := \sum_{n=1}^{\infty} \frac{g_0^{(n)}(0)}{n!} z^n$ associated to g_0 at the origin. Thus, g_0^* defines a holomorphic function on a neighborhood of 0. Let k be the first $n \in \mathbb{N}$ with $G^{(n)}(0) \neq 0$. It follows from $g_0 = G^N$ that Nk is the first $n \in \mathbb{N}$ such that $g_0^{(n)}(0) \neq 0$. Then $g_0^*(z) = z^{Nk} S(z)$ with S holomorphic on a neighborhood of 0 and $S(0) \neq 0$. Therefore there exists a holomorphic function Ψ on a neighborhood V of 0 with $\Psi(z)^N = S(z)$ (see [1, pp. 143-144]). Hence $g_0^*(z) = (z^k \Psi(z))^N$ in such neighborhood. Now, $g_0^* = (G^N)^* = (G^*)^N$, and $z^k \Psi(z)$ is a holomorphic branch of the Nth-root of g_0^* on V. Therefore, there is an Nth-root α of 1 such that $G^*(z) = \alpha z^k \Psi(z)$ on V. In particular, $R(G,0) = R(G^*,0) > 0$, which contradicts the conclusion of the preceding paragraph. Therefore, $R(g_0,0) = 0$ and the lemma is proved. □

Remark 2.2. Under the notation and assumptions of the preceding lemma, it can also be proved that if f is smooth on $\mathbb{T}$ and non-analytic at z_0, then g is also non-analytic at z_0. Anyway, this result will be not invoked along this paper.

We shall also make use of the following auxiliary assertion, which is useful to discover dense-lineability from mere lineability. The proof of its part (a) can be found in [19, Theorem 2.3] (see also [3, Section 7.3], [4, Theorem 2.2 and Remark 2.5] and [15, Lemma 2.1]), while the proof of (b) can be extracted from the one of Theorem 2.2 in [4], because in fact the condition $A + B \subset A$ is not needed in its full strength.

Lemma 2.3. *Let X be a metrizable separable topological vector space. Suppose that A is a subset of X. We have:*

(a) *If α is an infinite cardinal number such that A is α-lineable and there is a dense-lineable set $B \subset X$ such that $A + B \subset A$ and $A \cap B = \varnothing$, then A is α-dense-lineable in X.*
(b) *If there are an infinite dimensional vector space A_0 and a dense-lineable set $B \subset X$ such that $A_0 \setminus \{0\} \subset A$ and $(A_0 \setminus \{0\}) + B \subset A$, then A is dense-lineable in X.*

In order to construct large algebras of strange functions, we will employ the next lemma, whose proof can be found in [16] (see also [3, Section 7.5]). The lemma contains a complex version of a very useful method developed by Balcerzak *et al.* in [6, Proposition 7] (see also [8, Theorem 1.5 and Section 6] and [9]). By $\mathcal{E}$ we denote the family of exponential-like functions on $\mathbb{C}$, that is, the functions of the form

$$\varphi(z)=\sum_{j=1}^{m} a_j e^{b_j z}$$

for some $m \in \mathbb{N}$, some $a_1, \ldots, a_m \in \mathbb{C} \setminus \{0\}$ and some distinct $b_1, \ldots, b_m \in \mathbb{C} \setminus \{0\}$.

Lemma 2.4. *Let Ω be a nonempty set and let $\mathcal{F}$ be a family of functions from Ω into $\mathbb{C}$. Assume that there exists a function $f : \Omega \to \mathbb{C}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.*

By using Mergelyan's approximation theorem (see, for instance, [47, Theorem 20.5]) and Cauchy's inequalities, one can easily obtain the following well-known result.

Lemma 2.5. *Let $\mathcal{P}$ denote the family of restrictions to $\overline{\mathbb{D}}$ of all polynomials on $\mathbb{C}$. Assume that $f(z) := \sum_{l=0}^{\infty} a_l z^l \in H(\mathbb{D})$. We have:*

(a) *The set $\mathcal{P}$ is dense in $A(\mathbb{D})$.*
(b) *The set $\mathcal{P}$ is dense in $A^{\infty}(\mathbb{D})$.*
(c) *$f \in A^{\infty}(\mathbb{D})$ if and only if $\left(l^N a_l\right)_{l=1}^{\infty}$ is a bounded sequence for every $N \in \mathbb{N}$.*

3. Large spaces of nowhere hölderian functions

Let $\mathcal{NH}(\mathbb{T})$ denote the family of all functions $f \in A(\mathbb{D})$ such that the restriction $f|_{\mathbb{T}}$ is nowhere hölderian on $\mathbb{T}$. Although the non-vacuousness of $\mathcal{NH}(\mathbb{T})$ has been already established by Eskenazis and Makridis [28], we here exhibit an explicit example for the convenience of the reader. Let

$$f(z)=\sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}.$$

Since $\sum_{n=1}^{\infty} 1/n^2 < \infty$, the Weierstrass M-test for uniform convergence implies that $f \in A(\mathbb{D})$. Let $\alpha \in (0,1]$, $x_0 \in \mathbb{R}$ and $z_0 = e^{i\pi x_0} \in \mathbb{T}$. If $x \in \mathbb{R}$, then

$$f\left(e^{i\pi x}\right)=\sum_{n=1}^{\infty} \frac{e^{i2^n \pi x}}{n^2}=\sum_{n=1}^{\infty} \frac{\cos\left(2^n \pi x\right)}{n^2}+i \sum_{n=1}^{\infty} \frac{\sin\left(2^n \pi x\right)}{n^2}.$$

It is known that $u(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n \pi x)}{n^2}$ is nowhere hölderian (see [38]). Thus, there is a sequence $(x_m)_{m=1}^{\infty}$ in $\mathbb{R}$ such that $x_m \to x_0$ as $m \to \infty$ and $|u(x_m) - u(x_0)| > m\,|x_m - x_0|^{\alpha}$ for every $m \in \mathbb{N}$. Moreover,

$$\lim_{m \to \infty}\left|\frac{x_m-x_0}{e^{i\pi x_m}-e^{i\pi x_0}}\right|=\left|\frac{1}{i\pi e^{i\pi x_0}}\right|=\frac{1}{\pi}.$$

Hence there is $m_0 \in \mathbb{N}$ such that if $m \geq m_0$, then

$$\left|\frac{x_m-x_0}{e^{i\pi x_m}-e^{i\pi x_0}}\right|>\frac{1}{2\pi}.$$

Therefore, $z_m = e^{i\pi x_m} \in \mathbb{T}$, $z_m \to z_0$ as $m \to \infty$ and

$$\begin{aligned} |f(z_m) - f(z_0)| &\geq |\Re[f(z_m) - f(z_0)]| = |u(x_m) - u(x_0)| \\ &> m\,|x_m - x_0|^\alpha = m\left|e^{i\pi x_m} - e^{i\pi x_0}\right|^\alpha \cdot \left|\frac{x_m - x_0}{e^{i\pi x_m} - e^{i\pi x_0}}\right|^\alpha \\ &> \frac{m}{2\pi}\,|z_m - z_0|^\alpha \end{aligned}$$

for all $m \geq m_0$. This proves that f is nowhere hölderian on $\mathbb{T}$.

