Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations[*]

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October 14, 2018

Abstract

We first prove the existence and regularity of the trajectory attractor for a three-dimensional system of globally modified Navier-Stokes equations. Then we use the natural translation semigroup and trajectory attractor to construct the trajectory statistical solutions in the trajectory space. In our construction the trajectory statistical solution is an invariant Borel probability measure, which is supported by the trajectory attractor and is invariant under the action of the translation semigroup. As a byproduct of the regularity of the trajectory attractor, we obtain the asymptotic regularity of the trajectory statistical solution in the sense that it is supported by a set in the trajectory space in which all weak solutions are in fact strong solutions.

Keywords: Globally modified Navier-Stokes equations; Trajectory attractor; Trajectory statistical solution; Invariant measure; Asymptotic regularity

MSC2010: 35B41, 34D35, 76F20

1　Introduction

We study the following three-dimensional (3D) system of globally modified Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|_V)(u \cdot \nabla)u + \nabla p = g(x,t) \quad \text{in} \quad (0,+\infty) \times \Omega, \qquad (1.1)$$

$$\nabla \cdot u = 0, \qquad (1.2)$$

$$u = 0 \quad \text{on} \quad (0,+\infty) \times \partial \Omega, \qquad (1.3)$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \qquad (1.4)$$

where the unknown functions $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote, respectively, the velocity field of the fluid and the kinematic pressure; $g(x,t) = (g_1(x,t), g_2(x,t), g_3(x,t))$ is the density of volume forces and $\nu > 0$ is the viscosity coefficient. In (1.1), $\|u\|_V = \|\nabla u\|_{(L^2(\Omega))^3}$, and the function $F_N(\cdot) : (0,+\infty) \mapsto (0,1]$ is defined as

$$F_N(r) = \min\big\{1, N/r\big\}, \quad r \in (0,+\infty),$$

where $N > 0$ is given. In addition, $x = (x_1, x_2, x_3) \in \Omega$ and $\Omega \subset \mathbb{R}^3$ is an open bounded subset and smooth enough such that the Poincaré inequality holds true.

The above globally modified Navier-Stokes equations were initiated in the paper [3]. The modifying factor $F_N(\|u\|_V)$ depends on the norm $\|u\|_V = \|\nabla u\|_{(L^2(\Omega))^3}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions ([3,4]).

The globally modified Navier-Stokes equations are interesting themselves, but, more importantly, can be used to obtain useful information about the Navier-Stokes equations. We know that the Navier-Stokes equations are valid for accurate description of the motion of the fluid only in the case that $|\nabla u|$ is relatively small (see [18]). Obviously, if $\|\nabla u\|_{(L^2(\Omega))^3} \leqslant N$, then equations (1.1)-(1.2) reduce to the incompressible Navier-Stokes equations. So the number N is the threshold value for ∇u. Also, the globally modified Navier-Stokes equations were used as an intermediate step by Kloeden and Valero in [16] to prove that the attainability set of the weak solutions of the 3D Navier-Stokes equations which satisfy an energy constraint is compact and connected set in the weak topology.

Nowadays, the globally modified Navier-Stokes equations have been widely studied by several researchers in some articles, including the existence and uniqueness of weak and strong solutions [3, 5, 10, 15, 25, 27]; well-posedness and pullback attractor with infinite delays [23, 24], invariant measures and statistical solutions [4, 17, 22, 37], etc.

The invariant measures and statistical solutions have proven to be very useful in the understanding of turbulence in the case of Navier-Stokes equations (see Foias *et al.* [12]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Statistical solutions have

been introduced as a rigorous mathematical object to formalize the notion of ensemble average in the conventional statistical theory of turbulence. There are two main notions of statistical solutions, one introduced by Foias and Prodi [11] (will be called Foias-Prodi statistical solution) and the other by Vishik and Furshikov [29] (will be called Vishik-Furshikov statistical solution). The Foias-Prodi statistical solution defined in [11] is a family of Borel measures parametrized by the time variable and defined on the phase space of the Navier-Stokes equations, representing the probability distribution of the velocity field of the flow at each time. While the Vishik-Furshikov statistical solutions given in [29] is a single Borel measure on the space of trajectories, representing the probability distribution of the space-time velocity field.

We recognize that the Foias-Prodi statistical solutions are associated to some invariant measures defined on the phase space (independent of time t) of the addressed system, while the Vishik-Furshikov statistical solutions are associated to some invariant measures defined on the trajectory space (dependent of time t) of solutions. The invariant measures for well-posed dissipative systems were studied in a series of references (see [6, 19–22, 26, 33]). For instance, Łukaszewicz, Real and Robinson [21] used the notion of Generalized Banach limit to construct the invariant measures for general continuous dynamical systems on metric spaces. Later, Chekroun and Glatt-Holtz [6] improved the results of [21] to construct invariant measures for a broad class of dissipative autonomous dynamical systems. Recently, Łukaszewicz and Robinson [22] extended the result of [6] to construct invariant measures for dissipative non-autonomous dynamical systems.

The construction of invariant measures in [6] depends heavily on the existence of global attractor of the continuous semigroup generated by the solution operators, and in [22] the construction depends heavily on the pullback attractor of the continuous process associated to the solutions operators. The original motivation of this article is to use the natural translation semigroup and trajectory attractor to construct invariant space-time measure and Vishik-Furshikov statistical solution for the globally modified Navier-Stokes equations in trajectory space.

The first main result in this article is the existence and regularity of the trajectory attractor for the globally modified Navier-Stokes equations (1.1)-(1.2). For the existence we will employ the theory of Vishik and Chepyzhov [8, 31]. Consider the natural translation semigroup $\{S(t)\}_{t \geqslant 0}$ defined on the trajectory space $\mathcal{T}_H^{\mathrm{tr}}$ and on the regular trajectory space $\mathcal{T}_V^{\mathrm{tr}}$ (see notations in §2). We will first prove that $\{S(t)\}_{t \geqslant 0}$ possesses respectively a trajectory attractor $\mathcal{A}_H^{\mathrm{tr}}$ in $\mathcal{T}_H^{\mathrm{tr}}$ and $\mathcal{A}_V^{\mathrm{tr}}$ in $\mathcal{T}_V^{\mathrm{tr}}$, and then reveal the regularity and structure of the trajectory attractor by showing

$$\mathcal{A}_H^{\mathrm{tr}} = \mathcal{A}_V^{\mathrm{tr}} = \Pi_+ \mathcal{K}_X = \{u(\cdot)|_{[0,+\infty)} \,\big|\, u \in \mathcal{K}_X\}, \tag{1.5}$$

where $\mathcal{K}_X$, which is called the weak kernel (with $X = H$) or strong kernel (with $X = V$) of equations (1.1)-(1.2), is the set of all bounded complete trajectories of equations (1.1)-(1.2):

$$\mathcal{K}_X = \{u(t), t \in \mathbb{R} \,\big|\, u(t) \text{ is a weak solution of} (1.1) - (1.2) \text{ and } \|u(t)\|_X \leqslant C_u \text{ for } t \in \mathbb{R}\}.$$

Equality (1.5) implies the trajectory asymptotic smooth effect of the globally modified Navier-Stokes equations in the sense that all bounded weak solutions $u(t)$ are in fact strong solution when $t \in \mathbb{R}_+$.