In this section, it will be shown that $\mathcal{NH}(\mathbb{T})$ possesses maximal algebraic size, in several senses. Take into account that both $\mathrm{card}(A(\mathbb{D}))$ and $\dim(A(\mathbb{D}))$ equal $\mathfrak{c}$.

Theorem 3.1. *The set $\mathcal{NH}(\mathbb{T})$ is strongly $\mathfrak{c}$-algebrable. In particular, it is $\mathfrak{c}$-lineable.*

Proof. Just take any $f \in \mathcal{NH}(\mathbb{T})$ and apply Lemma 2.1(a) together with Lemma 2.4 with $\Omega = \overline{\mathbb{D}}$ and $\mathcal{F} = \mathcal{NH}(\mathbb{T})$. □

Remark 3.2. If one had required mere algebrability for $\mathcal{NH}(\mathbb{T})$, the proof would have been even easier: the family $\{\varphi \circ f : \varphi \text{ entire}, \varphi(0) = 0\}$ is an infinitely generated algebra all of whose nonzero members belong to $\mathcal{NH}(\mathbb{T})$.

Theorem 3.3. *The set $\mathcal{NH}(\mathbb{T})$ is $\mathfrak{c}$-dense-lineable.*

Proof. According to Lemma 2.5, the set $\mathcal{P}$ of restrictions to $\overline{\mathbb{D}}$ of all polynomials on $\mathbb{C}$ is dense in $A(\mathbb{D})$. In fact, $\mathcal{P}$ is dense-lineable, because it is a vector space. Consequently, the set of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is dense in $A(\mathbb{D})$ and thus $A(\mathbb{D})$ is a separable Banach space.

Let us fix $f \in \mathcal{NH}(\mathbb{T})$, $P \in \mathcal{P}$ and $z_0 \in \mathbb{T}$. The polynomial P is differentiable at z_0, so it is also α-hölderian at z_0 for any $\alpha \in (0,1]$. If $f + P$ were α-hölderian at z_0 for some α, then so would be $f = (f + P) - P$, which is absurd. Therefore, $f + P \in \mathcal{NH}(\mathbb{T})$, that is, $\mathcal{NH}(\mathbb{T}) + \mathcal{P} \subset \mathcal{NH}(\mathbb{T})$. Moreover, of course, $\mathcal{NH}(\mathbb{T}) \cap \mathcal{P} = \varnothing$. Now, the conclusion follows from Theorem 3.1 and Lemma 2.3 as soon as we take $\alpha = \mathfrak{c}$, $X = A(\mathbb{D})$, $A = \mathcal{NH}(\mathbb{T})$ and $B = \mathcal{P}$. □

Finally, we show the existence of large Banach spaces inside the disc algebra consisting, except for zero, of nowhere hölderian functions on the circle. To this end, let us recall (see, e.g., [41] and the references contained in it) that if $v : \mathbb{C} \to (0, +\infty)$ is a continuous function, then the weighted space of entire functions associated to v is defined as

$$H_v(\mathbb{C}) := \left\{ f \in H(\mathbb{C}) : \sup_{z \in \mathbb{C}} v(z) \cdot |f(z)| < \infty \right\}.$$

It is known that $H_v(\mathbb{C})$ is a Banach space when it is endowed with the norm

$$\|f\|_v := \sup_{z \in \mathbb{C}} v(z) \cdot |f(z)|$$

and that convergence in $\|\cdot\|_v$ implies uniform convergence on compacta in $\mathbb{C}$.

Theorem 3.4. *There exists a Banach space X satisfying the following properties:*

(a) $X \subset A(\mathbb{D})$ *and* $X \setminus \{0\} \subset \mathcal{NH}(\mathbb{T})$.
(b) X *is infinite dimensional.*
(c) *The norm topology of X is stronger than the one inherited from $A(\mathbb{D})$.*

Proof. Fix any $f \in \mathcal{NH}(\mathbb{T})$, for instance $f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}$, and set $v(z) := e^{-|z|}$. Consider the corresponding weighted Banach space $H_v(\mathbb{C})$ and define

$$X := \big\{\varphi \circ f : \varphi \in H_v(\mathbb{C}),\, \varphi(0) = 0\big\}.$$

Plainly, X is a vector space and $X \subset A(\mathbb{D})$. If $g \in X \setminus \{0\}$, then there is $\varphi \in H_v(\mathbb{C})$ such that $\varphi \neq 0$, $\varphi(0) = 0$ and $g = \varphi \circ f$. Then φ is a nonconstant holomorphic function on $\mathbb{C}$, so $g \in \mathcal{NH}(\mathbb{T})$ according to Lemma 2.1(a).

That $\dim(X) = \infty$ follows from the facts that the monomials $\varphi_k(z) = z^k$ $(k \in \mathbb{N})$ are linearly independent, belong to $H_v(\mathbb{C})$ and satisfy $\varphi_k(0) = 0$. From this one derives that the functions $f(z)^k$ $(k \in \mathbb{N})$ belong to X and are linearly independent. Indeed, if $\lambda_1 f(z) + \cdots + \lambda_p f^p(z) = 0$ on $\overline{\mathbb{D}}$, then $\lambda_1 z + \cdots + \lambda_p z^p = 0$ on the nonempty open set $f(\mathbb{D})$, so $\lambda_1 = 0, \ldots, \lambda_p = 0$. Hence (a) and (b) have been proved.

If $g \in X$ and $\varphi_1 \circ f = g = \varphi_2 \circ f$, where $\varphi_1, \varphi_2 \in H_v(\mathbb{C})$, then $\varphi_1 = \varphi_2$ on the nonempty open set $f(\mathbb{D})$, so $\varphi_1 = \varphi_2$ on the whole $\mathbb{C}$ by the identity principle. Therefore, the mapping

$$g = \varphi \circ f \in X \mapsto \|g\|_X := \|\varphi\|_v$$

is well defined, and it is in fact a norm on X. Since convergence in $H_v(\mathbb{C})$ implies uniform convergence on compacta, it follows that $Y := \{\varphi \in H_v(\mathbb{C}) : \varphi(0) = 0\}$ is a closed subspace of $H_v(\mathbb{C})$. Therefore, $(Y, \|\cdot\|_v)$ is a Banach space, which in turn implies that $\|\cdot\|_X$ is a complete norm on X.

In order to prove (c), consider the supremum norm $\|\cdot\|_\infty$ on $A(\mathbb{D})$. Since $f(\overline{\mathbb{D}})$ is compact, it follows that $\sup_{z \in f(\overline{\mathbb{D}})} |z| < \infty$. Let $K := e^{\sup_{z \in f(\overline{\mathbb{D}})} |z|}$. For each $g \in X$ there is $\varphi \in H_v(\mathbb{C})$ with $g = \varphi \circ f$ and then

$$\begin{aligned} \|g\|_\infty &= \|\varphi \circ f\|_\infty = \sup_{z \in f(\overline{\mathbb{D}})} |\varphi(z)| \\ &\leq \sup_{z \in f(\overline{\mathbb{D}})} (K\, e^{-|z|} |\varphi(z)|) = K \cdot \sup_{z \in f(\overline{\mathbb{D}})} e^{-|z|} |\varphi(z)| \\ &\leq K \cdot \sup_{z \in \mathbb{C}} e^{-|z|} |\varphi(z)| = K \cdot \|\varphi\|_v = K \cdot \|g\|_X. \end{aligned}$$

This completes the proof. $\square$

Remarks 3.5. 1. Concerning assertion (b) of Theorem 3.4, the norm topology of X is *strictly* stronger than the one inherited from $A(\mathbb{D})$. Indeed, for each $n \in \mathbb{N}$, we consider

$$P_n(z) = \sum_{k=1}^{n} \frac{z^{2k}}{k!}.$$

Then $P_n \in H_v(\mathbb{C})$, so every $g_n := P_n \circ f$ belongs to X. Moreover, $(g_n)_{n=1}^\infty$ is a Cauchy sequence for the supremum norm $\|\cdot\|_\infty$, because it converges to $e^{f^2} - 1$ in $A(\mathbb{D})$. However, $(g_n)_{n=1}^\infty$ is not a Cauchy sequence in $(X, \|\cdot\|_X)$, since $e^{z^2} - 1 \notin H_{e^{-|z|}}(\mathbb{C})$.

2. If in the proof of Theorem 3.4 we had defined $X := \{\varphi \circ f : \varphi \text{ entire}, \varphi(0) = 0\}$ endowed with the family of norms $\|g\|_n := \sup_{|z| \le n} |\varphi(z)|$ for $g = \varphi \circ f$ $(n = 1, 2, \dots)$ and the usual multiplication, then we would have obtained an *infinite dimensional Fréchet algebra* consisting, except for 0, of $\mathbb{T}$-nowhere hölderian $A(\mathbb{D})$-functions.

In view of the latter theorem, the following *question* arises naturally: *Is* $\mathcal{NH}(\mathbb{T})$ *spaceable* in $A(\mathbb{D})$*?*

4. Large spaces of Pringsheim singular functions

Let $\mathcal{PS}(\mathbb{T})$ denote the family of all functions $f \in A^\infty(\mathbb{D})$ such that the restriction $f|_\mathbb{T}$ is P-singular at every point of $\mathbb{T}$. Analogously to the preceding section, we will establish in the present one that $\mathcal{PS}(\mathbb{T})$ enjoys a large algebraic size. Again, recall that both $\text{card}(A^\infty(\mathbb{D}))$ and $\dim(A^\infty(\mathbb{D}))$ equal $\mathfrak{c}$.

We start with a proof of the topological genericity of $\mathcal{PS}(\mathbb{T})$ because, as far as we know, this fact has not been explicitly stated up to date. This property can be established, in fact, in a more general way. Let us recall that given a function $f : \mathbb{T} \to \mathbb{C}$, we have defined $f_0 : \mathbb{R} \to \mathbb{C}$ as

$$f_0(t) := f(e^{it}).$$

If $(c_n)_{n=1}^\infty$ is a sequence in $(0, +\infty)$, let us consider the set

$$\mathcal{A}((c_n), \mathbb{T}) := \{f \in A^\infty(\mathbb{D}) : \text{there are infinitely many } n \in \mathbb{N} \text{ such that } |f_0^{(n)}(t)| > c_n \text{ for all } t \in \mathbb{R}\}.$$

The following elementary lemma connecting two function spaces will be needed.

Lemma 4.1. *Let $C_{2\pi}(\mathbb{R})$ be the space of all 2π-periodic continuous functions from $\mathbb{R}$ into $\mathbb{C}$, endowed with the topology of uniform convergence on $\mathbb{R}$. Then, for every $k \in \mathbb{N}$, the mapping*

$$f \in A^\infty(\mathbb{D}) \to f_0^{(k)} \in C_{2\pi}(\mathbb{R})$$

is continuous.

Proof. Due to the linearity of the mapping, we only have to prove the continuity at zero. Let $(f_j)_{j=1}^\infty$ be a sequence in $A^\infty(\mathbb{D})$ such that $f_j \to 0$ as $j \to \infty$. This means that, for each $n \in \mathbb{N}_0$, we have that $f_j^{(n)} \to 0$ uniformly on $\overline{\mathbb{D}}$. Then $f_j \to 0$ uniformly on $\overline{\mathbb{D}}$, so

$$(f_j)_0(t) = f_j(e^{it}) \to 0$$

uniformly on $\mathbb{R}$. Also $f_j' \to 0$ uniformly on $\overline{\mathbb{D}}$ as $j \to \infty$. Then

$$((f_j)_0)' = ie^{it} f_j'(e^{it}) \to 0$$

uniformly on $\mathbb{R}$, since the function ie^{it} is bounded. Again, $f_j'' \to 0$ uniformly on $\overline{\mathbb{D}}$ as $j \to \infty$. Therefore

$$((f_j)_0)'' = -e^{it} f_j'(e^{it}) - e^{2it} f_j''(e^{it}) \to 0$$

uniformly on $\mathbb{R}$, since the functions $-e^{it}$ and $-e^{2it}$ are bounded. With this procedure, we obtain our result after a finite number of steps. □

Proposition 4.2.