The theory of trajectory attractor was studied systematically by Vishik and Chepyzhov in [8]. Now this theory has been applied to several concrete nonlinear partial differential equations, including the 3D Navier-Stokes equations [30, 31], 2D and 3D non-Newtonian fluids [34, 35], 3D dissipative Euler equations [9], the convective Brinkman-Forchheimer equations [36].

The second main result in this article is concerned with the existence of invariant Borel probability measure and trajectory statistical solution for the globally modified Navier-Stokes equations in the trajectory space. We want to remark that the approach of [6, 22] to construct invariant measure can not be applied directly here. But our ideas are originated from [6, 22]. The main difference is that we consider the invariant measure on the trajectory space $\mathcal{A}_H^{\mathrm{tr}}$ (or $\mathcal{A}_V^{\mathrm{tr}}$) other than on the phase space H. It is worth mentioning that our approach for constructing invariant measure and trajectory statistical solution can be raised to an abstract level for a broad class of evolutionary equations displaying the property of global existence of weak solutions without a known result of global uniqueness, such as the 2D dissipative Euler equations [9], the 3D MHD equations [7, 14, 32, 38] and the 3D convective Brinkman-Forchheimer equations [36]. The fundamental hypotheses on the concrete evolutionary equations is that the trajectory space is a metrizable normal topological space and the natural translation semigroup possesses a compact trajectory attractor.

The paper is arranged as follows. In section 2, we first introduce some notations and operators, then recall some known results concerning the existence, uniqueness and regularity of the solutions. In section 3, we prove the existence and regularity of the trajectory attractor. In section 4, we use the trajectory attractor and translation semigroup to construct the invariant Borel probability measure and trajectory statistical solution. Finally, we summarize the article and provide a remark in section 5.

2   Preliminaries

In this section, we first introduce some notations and operators. Then we present some preliminaries including the existence and uniqueness of solutions, and a compact embedding theorem which is essential in our applications.

In this paper we use the following notations:

$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$;

$\mathbb{L}^p(\Omega) = (L^p(\Omega))^3$–the 3D Lebesgue space with norm $\|\cdot\|_{\mathbb{L}^p(\Omega)}$; $\|\cdot\|_{\mathbb{L}^2(\Omega)} = \|\cdot\|$;

$\mathbb{H}^m(\Omega) =$ the 3D Sobolev space $\{\phi \in \mathbb{L}^2(\Omega)|\, \nabla^k \phi \in \mathbb{L}^2(\Omega), k \leqslant m\}$ with norm $\|\cdot\|_{\mathbb{H}^m(\Omega)}$;

$\mathbb{H}_0^1(\Omega) =$ closure of $\{\phi |\, \phi \in (\mathcal{C}_0^\infty(\Omega))^3\}$ in $\mathbb{H}^1(\Omega)$ with norm $\|\cdot\|_{\mathbb{H}^1(\Omega)}$;

$\mathcal{V} = \{\phi \in (\mathcal{C}_0^\infty(\Omega))^3 |\, \nabla \cdot \phi = 0\}$;

$H =$ closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ with norm $\|\cdot\|$; $H' =$ dual space of H;

$V =$ closure of $\mathcal{V}$ in $\mathbb{H}^1(\Omega)$ with norm $\|\cdot\|_V = \|\cdot\|_{\mathbb{H}^1(\Omega)}$;

$V'=$ dual space of V with norm $\|\cdot\|_{V'}$;

$(\cdot,\cdot)-$ the inner product in H, $\langle\cdot,\cdot\rangle-$ the dual pairing between V and V';

$\text{dist}_M(X,Y)-$ the Hausdorff semidistance between $X \subseteq M$ and $Y \subseteq M$ defined by

$$\text{dist}_M(X,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_M.$$

" $\longrightarrow$ " and " $\rightharpoonup$ " denote the strong and weak convergence, respectively.

To write problem (1.1)-(1.4) as a functional form, we next introduce some operators. Firstly, we consider the operator $A : V \mapsto V'$ defined as

$$\langle Au, v\rangle = (\nabla u, \nabla v), \quad u, v \in V.$$

Denoting $D(A) = \mathbb{H}^2(\Omega) \cap V$, then $Au = -P\Delta u$, $\forall u \in D(A)$, is the Stokes operator, where P is the Leray-Helmholtz projection from $\mathbb{L}^2(\Omega)$ onto H. Secondly, we define a continuous trilinear form

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x, \quad u, v, w \in \mathbb{H}_0^1(\Omega).$$

Note that $V \subseteq \mathbb{H}_0^1(\Omega)$ is a closed subspace, $b(u,v,w)$ is continuous on $V \times V \times V$, and

$$b(u,v,w) = -b(u,w,v), \quad b(u,v,v) = 0, \quad \forall u, v, w \in V.$$

For any $u, v \in V$,

$$\langle B(u,v), w\rangle = b(u,v,w), \quad \forall w \in V,$$

defines a continuous function $B(u,v)$ on $V \times V$. We further set

$$b_N(u,v,w) = F_N(\|v\|_V)b(u,v,w), \quad \forall u, v, w \in V.$$
$$\langle B_N(u,v), w\rangle = b_N(u,v,w), \quad \forall u, v, w \in V.$$

Note that the form $b_N(u,v,w)$ is linear in u and w, but it is nonlinear in v. Obviously,

$$b_N(u,v,v) = 0, \quad \forall u, v \in V. \tag{2.1}$$

For above introduced operators, we select the following estimations.

Lemma 2.1. ([4, 17]) *There exist two positive constants c_1 and c_2 depending only on Ω such that*

$$|b_N(u,v,w)| \leqslant c_1 N \|u\|_V \|w\|_V, \quad \forall u, v, w \in V, \tag{2.2}$$
$$|b_N(u,v,w)| \leqslant c_2 N \|Au\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H. \tag{2.3}$$

With the above notations, we can express, excluding the pressure p, the functional form of problem (1.1)-(1.4) in the solenoidal vector field as following (see e.g. [4, 17]):

$$u'(t) + \nu Au(t) + B_N(u(t), u(t)) = g(t) \text{ in } \mathcal{D}'(0, +\infty; V'), \tag{2.4}$$
$$u(x,0) = u_0, \quad x \in \Omega, \tag{2.5}$$

where $u'(t) = \dfrac{\partial u(t)}{\partial t}$.

We next specify the definition of solutions to problem (2.4)-(2.5).