(a) *For each sequence* $(c_n)_{n=1}^\infty \subset (0, +\infty)$*, the set* $\mathcal{A}((c_n), \mathbb{T})$ *is residual in* $A^\infty(\mathbb{D})$*.*
(b) *The set* $\mathcal{PS}(\mathbb{T})$ *is residual in* $A^\infty(\mathbb{D})$*.*

Proof. (a) Let us fix $(c_n)_{n=1}^\infty \subset (0, +\infty)$. Firstly, we show that $\mathcal{A}((c_n), \mathbb{T})$ is a G_δ subset of $A^\infty(\mathbb{D})$ or, equivalently, that the set

$$\mathcal{B} := A^\infty(\mathbb{D}) \setminus \mathcal{A}((c_n), \mathbb{T})$$

is an F_σ subset. For this, observe that $\mathcal{B} = \bigcup_{N\in\mathbb{N}} A_N$, where

$$A_N := \bigcap_{\substack{k>N \\ k\in\mathbb{N}}} B_k$$

and

$$B_k := \{f \in A^\infty(\mathbb{D}) : \text{ there exists } t \in [0, 2\pi] \text{ such that } |f_0^{(k)}(t)| \le c_k\}.$$

From Lemma 4.1 and the continuity of each evaluation map $g \in C_{2\pi}(\mathbb{R}) \mapsto g(t) \in \mathbb{C}$ $(t \in \mathbb{R})$, we get the continuity of each map

$$\Phi_k : (f, t) \in A^\infty(\mathbb{D}) \times [0, 2\pi] \longmapsto f_0^{(k)}(t) \in \mathbb{C} \quad (k \in \mathbb{N}).$$

Hence the set $\Phi_k^{-1}(\{w : |w| \le c_k)$ is closed in $A^\infty(\mathbb{D}) \times [0, 2\pi]$. Consequently, its projection on $A^\infty(\mathbb{D})$ is closed, because it is a projection that is parallel to $[0, 2\pi]$, which is compact. But such projection is precisely B_k, so B_k is closed. Since A_N is an intersection of certain sets B_k, we obtain that each A_N is also closed. It follows that $\mathcal{B}$ is an F_σ set, as required.

Finally, we will prove that $\mathcal{A}((c_n), \mathbb{T})$ is dense in $A^\infty(\mathbb{D})$. Since $\mathcal{A}((c_n), \mathbb{T}) = \bigcap_{n\in\mathbb{N}}(A^\infty(\mathbb{D}) \setminus A_n)$, by Baire's category theorem (see, e.g., [44]), it is enough to show that each set

$$A^\infty(\mathbb{D}) \setminus A_n = \{f \in A^\infty(\mathbb{D}) : \text{there is } k > n \text{ such that} \\ |f_0^{(k)}(t)| > c_k \text{ for all } t \in \mathbb{R}\}$$

is dense in $A^\infty(\mathbb{D})$. To this end, we fix an $n \in \mathbb{N}$. We also fix a function $g(z) = \sum_{l=0}^\infty a_l z^l \in A^\infty(\mathbb{D})$, $\varepsilon > 0$ and $N \in \mathbb{N}$ and consider the basic open neighborhood of g

$$S = S(g, \varepsilon, N) := \big\{ f \in A^\infty(\mathbb{D}) : \, |f^{(j)}(z) - g^{(j)}(z)| < \varepsilon \\ \text{for all } z \in \overline{\mathbb{D}} \text{ and all } j \in \{0, 1, \dots, N\} \big\}.$$

Our goal is to show that $S \cap (A^\infty(\mathbb{D}) \setminus A_n) \neq \varnothing$. Let us choose any natural number $k > \max\{n, N\}$. According to Lemma 2.5(c), the sequence $\big(l^{k+2} a_l\big)_{l=1}^\infty$ is bounded, so the series $\sum_{l=1}^\infty l^k \, |a_l|$ converges. Thus, we can choose another natural number m satisfying

$$m > \max \left\{ N, \frac{2}{\varepsilon}\big(c_k + \sum_{l=0}^\infty l^k |a_l|\big) \right\}.$$

Now, we define the function

$$f(z) := g(z) + \frac{\varepsilon \, z^m}{2m(m-1)(m-2)\cdots(m-N+1)}.$$

Clearly, $f \in A^\infty(\mathbb{D})$. For every $j \in \{0, 1, \dots, N\}$ and every $z \in \overline{\mathbb{D}}$, we estimate

$$|f^{(j)}(z) - g^{(j)}(z)| = \left| \frac{\varepsilon \, m(m-1)(m-2)\cdots(m-j+1) z^{m-j}}{2\, m(m-1)(m-2)\cdots(m-N+1)} \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

This shows that $f \in S$. Now, fix $t \in \mathbb{R}$ and estimate

$$\begin{aligned} |f_0^{(k)}(t)| &= \left| \frac{\varepsilon \, (im)^k e^{imt}}{2m(m-1)(m-2)\cdots(m-N+1)} + \sum_{l=0}^\infty a_l (il)^k e^{ilt} \right| \\ &\geq \frac{\varepsilon \, m^k}{2m(m-1)(m-2)\cdots(m-N+1)} - \sum_{l=0}^\infty l^k |a_l| \\ &\geq \frac{\varepsilon}{2} \, m^{k-N} - \sum_{l=0}^\infty l^k |a_l| \geq \frac{\varepsilon \, m}{2} - \sum_{l=0}^\infty l^k |a_l| > c_k. \end{aligned}$$

Consequently, f does not belong to A_n, so $S \cap (A^\infty(\mathbb{D}) \setminus A_n) \neq \varnothing$, as required.

(b) Fix $f \in \mathcal{A}((n!\, n^n), \mathbb{T})$. Then $f \in A^\infty(\mathbb{D})$ and there is an infinite subset $M \subset \mathbb{N}$ such that $|f_0^{(n)}(t)| > n!\, n^n$ for all $n \in M$. Therefore,

$$\limsup_{n \to \infty} \left| \frac{f_0^{(n)}(t)}{n!} \right|^{1/n} \geq \limsup_{n \to \infty} n = +\infty.$$

This implies that $R(f_0,t)=0$ for all $t\in\mathbb{R}$. In other words, $f\in\mathcal{PS}(\mathbb{T})$. Thus, $\mathcal{A}((n!\,n^n),\mathbb{T})\subset\mathcal{PS}(\mathbb{T})$. From part (a), $\mathcal{A}((n!\,n^n),\mathbb{T})$ is residual. Consequently, the bigger set $\mathcal{PS}(\mathbb{T})$ is also residual. □

Remark 4.3. With only trivial changes in the proof of assertion (a) in the last proposition, it can be shown that, for each sequence $(c_n)_{n=1}^\infty\subset(0,+\infty)$ and each *infinite subset* $M\subset\mathbb{N}$, the set

$$\mathcal{A}((c_n),M,\mathbb{T}) := \{f\in A^\infty(\mathbb{D}) : \text{there are infinitely many } n\in M \text{ such that } |(f_0)^{(n)}(t)|>c_n \text{ for all } t\in\mathbb{R}\}$$

is residual in $A^\infty(\mathbb{D})$.

Theorem 4.4. *The set* $\mathcal{PS}(\mathbb{T})$ *is strongly* $\mathfrak{c}$*-algebrable.*

Proof. Similarly to the proof of Theorem 3.1, take any $f\in\mathcal{PS}(\mathbb{T})$ and apply Lemma 2.1(b) together with Lemma 2.4 with $\Omega=\overline{\mathbb{D}}$ and $\mathcal{F}=\mathcal{PS}(\mathbb{T})$. □

We will show that dense-lineability holds for families of smooth functions whose restrictions to the circle have large derivatives of many orders at all points.

Theorem 4.5.

(a) *For each sequence* $(c_n)_{n=1}^\infty\subset(0,+\infty)$*, the set* $\mathcal{A}((c_n),\mathbb{T})$ *is dense-lineable in* $A^\infty(\mathbb{D})$.

(b) *The set* $\mathcal{PS}(\mathbb{T})$ *is* $\mathfrak{c}$*-dense-lineable in* $A^\infty(\mathbb{D})$.

Proof. (a) According to Proposition 4.2(a), there exists a function $f_1\in\mathcal{A}(n(c_n+n^n),\mathbb{T})$, so there is an infinite subset $M_1\subset\mathbb{N}$ such that

$$|(f_1)_0^{(n)}(t)|>n(c_n+n^n)$$

for all $n\in M_1$ and all $t\in\mathbb{R}$. Now, by Proposition 4.2(a) combined with Remark 4.3, there exist $f_2\in A^\infty(\mathbb{D})$ and an infinite set $M_2\subset M_1$ such that

$$|(f_2)_0^{(n)}(t)|>n^2(c_n+n^n)$$

for all $n\in M_2$ and all $t\in\mathbb{R}$. Proceeding recursively, we can find a sequence $(f_k)_{k=1}^\infty$ in $A^\infty(\mathbb{D})$ as well as a nested sequence of infinite sets

$$\mathbb{N}\supset M_1\supset M_2\supset\cdots\supset M_k\supset\cdots$$

satisfying

$$|(f_k)_0^{(n)}(t)|>n^k(c_n+n^n)$$

for all $k\in\mathbb{N}$, all $n\in M_k$ and all $t\in\mathbb{R}$. Consider the set

$$A_0:=\operatorname{span}\{f_k:\,k\in\mathbb{N}\},$$

which is a vector subspace of $A^\infty(\mathbb{D})$. As in the case of $A(\mathbb{D})$, we obtain from Lemma 2.5 that the set $\mathcal{P}$ of restrictions to $\overline{\mathbb{D}}$ of all polynomials on $\mathbb{C}$ is dense in $A^\infty(\mathbb{D})$ and that $A^\infty(\mathbb{D})$ is separable. Hence $\mathcal{P}$ is dense-lineable. Since $0\in\mathcal{P}$, we get

$$A_0\setminus\{0\}\subset(A_0\setminus\{0\})+\mathcal{P}.$$

Therefore, in order to apply Lemma 2.3(b) to the space $X := A^\infty(\mathbb{D})$, the vector subspace A_0 and the sets $B := \mathcal{P}$ and $A := \mathcal{A}((c_n), \mathbb{T})$, it is enough to show that the f_n's are linearly independent and that

$$(A_0 \setminus \{0\}) + \mathcal{P} \subset \mathcal{A}((c_n), \mathbb{T}).$$

In turn, since $0 \in \mathcal{P}$ but $0 \notin \mathcal{A}((c_n), \mathbb{T})$, it suffices to prove that for any finite nontrivial linear combination F of the f_n's (say, $F = \lambda_1 f_1 + \cdots + \lambda_p f_p$ with $p \in \mathbb{N}$ and $\lambda_p \neq 0$) and any $P(z) = \sum_{j=1}^m a_j z^j \in \mathcal{P}$, we have

$$h := F + P \in \mathcal{A}((c_n), \mathbb{T}).$$

To this end, note that for $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$h_0^{(n)}(t) = F_0^{(n)}(t) + P_0^{(n)}(t) = \lambda_1 (f_1)_0^{(n)}(t) + \cdots + \lambda_p (f_p)_0^{(n)}(t) + P_0^{(n)}(t).$$

Since $P_0^{(n)}(t) = \sum_{j=1}^m a_j (ij)^n e^{ijt}$, by the triangle inequality we therefore obtain

$$|h_0^{(n)}(t)| \geq \Big(|\lambda_p| n^p - \sum_{k=1}^{p-1} |\lambda_k| n^k \Big)(c_n + n^n) - \Big(\sum_{j=1}^m |a_j| \Big) m^n$$

for all $t \in \mathbb{R}$ and all $n \in M_p$. Now, one can choose $n_0 \in \mathbb{N}$ such that

$$|\lambda_p| n^p - \sum_{k=1}^{p-1} |\lambda_k| n^k > 1 \quad \text{and} \quad n^n > \Big(\sum_{j=1}^m |a_j| \Big) m^n \quad \text{for all } n > n_0.$$

Hence the infinite set $\widetilde{M} := \{n \in M_p : n > n_0\}$ satisfies $|h_0^{(n)}(t)| > c_n$ $(t \in \mathbb{R})$ provided that $n \in \widetilde{M}$. Consequently, $h \in A$, as required.