Definition 2.1. *Let $u_0 \in H$ and $g \in L^2(0,T;H)$ for any $T > 0$. A weak solution of problem (2.4)-(2.5) is any function $u(x,t) \in L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, with $u(x,t)|_{t=0} = u_0$, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),\phi) + \nu\langle Au(t),\phi\rangle + \langle B_N(u(t),u(t)),\phi\rangle = (g(t),\phi), \quad \forall\,\phi \in V,$$

holds in the distribution sense of $\mathcal{D}'(0,+\infty)$. If $u(x,t)$ is a weak solution and furthermore $u(x,t) \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ for all $T > 0$, then $u(x,t)$ is called a strong solution of problem (2.4)-(2.5).

For the existence and uniqueness of solutions to problem (2.4)-(2.5), we have the following result. We want to remark that the result of Lemma 2.2 (1) can be found in [27, Theorem 1.1] and [3, Theorem 7(b), (c)], and the result of Lemma 2.2 (2) is from [3, Theorem 7(a)].

Lemma 2.2. *([3,27]) Suppose that $g(t) \in L^2(0,T;H)$ for all $T > 0$ and let $u_0 \in H$ be given. Then*

(1) *If $u_0 \notin V$, there exists a unique weak solution u of problem (2.4)-(2.5) satisfying*

$$u \in \mathcal{C}([\varepsilon,T];V) \cap L^2(\varepsilon,T;D(A)), \quad for\ all \quad T > \varepsilon > 0, \forall\,\varepsilon > 0. \tag{2.6}$$

(2) *If $u_0 \in V$, there exists a unique strong solution u of problem (2.4)-(2.5) satisfying*

$$u \in \mathcal{C}([0,T];V) \cap L^2(0,T;D(A)), \quad for\ all \quad T > 0. \tag{2.7}$$

We conclude this section with the following useful results.

Lemma 2.3. *([8]) Let Y be a Banach space and $E \hookrightarrow E_0 \subseteq Y$. Also let the embedding $E \hookrightarrow E_0$ be compact. Set*

$$W_{\infty,p}(0,T;E,Y) = \big\{\phi(t), t \in [0,T]\big|\ \phi(t) \in L^\infty(0,T;E),\ \phi'(t) \in L^p(0,T;Y)\big\},$$

with the norm $\|\phi\|_{W_{\infty,p}} = \operatorname*{esssup}_{t\in[0,T]}\{\|\phi(t)\|_E\} + \big(\int_0^T \|\phi'\|_Y^p\big)^{\frac{1}{p}}$, where $p > 1$. Then $W_{\infty,p}(0,T;E,Y) \hookrightarrow \mathcal{C}([0,T];E_0)$ with compact embedding.

Lemma 2.4. *([8]) Let $y(s), K(s) \in L^1_{loc}(0,+\infty)$ and let*

$$-\int_0^{+\infty} y(s)\phi'(s)\mathrm{d}s + \beta \int_0^{+\infty} y(s)\phi(s)\mathrm{d}s \leqslant \int_0^{+\infty} K(s)\phi(s)\mathrm{d}s$$

holds for any $\phi(s) \in \mathcal{C}_0^\infty(\mathbb{R}_+), \phi(s) \geqslant 0$, where $\beta \in \mathbb{R}$. Then for any $t, \tau \in \mathbb{R}_+, t \geqslant \tau$ there holds

$$y(t)e^{\beta t} - y(\tau)e^{\beta \tau} \leqslant \int_\tau^t K(s)e^{\beta s}\mathrm{d}s.$$

3 Existence and regularity of trajectory attractor

In this section, we will introduce the definitions of trajectory space $\mathcal{T}_H^{\text{tr}}$ and regular trajectory space $\mathcal{T}_V^{\text{tr}}$ for equation (2.4), as well as the definitions of trajectory attractor $\mathcal{A}_H^{\text{tr}}$ in $\mathcal{T}_H^{\text{tr}}$ and $\mathcal{A}_V^{\text{tr}}$ in $\mathcal{T}_V^{\text{tr}}$, respectively. The main result of this section is the existence and regularity of trajectory attractor.

We set $H^\eta = (-\Delta)^{-\eta/2} H$ in this paper, where $\eta > 0$ and the Laplace operator Δ is taken with zero boundary condition $u\big|_{\partial\Omega} = 0$. We use $H^{-\eta}$ to denote the dual space of H^η. In the whole article, we take $0 < \eta < 1$ and thus the embeddings $H \hookrightarrow H^{-\eta}$ and $V \hookrightarrow H^{1-\eta}$ are compact. Therefore, we can use the compact embedding result in Lemma 2.3. In addition, we use Π_+ to denote the restriction operator (with respect to time variable) to the semi-infinite interval $\mathbb{R}_+$. Analogously, Π_T stands for the restriction operator to the interval $[0, T]$. For example, if $u(\cdot) \in \mathcal{C}(\mathbb{R}_+; H^{-\eta}) \cap L^\infty(\mathbb{R}_+; H)$, then $\Pi_T u(\cdot) \in \mathcal{C}([0, T]; H^{-\eta}) \cap L^\infty(0, T; H)$; $\Pi_T u(t) = u(t)$ if $t \in [0, T]$.

Definition 3.1.

(1) (Trajectory space) *The trajectory space $\mathcal{T}_H^+$ of equation (2.4) consists of functions $u \in L^\infty(\mathbb{R}_+; H) \cap L^2_{loc}(\mathbb{R}_+; V)$ such that for any $T > 0$ the function $\Pi_T u(t)$ is a weak solution of (2.4) on $[0, T]$.*

(2) (Regular trajectory space) *The regular trajectory space $\mathcal{T}_V^+$ of equation (2.4) consists of functions $u \in L^\infty(\mathbb{R}_+; V) \cap L^2_{loc}(\mathbb{R}_+; D(A))$ such that for any $T > 0$ the function $\Pi_T u(t)$ is a strong solution of (2.4) on $[0, T]$.*

In the sequel, we let $\mathcal{C}([a, b]; H^{-\eta})$ be the space of continuous $H^{-\eta}$-valued functions on $[a, b]$ endowed with the metric

$$\mathrm{d}_{\mathcal{C}([a,b];H^{-\eta})}(u, v) = \sup_{t \in [a,b]} \mathrm{d}_{H^{-\eta}}(u(t), v(t)).$$

Also we denote by $\mathcal{C}([a, +\infty); H^{-\eta})$ the space of continuous $H^{-\eta}$-valued functions on $[a, +\infty)$ endowed with the metric

$$\mathrm{d}_{\mathcal{C}([a,+\infty);H^{-\eta})}(u, v) = \sum_{T \in \mathbb{N}} \frac{1}{2^T} \frac{\sup\{\mathrm{d}_{H^{-\eta}}(u(t), v(t)) : a \leqslant t \leqslant a + T\}}{1 + \sup\{\mathrm{d}_{H^{-\eta}}(w(t), \phi(t)) : a \leqslant t \leqslant a + T\}}. \tag{3.1}$$