(b) Note that since $\mathcal{A}((n^n\, n!), \mathbb{T}) \subset \mathcal{PS}(\mathbb{T})$, it follows from (a) the dense-lineability of $\mathcal{PS}(\mathbb{T})$. But we demand more than this, namely, the $\mathfrak{c}$-dense-lineability of this set. To show this, we proceed as in the proof of Theorem 3.3. By Theorem 4.4, $\mathcal{PS}(\mathbb{T})$ is $\mathfrak{c}$-lineable. Setting $X = A^\infty(\mathbb{D})$, $A = \mathcal{PS}(\mathbb{T})$, $B = \mathcal{P}$ and $\alpha = \mathfrak{c}$, it is evident that B is dense-lineable, $A \cap B = \varnothing$ and $A + B \subset A$. Consequently, the conditions in Lemma 2.3(a) are fulfilled, so the result follows. □

Theorem 4.6. *There exists a Banach space X satisfying the following properties:*

(a) *$X \subset A^\infty(\mathbb{D})$ and $X \setminus \{0\} \subset \mathcal{PS}(\mathbb{T})$.*
(b) *X is infinite dimensional.*
(c) *The norm topology of X is stronger than the one inherited from $A^\infty(\mathbb{D})$.*

Proof. To obtain (a) and (b), we follow steps similar to those in the proof of Theorem 3.4, so the details are left to the interested reader (if any). Fix any $f \in \mathcal{PS}(\mathbb{T})$ and set $v(z) := e^{-|z|}$. Define

$$X := \big\{ \varphi \circ f : \varphi \in H_v(\mathbb{C}),\, \varphi(0) = 0 \big\},$$

that is a vector space contained in $A^\infty(\mathbb{D})$. If $g \in X \setminus \{0\}$, then there is $\varphi \in H_v(\mathbb{C})$ such that $\varphi \neq 0$, $\varphi(0) = 0$ and $g = \varphi \circ f$. Then φ is a nonconstant holomorphic function on $\mathbb{C}$, so $g \in \mathcal{PS}(\mathbb{T})$ according to Lemma 2.1(b). If $g = \varphi \circ f \in X$ (with $\varphi \in H_v(\mathbb{C})$, $\varphi(0) = 0$), the mapping $\|g\|_X := \|\varphi\|_v$ is a well-defined, complete norm on X. That $\dim(X) = \infty$ follows from the fact the functions $f(z)^k$ ($k \in \mathbb{N}$) belong to X and are linearly independent. Then (a) and (b) are proved.

To prove (c), consider the family of seminorms

$$\|g\|_N := \sup_{z \in \overline{\mathbb{D}}} |g^{(N)}(z)| \qquad (N \in \mathbb{N}_0),$$

which define the topology of $A^\infty(\mathbb{D})$. Fix $N \in \mathbb{N}_0$. Our goal is to exhibit a constant $K \in (0, +\infty)$, depending only on N, such that

$$\|g\|_N \le K \cdot \|g\|_X \ \text{ for all } \ g \in X.$$

If $N = 0$, this is done at the end of the proof of Theorem 3.4. Thus, we can assume $N \in \mathbb{N}$. Firstly, Harutyunyan and Lusky [36] proved in 2008 that if $a > 0$ and $v_a(z) := e^{-a|z|}$, then the differentiation operator $D : \varphi \mapsto \varphi'$ is well-defined and bounded on $H_{v_a}(\mathbb{C})$. In particular, each mapping $D^j : \varphi \mapsto \varphi^{(j)}$ ($j = 0, 1, \dots, N$) is well-defined and continuous from our $H_v(\mathbb{C})$ into itself, so that there are constants $\alpha_j \in (0, +\infty)$ ($j = 0, 1, \dots, N$) such that

$$\|\varphi^{(j)}\|_v \le \alpha_j \|\varphi\|_v \ \text{ for all } \ \varphi \in H_v(\mathbb{C}).$$

Now, assume that $g \in X$. Then there is $\varphi \in H_v(\mathbb{C})$ with $g = \varphi \circ f$. According to Faà di Bruno's formula (see, e.g., [25]), we have

$$g^{(N)} = \sum_{j=0}^{N} (\varphi^{(j)} \circ f) \cdot B_{N,j}(f', f'', \dots, f^{(N-j+1)}),$$

where $B_{N,j}$ ($0 \le j \le N$) are polynomials in $N - j + 1$ variables, depending only on N. Denote

$$\beta := e^{\sup_{z \in f(\overline{\mathbb{D}})} |z|}$$

and

$$\gamma := \sup_{0 \le j \le N} \sup_{z \in \overline{\mathbb{D}}} |B_{N,j}(f'(z), f''(z), \dots, f^{(N-j+1)}(z))|.$$

We have:

$$\begin{aligned}
\|g\|_N &= \sup_{z \in \overline{\mathbb{D}}} |g^{(N)}(z)| \le \gamma \cdot \sum_{j=0}^{N} \sup_{z \in \overline{\mathbb{D}}} |\varphi^{(j)}(f(z))| \\
&\le \gamma \cdot \sum_{j=0}^{N} \beta \cdot \sup_{z \in f(\overline{\mathbb{D}})} e^{-|z|} |\varphi^{(j)}(z)| \le \beta\, \gamma \cdot \sum_{j=0}^{N} \sup_{z \in \mathbb{C}} e^{-|z|} |\varphi^{(j)}(z)| \\
&= \beta\, \gamma \cdot \sum_{j=0}^{N} \|\varphi^{(j)}\|_v \le \beta\, \gamma \cdot \sum_{j=0}^{N} \alpha_j \|\varphi\|_v = K \cdot \|g\|_X,
\end{aligned}$$

where $K = \beta\,\gamma \cdot \sum_{j=0}^{N} \alpha_j$. This proves that the topology on X is stronger than the one inherited from $A^\infty(\mathbb{D})$, so the proof is finished. $\square$

Remarks 4.7. 1. By taking into account that convergence in $H_v(\mathbb{C})$ implies uniform convergence on compacta in $\mathbb{C}$ of derivatives of all orders, we can use the same example of Remark 3.5.1 (with $f \in \mathcal{PS}(\mathbb{T})$ this time) to show that the norm topology of the space X in Theorem 4.6 is *strictly* stronger than the one inherited from $A^\infty(\mathbb{D})$.

2. Let $H_e(\mathbb{D})$ be the class of functions in $H(\mathbb{D})$ that cannot be continued holomorphically beyond $\mathbb{T}$. In [13] and [16] the dense-lineability and the strong $\mathfrak{c}$-algebrability of $H_e(\mathbb{D}) \cap A^\infty(\mathbb{D})$ were proved. But, plainly,

$$\mathcal{PS}(\mathbb{T}) \subset \{f \in A^\infty(\mathbb{D}) : \, f_0 \text{ is not analytic at any } t \in \mathbb{R}\} \subset H_e(\mathbb{D}) \cap A^\infty(\mathbb{D}).$$

Hence Theorems 4.4 and 4.5 above provide two new approaches to get these properties.

2. Despite the guarantee of a large supply of functions in $\mathcal{PS}(\mathbb{T})$ given in Proposition 4.2 and Theorems 4.4-4.5-4.6, it might be of some interest to exhibit an explicit member of such special family. Since $\mathcal{A}((n!\,n^n),\mathbb{T}) \subset \mathcal{PS}(\mathbb{T})$, it is enough to give an example of a function $f \in \mathcal{A}((c_n),\mathbb{T})$ for every prescribed sequence $(c_n)_{n=1}^\infty \subset (0,+\infty)$. To do this, notice that if we replace each c_n by any natural number bigger than c_n, then we can assume $(c_n)_{n=1}^\infty \subset \mathbb{N}$. Let us define recursively the sequence $(b_n)_{n=1}^\infty$ of natural numbers as $b_1 := 2 + c_1$ and

$$b_k = c_k + 2 + \sum_{j=1}^{k-1} b_j^{k+1-j}$$

for $k \geq 2$. It is easy to see that $(b_k)_{k=1}^\infty$ is increasing and $b_k > 2$ for all $k \in \mathbb{N}$. Define the function $f : \overline{\mathbb{D}} \to \mathbb{C}$ by the lacunary power series

$$f(z) = \sum_{k=1}^\infty b_k^{1-k} z^{b_k}.$$

For every fixed $N \in \mathbb{N}$ and every $k \geq N+1$, we have that $b_k^{N+1-k} \leq 2^{N+1-k}$. The majoring sequence $\left(2^{N+1-k}\right)_{k=1}^\infty$ tends to 0, so $\left(b_k^N b_k^{1-k}\right)_{k=1}^\infty$ is bounded. This entails $f \in A^\infty(\mathbb{D})$ by using Lemma 2.5(c). Finally, observe that

$$f_0(t) = f(e^{it}) = \sum_{k=1}^\infty b_k^{1-k} e^{ib_k t} \qquad (t \in \mathbb{R})$$

and this function is proved to satisfy $|f_0^{(n)}(t)| > c_n$ $(n \in \mathbb{N},\, t \in \mathbb{R})$ in [12, Section 2]. Then $f \in \mathcal{A}((c_n),\mathbb{T})$.

Similarly to the end of Section 3, we conclude with a natural *question:*

Is $\mathcal{PS}(\mathbb{T})$ *spaceable* in $A^\infty(\mathbb{D})$*?*

Note that $\mathcal{PS}(\mathbb{T})$ is *not* spaceable *in* $A(\mathbb{D})$. Indeed, if $\mathcal{PS}(\mathbb{T}) \cup \{0\}$ contained a closed infinite dimensional vector subspace M of $A(\mathbb{D})$, then $M_0 := \{f_0|_{[0,2\pi]} : f \in M\}$ would be a closed infinite dimensional vector subspace of $C([0,2\pi])$ (use the maximum modulus principle) consisting of differentiable functions, which would contradict a celebrated theorem due to Gurariy [34].

Acknowledgements. The first author was supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 Grant P08-FQM-03543 and by MEC Grant MTM2015-65242-C2-1-P. The second author was supported by MINECO Project MTM2016-75963-P. The third and fourth authors were supported by MEC Grant MTM2015-65825-P.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, London, 1979.
[2] A. Aizpuru, C. Pérez-Eslava, and J. B. Seoane-Sepúlveda, *Linear structure of sets of divergent sequences and series*, Linear Algebra Appl. **418** (2006), no. 2-3, 595–598, DOI 10.1016/j.laa.2006.02.041.
[3] R. Aron, L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Lineability: The search for linearity in Mathematics*, Monographs and Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2016.
[4] R. M. Aron, F. J. García-Pacheco, D. Pérez-García, and J. B. Seoane-Sepúlveda, *On dense-lineability of sets of functions on* $\mathbb{R}$, Topology **48** (2009), no. 2-4, 149–156.
[5] H. Auerbach and S. Banach, *Über die Höldersche Bedingung*, Studia Math. **3** (1931), 180–184 (in German).
[6] M. Balcerzak, A. Bartoszewicz, and M. Filipczak, *Nonseparable spaceability and strong algebrability of sets of continuous singular functions*, J. Math. Anal. Appl. **407** (2013), no. 2, 263–269.
[7] S. Banach, *Über die Bairesche Kategorie gewisser Funktionenmengen*, Studia Math. **3** (1931), 174–179 (in German).
[8] A. Bartoszewicz, M. Bienias, M. Filipczak, and S. Głąb, *Strong $\mathfrak{c}$-algebrability of strong Sierpiński-Zygmund, smooth nowhere analytic and other sets of functions*, J. Math. Anal. Appl. **412** (2014), no. 2, 620–630.
[9] A. Bartoszewicz and S. Głąb, *Large function algebras with certain topological properties*, J. Funct. Spaces **2015**, Article ID 761924, 7 pages.
[10] F. Bastin, J. A. Conejero, C. Esser, and J. B. Seoane-Sepúlveda, *Algebrability and nowhere Gevrey differentiability*, Israel J. Math. **205** (2015), no. 1, 127–143.
[11] F. Bayart and L. Quarta, *Algebras in sets of queer functions*, Israel J. Math. **158** (2007), no. 1, 285–296.
[12] L. Bernal-González, *Funciones con derivadas sucesivas grandes y pequeñas por doquier*, Collect. Math. **38** (1987), 117–122 (in Spanish).
[13] ———, *Linear Kierst-Szpilrajn theorems*, Studia Math. **166** (2005), no. 1, 55–69.
[14] ———, *Lineability of sets of nowhere analytic functions*, J. Math. Anal. Appl. **340** (2008), no. 2, 1284–1295.
[15] ———, *Algebraic genericity of strict-order integrability*, Studia Math. **199** (2010), no. 3, 279–293.
[16] ———, *Vector spaces of non-extendable holomorphic functions*, J. Anal. Math., to appear.
[17] L. Bernal-González and M. C. Calderón-Moreno, *Large algebras of singular functions vanishing on prescribed sets*, Results Math. **71** (2017), no. 3, 1207–1221.