We will adopt the topology of $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$ as the topology of the trajectory space $\mathcal{T}_H^{\text{tr}}$, where $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$ is endowed with the compact open topology. Endowed with this topology, the trajectory space $\mathcal{T}_H^{\text{tr}}$ is metrizable. Similarly, the regular trajectory space $\mathcal{T}_V^{\text{tr}}$ is also metrizable with the following metric

$$\mathrm{d}_{\mathcal{C}([0,+\infty);H^{1-\eta})}(u, v) = \sum_{T \in \mathbb{N}} \frac{1}{2^T} \frac{\sup\{\mathrm{d}_{H^{1-\eta}}(u(t), v(t)) : 0 \leqslant t \leqslant T\}}{1 + \sup\{\mathrm{d}_{H^{1-\eta}}(w(t), \phi(t)) : 0 \leqslant t \leqslant T\}}. \tag{3.2}$$

In addition, the natural translation semigroup $\{S(t)\}_{t \geqslant 0}$ on $\mathcal{T}_H^{\text{tr}}$ and $\mathcal{T}_V^{\text{tr}}$ is defined as

$$S(t)u(\cdot) = u(t + \cdot), \, \forall \, u \in \mathcal{T}_H^{\text{tr}} \text{ or } \mathcal{T}_V^{\text{tr}}.$$

We next specify the definition of trajectory attractor.

Definition 3.2.

(1) *A set $\mathcal{A}_H^{\mathrm{tr}} \subseteq \mathcal{T}_H^+$ is called the trajectory attractor of equation (2.4) with respect to the topology $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$ if*

(i) *$\mathcal{A}_H^{\mathrm{tr}}$ is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$ and bounded in $L^\infty(\mathbb{R}_+; H)$;*

(ii) *$S(t)\mathcal{A}_H^{\mathrm{tr}} = \mathcal{A}_H^{\mathrm{tr}}, \ \forall\, t \geqslant 0$;*

(iii) *For any bounded (in $L^\infty(\mathbb{R}_+; H)$ norm) set $\mathcal{B}_H \subset \mathcal{T}_H^+$ and $\forall\, T \geqslant 0$,*

$$\lim_{t \to +\infty} \mathrm{dist}_{\mathcal{C}([0,T]; H^{-\eta})}\big(\Pi_T S(t)\mathcal{B}_H, \Pi_T \mathcal{A}_H^{\mathrm{tr}}\big) = 0.$$

(2) *A set $\mathcal{A}_V^{\mathrm{tr}} \subseteq \mathcal{T}_V^+$ is called the trajectory attractor of equation (2.4) with respect to the topology $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$ if*

(i) *$\mathcal{A}_V^{\mathrm{tr}}$ is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$ and bounded in $L^\infty(\mathbb{R}_+; V)$;*

(ii) *$S(t)\mathcal{A}_V^{\mathrm{tr}} = \mathcal{A}_V^{\mathrm{tr}}, \ \forall\, t \geqslant 0$;*

(iii) *For any bounded (in $L^\infty(\mathbb{R}_+; V)$ norm) set $\mathcal{B}_V \subset \mathcal{T}_V^+$ and $\forall\, T \geqslant 0$,*

$$\lim_{t \to +\infty} \mathrm{dist}_{\mathcal{C}([0,T]; H^{1-\eta})}\big(\Pi_T S(t)\mathcal{B}_V, \Pi_T \mathcal{A}_V^{\mathrm{tr}}\big) = 0.$$

We remark that a set $\mathcal{A}_H^{\mathrm{tr}}$ satisfying property (1)(iii) of Definition 3.2 is called a trajectory attracting set of $\{S(t)\}_{t \geqslant 0}$ in $\mathcal{T}_H^+$. Similarly, a set $\mathcal{A}_V^{\mathrm{tr}}$ satisfying property (2)(iii) of Definition 3.2 is called a trajectory attracting set of $\{S(t)\}_{t \geqslant 0}$ in $\mathcal{T}_V^+$. To prove the existence of the trajectory attractor in $\mathcal{T}_H^+$ and $\mathcal{T}_V^+$ respectively, we shall establish, according to the theory of Vishik and Chepyzhov [8], that $\mathcal{T}_H^+$ and $\mathcal{T}_V^+$ are translation invariant under the action of $\{S(t)\}_{t \geqslant 0}$, and $\{S(t)\}_{t \geqslant 0}$ possesses a compact attracting set in $\mathcal{T}_H^+$ and $\mathcal{T}_V^+$, respectively.

From Lemma 2.2 and Definitions 3.2, we are ready to establish

Lemma 3.1. *Let $g \in L^2(0, T; H)$ for any $T > 0$.*

(1) (i) *For any $u_0 \in H$, there exists a trajectory $u(t) \in \mathcal{T}_H^+$ such that $u(0) = u_0$;*

(ii) *$\mathcal{T}_H^+$ is translation invariant under the action of $\{S(t)\}_{t \geqslant 0}$, i.e.,*

$$S(t)\mathcal{T}_H^+ \subseteq \mathcal{T}_H^+, \ \ \forall\, t \geqslant 0. \tag{3.3}$$

(2) (i) *For any $u_0 \in V$, there exists a trajectory $u(t) \in \mathcal{T}_V^+$ such that $u(0) = u_0$;*

(ii) *$\mathcal{T}_V^+$ is translation invariant under the action of $\{S(t)\}_{t \geqslant 0}$, i.e.,*

$$S(t)\mathcal{T}_V^+ \subseteq \mathcal{T}_V^+, \ \ \forall\, t \geqslant 0. \tag{3.4}$$

Further, we have the following inclusion relations. This is the reason that we endow $\mathcal{T}_H^+$ with the topology of $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$, and endow $\mathcal{T}_V^+$ with the topology of $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$.

Lemma 3.2. *Let* $g \in L^2(0, T; H)$ *for any* $T > 0$.

(1) $\mathcal{T}_H^+ \subseteq \mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta}) \cap L^\infty(\mathbb{R}_+; H)$.

(2) $\mathcal{T}_V^+ \subseteq \mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta}) \cap L^\infty(\mathbb{R}_+; V)$.