[18] L. Bernal-González, J. López-Salazar, and J. B. Seoane, *On Weierstrass' monsters in the disc algebra*, Preprint (2017).

[19] L. Bernal-González and M. Ordóñez Cabrera, *Lineability criteria, with applications*, J. Funct. Anal. **266** (2014), no. 6, 3997–4025.

[20] L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 1, 71–130.

[21] D. Cariello and J. B. Seoane-Sepúlveda, *Basic sequences and spaceability in ℓ_p spaces*, J. Funct. Anal. **266** (2014), no. 6, 3797–3814, DOI 10.1016/j.jfa.2013.12.011.

[22] H. Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, 6th ed., Hermann, Paris, 1997 (in French).

[23] F. S. Cater, *Differentiable, nowhere analytic functions*, Amer. Math. Monthly **91** (1984), no. 10, 618–624.

[24] K. C. Ciesielski, J. L. Gámez-Merino, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Lineability, spaceability, and additivity cardinals for Darboux-like functions*, Linear Algebra Appl. **440** (2014), 307–317, DOI 10.1016/j.laa.2013.10.033.

[25] A. D. D. Craik, *Prehistory of Faà di Bruno's Formula*, Amer. Math. Monthly **112** (2005), no. 2, 217–234.

[26] R. B. Darst, *Most infinitely differentiable functions are nowhere analytic*, Canad. Math. Bull. **16** (1973), 597–598.

[27] A. Eskenazis, *Topological genericity of nowhere differentiable functions in the disc algebra*, Arch. Math. (Basel) **103** (2014), no. 1, 85–92.

[28] A. Eskenazis and K. Makridis, *Topological genericity of nowhere differentiable functions in the disc and polydisc algebras*, J. Math. Anal. Appl. **420** (2014), no. 1, 435–446.

[29] C. Esser, *Generic results in classes of ultradifferentiable functions*, J. Math. Anal. Appl. **413** (2014), no. 1, 378–391.

[30] V. P. Fonf, V. I. Gurariy, and M. I. Kadets, *An infinite dimensional subspace of $C[0,1]$ consisting of nowhere differentiable functions*, C. R. Acad. Bulgare Sci. **52** (1999), no. 11-12, 13–16.

[31] J. L. Gámez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *Lineability and additivity in $\mathbb{R}^{\mathbb{R}}$*, J. Math. Anal. Appl. **369** (2010), no. 1, 265–272.

[32] J. L. Gámez-Merino, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *A characterization of continuity revisited*, Amer. Math. Monthly **118** (2011), no. 2, 167–170, DOI 10.4169/amer.math.monthly.118.02.167.

[33] J. L. Gámez-Merino and J. B. Seoane-Sepúlveda, *An undecidable case of lineability in $\mathbb{R}^{\mathbb{R}}$*, J. Math. Anal. Appl. **401** (2013), no. 2, 959–962, DOI 10.1016/j.jmaa.2012.10.067.

[34] V. I. Gurariĭ, *Subspaces and bases in spaces of continuous functions*, Dokl. Akad. Nauk SSSR **167** (1966), 971–973 (Russian).

[35] ———, *Linear spaces composed of everywhere nondifferentiable functions*, C. R. Acad. Bulgare Sci. **44** (1991), no. 5, 13–16 (in Russian).

[36] A. Harutyunyan and W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. **184** (2008), 233–247.

[37] S. Hencl, *Isometrical embeddings of separable Banach spaces into the set of nowhere approximatively differentiable and nowhere Hölder functions*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3505–3511.

[38] B. R. Hunt, *The prevalence of continuous nowhere differential functions*, Proc. Amer. Math. Soc. **122** (1994), no. 3, 711–717.

[39] P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *On Weierstrass' Monsters and lineability*, Bull. Belg. Math. Soc. Simon Stevin **20** (2013), no. 4, 577–586.

[40] K. Kavvadias and K. Makridis, *Nowhere differentiable functions with respect to the position*, arXiv:1701.04875v1 [math.CV], Preprint (2017).

[41] W. Lusky, *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc. **51** (1995), 309–320.
[42] S. Mazurkiewicz, *Sur les fonctions non dérivables*, Studia Math. **3** (1931), 92-94.
[43] D. Morgenstern, *Unendlich oft differenzierbare nicht-analytische Funktionen*, Math. Nachr. **12** (1954), 74 (in German).
[44] J. C. Oxtoby, *Measure and Category,* 2nd edition, Springer-Verlag, New York, 1980.
[45] T. I. Ramsamujh, *Nowhere analytic C^∞ functions*, J. Math. Anal. Appl. **160** (1991), 263–266.
[46] L. Rodríguez-Piazza, *Every separable Banach space is isometric to a space of continuous nowhere differentiable functions*, Proc. Amer. Math. Soc. **123** (1995), no. 12, 3649–3654.
[47] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.
[48] H. Salzmann and K. Zeller, *Singularitäten unendlicht oft differenzierbarer Funktionen*, Math. Z. **62** (1955), 354–367 (in German).
[49] J. Siciak, *Regular and singular points of C^∞ functions*, Zesz. Nauk. Politech. Sl., Math.-Fiz. **48** (1986), no. 853, 127–146 (in Polish).
[50] K. Weierstrass, *Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen*, Gelesen Akad. Wiss, 18 Juli 1872 (in German).

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Avenida Reina Mercedes, Sevilla, 41080, Spain
E-mail address: lbernal@us.es

Departamento de Análisis Matemático
Universidad de la Laguna
La Laguna (Tenerife), 38271, Spain
E-mail address: abonilla@ull.es

Departamento de Matemática Aplicada a las Tecnologías de la Información y las Comunicaciones
Escuela Técnica Superior de Ingeniería y Sistemas de Telecomunicación
Universidad Politécnica de Madrid
Carretera de Valencia, Km. 7.
Madrid, 28031, Spain
E-mail address: jeronimo.lopezsalazar@upm.es

Instituto de Matemática Interdisciplinar (IMI)
Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
Plaza de Ciencias 3
Madrid, 28040, Spain
E-mail address: jseoane@ucm.es