Proof. To prove item Lemma 3.2(1), we first verify that

$$t \mapsto Au \in L^2(0, T; V'), \ \forall u \in L^\infty(0, T; H) \cap L^2(0, T; V), \forall T > 0; \tag{3.5}$$

$$t \mapsto B_N(u, u) \in L^2(0, T; V'), \ \forall u \in L^\infty(0, T; H) \cap L^2(0, T; V), \forall T > 0. \tag{3.6}$$

In fact, for any $v \in V$,

$$|\langle Au, v \rangle| = |\langle \nabla u, \nabla v \rangle| \leqslant \|u\|_V \|v\|_V, \ \forall u \in V,$$

thus

$$\int_0^T \|Au(t)\|_{V'}^2 \mathrm{d}t \leqslant \|u\|_{L^2(0,T;V)}^2, \ \forall u \in L^2(0, T; V), \tag{3.7}$$

and (3.5) is proved. Also, by (2.2),

$$\int_0^T \|B_N(u(t), u(t))\|_{V'}^2 \mathrm{d}t \leqslant c_1 N \|u\|_{L^2(0,T;V)}^2 \tag{3.8}$$

and (3.6) is established. Since $g \in L^2(0, T; H)$ for any $T > 0$, we conclude from (3.5)-(3.6) and equation (2.4) that $\partial_t u(\cdot) \in L^2_{loc}(\mathbb{R}_+; V')$. Since $H \hookrightarrow H^{-\eta} \subseteq V'$ and the embedding $H \hookrightarrow H^{-\eta}$ is compact, we infer from Lemma 2.3 that $u(\cdot) \in \mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$. The inclusion relation in item Lemma 3.2(1) follows.

To prove item Lemma 3.2(2), we check that

$$t \mapsto Au \in L^2(0, T; H), \ \forall u \in L^\infty(0, T; V) \cap L^2(0, T; D(A)), \forall T > 0; \tag{3.9}$$

$$t \mapsto B_N(u, u) \in L^2(0, T; H), \ \forall u \in L^\infty(0, T; V) \cap L^2(0, T; D(A)), \forall T > 0. \tag{3.10}$$

Indeed, for any $u \in D(A) = \mathbb{H}^2(\Omega) \cap V$, we have $Au \in H$ and $\|Au\| \leqslant \|u\|_{D(A)}$. Thus

$$|(Au, v)| \leqslant \|u\|_{D(A)} \|v\|, \ \forall v \in H,$$

and, consequently,

$$\int_0^T \|Au(t)\|^2 \mathrm{d}t \leqslant \int_0^T \|u(t)\|_{D(A)}^2 \mathrm{d}t = \|u\|_{L^2(0,T;D(A))}^2, \ \forall u \in L^2(0, T; D(A)). \tag{3.11}$$

We obtain (3.9). At the same time, (2.3) implies that

$$\|B_N(u, u)\| \leqslant c_2 N \|Au\| \leqslant c_2 N \|u\|_{D(A)}, \ \forall u \in D(A),$$

and thus

$$\int_0^T \|B_N(u(t), u(t))\|^2 \mathrm{d}t \leqslant c_2 N \|u\|_{L^2(0,T;D(A))}^2. \tag{3.12}$$

We obtain (3.10). Note that $g \in L^2(0, T; H)$ for any $T > 0$, we infer from (3.9)-(3.10) and equation (2.4) that $\partial_t u(\cdot) \in L^2_{loc}(\mathbb{R}_+; H)$. Since $V \hookrightarrow H^{1-\eta} \subseteq H$ and the embedding $V \hookrightarrow H^{1-\eta}$ is compact, we infer also from Lemma 2.3 that $u(\cdot) \in \mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$. The proof of Lemma 3.2 is completed. $\qquad \square$

We next estimate the solutions in $\mathcal{T}_H^+$ and $\mathcal{T}_V^+$. Let λ_1 be the first eigenvalue of the Stokes operator A. Also, we will employ the notation $a \lesssim b$ to mean that $a \leqslant cb$ for a universal constant $c > 0$ that only depends on the parameters coming from the problem.

Lemma 3.3. *Suppose $g \in H$ and set $\gamma = \nu\lambda_1$.*

(1) *For any trajectory $u \in \mathcal{T}_H^+$, there exists a positive constant R_H which is independent of u, such that*

$$\|S(t)u\|_{L^\infty(\mathbb{R}_+;H)} + \|S(t)u\|_{L^2(0,1;V)} + \|S(t)\partial_s u\|_{L^2(0,1;V')}$$

$$= \operatorname*{esssup}_{s \geqslant t} \|u(s)\| + \Big(\int_t^{t+1} \|u(s)\|_V^2 \mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\int_t^{t+1} \|\partial_s u(s)\|_{V'}^2 \mathrm{d}s\Big)^{\frac{1}{2}}$$

$$\lesssim \|u\|_{L^\infty(0,1;H)} e^{-\frac{\gamma t}{2}} + R_H, \quad \forall\, t \geqslant 1. \tag{3.13}$$

(2) *For any trajectory $u \in \mathcal{T}_V^+$, there exists a positive constant R_V which is independent of u, such that*

$$\|S(t)u\|_{L^\infty(\mathbb{R}_+;V)} + \|S(t)u\|_{L^2(0,1;D(A))} + \|S(t)\partial_s u\|_{L^2(0,1;H)}$$

$$= \operatorname*{esssup}_{s \geqslant t} \|u(s)\|_V + \Big(\int_t^{t+1} \|u(s)\|_{D(A)}^2 \mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\int_t^{t+1} \|\partial_s u(s)\|^2 \mathrm{d}s\Big)^{\frac{1}{2}}$$

$$\lesssim \|u\|_{L^\infty(0,1;V)} (1 + t + t^2)^{1/2} e^{-\frac{\gamma t}{2}} + R_V, \quad \forall\, t \geqslant 1. \tag{3.14}$$

Proof. We first prove (3.13). Let $u \in \mathcal{T}_H^+$. Using u to take dual pairing $\langle\cdot,\cdot\rangle$ with equation (2.4) and also using (2.1) yield the following energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \nu\|\nabla u(t)\|^2 = (g, u(t)), \forall\, t > 0. \tag{3.15}$$

By the Schwartz inequality and the following Poincaré inequality

$$\lambda_1\|\phi\|^2 \leqslant \|\nabla\phi\|^2, \,\forall\, \phi \in \mathbb{H}_0^1(\Omega), \tag{3.16}$$

we deduce from (3.15) that $\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \nu\|\nabla u(t)\|^2 \leqslant \dfrac{\|g\|^2}{2\nu\lambda_1} + \dfrac{\nu\lambda_1}{2}\|u(t)\|^2$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \nu\|\nabla u(t)\|^2 \leqslant \frac{\|g\|^2}{\nu\lambda_1}, \,\forall\, t > 0. \tag{3.17}$$

Note that the energy inequality (3.17) is interpreted in the following sense

$$-\int_0^T \|u(t)\|^2\psi'(t)\mathrm{d}t + \nu\int_0^T \|\nabla u(t)\|^2\psi(t)\mathrm{d}t \leqslant \int_0^T \frac{\|g\|^2}{\nu\lambda_1}\psi(t)\mathrm{d}t,$$

for any $T > 0$ and any $\psi \in \mathcal{C}_0^\infty([0,T])$. Set $\gamma = \nu\lambda_1$, then (3.17) and above inequality imply

$$-\int_0^{+\infty} \|u(t)\|^2\psi'(t)\mathrm{d}t + \gamma\int_0^{+\infty} \|u(t)\|^2\psi(t)\mathrm{d}t$$

$$\leqslant \int_0^{+\infty} \Big(\frac{\|g\|^2}{\gamma} - \nu(\|\nabla u(t) - \lambda_1\|u(t)\|^2)\Big)\psi(t)\mathrm{d}t, \,\forall\, \psi \in \mathcal{C}_0^\infty(\mathbb{R}_+). \tag{3.18}$$

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Now (3.16), (3.18) and Lemma 2.4 yield

$$\|u(t)\|^2 e^{\gamma t} - \|u(\tau)\|^2 e^{\gamma \tau} \leqslant \frac{1}{\gamma} \int_\tau^t \|g\|^2 e^{\gamma s} \mathrm{d}s, \ \forall\, t \geqslant \tau, t, \tau \in \mathbb{R}_+,$$

which implies that

$$\|u(t)\| \leqslant \|u(\tau)\| e^{\gamma(\tau - t)/2} + \frac{\|g\|}{\gamma} \leqslant e^{\gamma/2} \sup_{\tau \in (0,1)} \|u(\tau)\| e^{-\gamma t/2} + \frac{\|g\|}{\gamma}$$

$$\lesssim \|u\|_{L^\infty(0,1;H)} e^{-\gamma t/2} + \|g\|, \ \forall\, t \geqslant 1. \tag{3.19}$$

Now, by (3.16) and (3.17),

$$\|S(t)u(s)\|_{L^2(0,1;V)}^2 = \int_t^{t+1} \|u(s)\|_V^2 \mathrm{d}s = \int_t^{t+1} (\|u(s)\|^2 + \|\nabla u(s)\|^2) \mathrm{d}s$$

$$\lesssim \int_t^{t+1} \|\nabla u(s)\|^2 \mathrm{d}s \lesssim \|u(t)\|^2 + \|g\|^2,$$

which, together with (3.19) gives directly that

$$\|S(t)u(s)\|_{L^2(0,1;V)} \lesssim \|u\|_{L^\infty(0,1;H)} e^{-\gamma t/2} + \|g\|. \tag{3.20}$$

It then follows from (2.4), (3.7), (3.8) and (3.20) that

$$\Big(\int_t^{t+1} \|\partial_s u(s)\|_{V'}^2 \mathrm{d}s\Big)^{1/2}$$

$$\lesssim \Big(\int_t^{t+1} \|Au(s)\|_{V'}^2 \mathrm{d}s\Big)^{1/2} + \Big(\int_t^{t+1} \|B_N(u(s), u(s))\|_{V'}^2 \mathrm{d}s\Big)^{1/2} + \|g\|$$

$$\lesssim \|u(s)\|_{L^2(t,t+1;V)} + \|g\| = \|S(t)u(s)\|_{L^2(0,1;V)} + \|g\|. \tag{3.21}$$

We then obtain (3.13) from (3.19)-(3.21). Obviously the positive constant R_H depends only on ν, λ_1, N and $\|g\|$.

Next we prove (3.14). Let $u \in \mathcal{T}_V^+$ corresponding to the initial value $u_0 \in V$. Using Au to take inner product $(\cdot, \cdot)$ with equation (2.4) yields the following entropy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2 + \nu\|Au(t)\|^2 + b_N(u(t), u(t), Au(t)) = (g, Au(t)), \ \forall\, t > 0. \tag{3.22}$$

Recall that we have the following estimate (cf. [3, (2.7)])

$$|b(u, v, w)| \leqslant c_2 \|u\|_V^{1/2} \|Au\|^{1/2} \|v\|_V \|w\|, \ \forall\, u \in D(A),\, v \in V,\, w \in H. \tag{3.23}$$

By Young's inequality,

$$|b_N(u, u, Au)| \leqslant c_2 \frac{N}{\|u\|_V} \|u\|_V^{3/2} \|Au\|^{3/2} \lesssim \frac{\nu}{4}\|Au\|^2 + \|u\|_V^2, \tag{3.24}$$

$$|(g, Au(t))| \lesssim \frac{\nu}{4}\|Au\|^2 + \|g\|^2. \tag{3.25}$$

Notice that $\|\nabla u\|$ is a norm in V equivalent to the usual norm $\|u\|_V$. We then obtain from (3.22) and (3.24)-(3.25) the following entropy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_V^2 + \nu\|Au(t)\|^2 \lesssim \|u\|_V^2 + \|g\|^2. \tag{3.26}$$

Note also that the entropy inequality (3.26) is interpreted in the following sense

$$-\int_0^T \|u(t)\|_V^2 \psi'(t)\mathrm{d}t + \nu\int_0^T \|Au(t)\|^2 \psi(t)\mathrm{d}t \lesssim \int_0^T (\|g\|^2 + \|u(t)\|_V^2)\psi(t)\mathrm{d}t,$$

for any $T > 0$ and any $\psi \in \mathcal{C}_0^\infty([0,T])$. Hence, using the Poincaré type inequality $\|Au(t)\|^2 \geqslant \lambda_1\|u(t)\|_V^2$,

$$-\int_0^{+\infty} \|u(t)\|_V^2 \psi'(t)\mathrm{d}t + \gamma\int_0^{+\infty} \|u(t)\|_V^2 \psi'(t)\mathrm{d}t$$

$$\lesssim -\int_0^{+\infty} (\|Au(t)\|^2 - \lambda_1\|u(t)\|_V^2]\psi(t)\mathrm{d}t + \int_0^{+\infty} (\|g\|^2 + \|u(t)\|_V^2)\psi(t)\mathrm{d}t$$

$$\lesssim \int_0^{+\infty} (\|g\|^2 + \|u(t)\|_V^2)\psi(t)\mathrm{d}t, \ \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}_+).$$

Again by Lemma 2.4,

$$\|u(t)\|_V^2 e^{\gamma t} - \|u(\tau)\|_V^2 e^{\gamma \tau} \lesssim \int_\tau^t (\|g\|^2 + \|u(s)\|_V^2)e^{\gamma s}\mathrm{d}s, \ \forall t \geqslant \tau, t, \tau \in \mathbb{R}_+,$$

which gives

$$\|u(t)\|_V^2 \lesssim \|u(\tau)\|_V^2 e^{\gamma \tau}e^{-\gamma t} + e^{-\gamma t}\int_\tau^t (\|g\|^2 + \|u(s)\|_V^2)e^{\gamma s}\mathrm{d}s$$

$$\lesssim \|u\|_{L^\infty(0,1;V)}^2 e^{-\gamma t} + \|g\|^2 + \int_\tau^t \|u(s)\|_V^2 e^{\gamma s}\mathrm{d}s. \tag{3.27}$$

Inserting the following inequality (see [3, (4.41)])

$$\|u(t)\|_V^2 \leqslant (\|u_0\|_V^2 + C^{(N)}t\|u_0\|^2)e^{-\gamma t} + \frac{\|g\|^2}{\nu^2\lambda_1}(2 + C^{(N)}\nu^2\lambda_1)$$

into (3.27) yields

$$\|u(t)\|_V^2 \lesssim \|u\|_{L^\infty(0,1;V)}^2 e^{-\gamma t} + \|g\|^2$$

$$+ e^{-\gamma t}\int_\tau^t \left((\|u\|_{L^\infty(0,1;V)}^2 + s\|u\|_{L^\infty(0,1;V)}^2)e^{-\gamma s} + \|g\|^2\right)e^{\gamma s}\mathrm{d}s$$

$$\lesssim \|u\|_{L^\infty(0,1;V)}^2(1 + t + t^2)e^{-\gamma t} + R_V^2, \ \forall t \geqslant \tau \geqslant 0, \tag{3.28}$$

where R_V is a positive constant depends only on ν, λ_1, c_2, N and $\|g\|$. Now by (3.16), (3.20), (3.26) and (3.28),

$$\|S(t)u(s)\|_{L^2(0,1;D(A))}^2 = \int_t^{t+1} \|u(s)\|_{D(A)}^2\mathrm{d}s \lesssim \int_t^{t+1} \|Au(s)\|\mathrm{d}s$$

$$\lesssim (\|u(t)\|_V^2 + \|g\|^2 + \int_t^{t+1} \|u(s)\|_V^2\mathrm{d}s)$$

$$\lesssim \|u(t)\|_V^2 + \|g\|^2 + (\|u\|_{L^\infty(0,1;H)}^2 e^{-\gamma t} + R_H^2)$$

$$\lesssim \|u\|_{L^\infty(0,1;V)}^2(1 + t + t^2)e^{-\gamma t} + R_V^2, \ \forall t \geqslant 1. \tag{3.29}$$

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We then deduce from (3.12) and (3.29) that

$$\|S(t)B_N(u(s),u(s))\|_{L^2(0,1;H)}^2 = \int_t^{t+1} \|B_N(u(s),u(s))\|^2 \mathrm{d}s \lesssim \int_t^{t+1} \|u(s)\|_{D(A)}^2 \mathrm{d}s$$
$$\lesssim \|u\|_{L^\infty(0,1;V)}^2 (1+t+t^2)e^{-\gamma t} + R_V^2, \quad \forall\, t \geqslant 1. \quad (3.30)$$

Taking (2.4) and (3.28)-(3.30) into account, we have

$$\|S(t)\partial_s u(s)\|_{L^2(0,1;H)} \lesssim \|S(t)Au(s)\|_{L^2(0,1;H)} + \|S(t)B_N(u(s),u(s))\|_{L^2(0,1;H)} + \|g\|$$
$$\lesssim \|u\|_{L^\infty(0,1;V)}(1+t+t^2)^{1/2}e^{-\frac{\gamma t}{2}} + R_V, \quad \forall\, t \geqslant 1. \quad (3.31)$$

Therefore, (3.14) is obtained from (3.28), (3.29) and (3.31). The proof of Lemma 3.3 is completed. □

We next use the estimates obtained in Lemma 3.3 to construct the trajectory attracting set for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}_H^+$ and $\mathcal{T}_V^+$, respectively.

Lemma 3.4. *Suppose $g \in H$.*

(1) *There exists a bounded (in the norm of $L^\infty(\mathbb{R}_+;H)$) trajectory absorbing set $\Lambda_H \subset \mathcal{T}_H^+$, i.e., for any bounded (in the norm of $L^\infty(\mathbb{R}_+;H)$) subset $\mathcal{B}_H \subset \mathcal{T}_H^+$, there exists a time $t_H = t_H(\mathcal{B})$ such that $S(t)u \in \Lambda_H$, $\forall\, u \in \mathcal{B}_H$, $\forall\, t \geqslant t_H$.*

(2) *There exists a bounded (in the norm of $L^\infty(\mathbb{R}_+;V)$) trajectory absorbing set $\Lambda_V \subset \mathcal{T}_V^+$, i.e., for any bounded (in the norm of $L^\infty(\mathbb{R}_+;V)$) subset $\mathcal{B}_V \subset \mathcal{T}_V^+$, there exists a time $t_V = t_V(\mathcal{B})$ such that $S(t)u \in \Lambda_V$, $\forall\, u \in \mathcal{B}_V$, $\forall\, t \geqslant t_V$.*

Proof. Set

$$\Lambda_H = \left\{ u \in \mathcal{T}_H^+ \ \Big| \ \sup_{t\geqslant 0} \left\{ \|u\|_{L^\infty(t,t+1;H)} + \|\partial_s u\|_{L^2(t,t+1;V')} \right\} \leqslant 2R_H \right\}, \quad (3.32)$$

$$\Lambda_V = \left\{ u \in \mathcal{T}_V^+ \ \Big| \ \sup_{t\geqslant 0} \left\{ \|u\|_{L^\infty(t,t+1;V)} + \|\partial_s u\|_{L^2(t,t+1;H)} \right\} \leqslant 2R_V \right\}, \quad (3.33)$$

where R_H and R_V are the positive constants come from Lemma 3.3.

We claim that Λ_H is a bounded trajectory absorbing set (thus is a bounded trajectory attracting set) for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}_H^+$. Indeed, let $\mathcal{B}_H$ be a bounded (in $L^\infty(\mathbb{R}_+;H)$ norm) subset of $\mathcal{T}_H^+$. Then from (3.13) we infer that for $\forall\, u \in \mathcal{B}_H \subset \mathcal{T}_H^+$, there exists a $t_0 = t_0(\mathcal{B}_H) \geqslant 1$ such that $\|u\|_{L^\infty(0,1;H)} e^{-\gamma t/2} \leqslant R_H$ as long as $t \geqslant t_0$. Therefore,

$$\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^2(t,t+1;V')} \leqslant 2R_H, \quad \forall\, t \geqslant t_0 + 1, \quad (3.34)$$

and $S(t)\mathcal{B}_H \subseteq \Lambda_H$, $\forall\, t \geqslant t_H = t_0 + 1$, which implies that Λ_H is a trajectory absorbing set for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}_H^+$. Obviously, Λ_H is bounded (in $L^\infty(\mathbb{R}_+;H)$ norm) in $\mathcal{T}_H^+$.

Similarly, Λ_V is a bounded trajectory absorbing set (thus is a bounded trajectory attracting set) for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}_V^+$. The proof is completed. □

To prove the existence of trajectory attractor for $\{S(t)\}_{t\geqslant 0}$ in the trajectory space $\mathcal{T}_H^+$, we now need establish that the trajectory absorbing set Λ_H is compact in $\mathcal{C}_{loc}(\mathbb{R}_+;H^{-\eta})$.

Lemma 3.5. *Suppose $g \in H$. Then the trajectory absorbing set Λ_H is compact in* $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$.

Proof. Step One We prove that $\mathcal{T}_H^+$ is closed in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$. Let $\{u_n\}$ be a bounded (in the norm of $L^\infty(\mathbb{R}_+; H)$) sequence in $\mathcal{T}_H^+$ and there exists a function $u^* \in \mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$ such that

$$u_n \longrightarrow u^* \quad \text{in the topology of} \quad \mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta}) \text{ as } n \to \infty. \tag{3.35}$$

We shall prove that $u^* \in \mathcal{T}_H^+$. To this end, we need to show that $u^* \in L^\infty(\mathbb{R}_+; H) \cap L_{loc}^2(\mathbb{R}_+; V)$, and $\forall T > 0$, $\Pi_T u^*(t)$ is a weak solution of (2.4) on the interval $[0, T]$. Indeed, since $\{u_n\} \subset \mathcal{T}_H^+$ and is bounded in $L^\infty(\mathbb{R}_+; H)$, by (3.13) we conclude that $\{u_n\}$ is bounded in $L_{loc}^2(\mathbb{R}_+; V)$ and $\{\partial_t u_n\}$ is bounded in $L_{loc}^2(\mathbb{R}_+; V')$. Using the diagonal procedure we deduce that there exist a function $u \in L^\infty(\mathbb{R}_+; H) \cap L_{loc}^2(\mathbb{R}_+; V)$ and a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that

$$\Pi_T u_{n'} \rightharpoonup \Pi_T u \quad \text{weakly in } L^2(0, T; V) \text{ as } n' \to \infty, \tag{3.36}$$

$$u_{n'} \rightharpoonup u \quad \text{weakly star in } L^\infty(\mathbb{R}_+; H) \text{ as } n' \to \infty, \tag{3.37}$$

$$\partial_t \Pi_T u_{n'} \rightharpoonup \partial_t \Pi_T u \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty. \tag{3.38}$$

Obviously, $\partial_t u \in L_{loc}^2(\mathbb{R}_+; V')$. By Lemma 2.3 we obtain $\Pi_T u \in \mathcal{C}([0, T]; H^{-\eta})$ since the embedding $H \hookrightarrow H^{-\eta}$ is compact. From (3.35) and the uniqueness of limit we have $u = u^*$. Next we verify that $\Pi_T u^*$ is a weak solution of (2.4) on the interval $[0, T]$. For this purpose, we check the following convergent relations

$$A\Pi_T u_{n'} \rightharpoonup A\Pi_T u^* \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty, \tag{3.39}$$

$$B_N(\Pi_T u_{n'}, \Pi_T u_{n'}) \rightharpoonup B_N(\Pi_T u^*, \Pi_T u^*) \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty. \tag{3.40}$$

In fact, the convergence of (3.39) is classical (see e.g. [28]), while the derivation of (3.40) is almost the same as that as [3, (3.20)]. With the convergent relations (3.38)-(3.40) in hand, we can pass the limit in equation (2.4) as $n' \to \infty$. This shows that $\Pi_T u^*$ is a weak solution of equation (2.4) on $[0, T]$.

Step Two We prove that the trajectory absorbing set Λ_H is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{-\eta})$. Indeed, from (3.32) we see that $\Pi_T \Lambda_H$ is bounded in $W_{\infty, 2}(0, T; H, V')$ and thus $\Pi_T \Lambda_H$ is pre-compact in $\mathcal{C}([0, T]; H^{-\eta})$ (thanks to Lemma 2.3). Hence, it suffices to show that $\Pi_T \Lambda_H$ is closed in $\mathcal{C}([0, T]; H^{-\eta})$ for any $T > 0$. Assume that $\{u_n\} \subset \Lambda_H$ and $\Pi_T u_n \longrightarrow \Pi_T u$ in the metric of $\mathcal{C}([0, T]; H^{-\eta})$ as $n \to \infty$. we see from *Step one* that $u \in \mathcal{T}_H^+$. Moreover, in the proof of *Step one* we know $\partial_t \Pi_T u_n \rightharpoonup \partial_t \Pi_T u$ weakly in $L^2(0, T; V')$ and $u_n \rightharpoonup u$ weakly star in $L^\infty(\mathbb{R}_+; H)$ as $n \to \infty$. Thus

$$\|u\|_{L^\infty(t, t+1; H)} + \|\partial_t u\|_{L^2(t, t+1; V')}$$
$$\leqslant \liminf_{n \to \infty} \|u_n\|_{L^\infty(t, t+1; H)} + \liminf_{n \to \infty} \|\partial_t u_n\|_{L^2(t, t+1; V')} \leqslant 2R_H, \ \forall \, t \geqslant 0.$$

Therefore $u \in \Lambda_H$ and $\Pi_T u \in \mathcal{C}([0, T]; H^{-\eta})$. The proof is completed. $\qquad\square$

Similarly, in order to obtain the existence of trajectory attractor for $\{S(t)\}_{t \geqslant 0}$ in the regular trajectory space $\mathcal{T}_V^+$, we need prove that the trajectory absorbing set Λ_V is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$.

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Lemma 3.6. *Suppose $g \in H$. Then the trajectory absorbing set Λ_V is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$.*

Proof. The procedure of the proof is analogous to that of Lemma 3.5, which is divided into two steps.

Step One We prove that $\mathcal{T}_V^+$ is closed in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$. Let $\{u_n\}$ be a bounded (in the norm of $L^\infty(\mathbb{R}_+; V)$) sequence in $\mathcal{T}_V^+$ and there exists a function $u^* \in \mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$ such that

$$u_n \longrightarrow u^* \quad \text{in the topology of } \mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta}) \text{ as } n \to \infty. \tag{3.41}$$

We next prove that $u^* \in \mathcal{T}_V^+$. To this end, we need to show that $u^* \in L^\infty(\mathbb{R}_+; V) \cap L^2_{loc}(\mathbb{R}_+; D(A))$, and $\forall T > 0$, $\Pi_T u^*(t)$ is a strong solution of (2.4) on the interval $[0, T]$. In fact, since $\{u_n\} \subset \mathcal{T}_V^+$ and is bounded in $L^\infty(\mathbb{R}_+; V)$, by (3.14) we conclude that $\{u_n\}$ is bounded in $L^2_{loc}(\mathbb{R}_+; D(A))$ and $\{\partial_t u_n\}$ is bounded in $L^2_{loc}(\mathbb{R}_+; H)$. Using the diagonal procedure we see that there exist a function $u \in L^\infty(\mathbb{R}_+; V) \cap L^2_{loc}(\mathbb{R}_+; D(A))$ and a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that

$$\Pi_T u_{n'} \rightharpoonup \Pi_T u \text{ weakly in } L^2(0, T; D(A)) \text{ as } n' \to \infty, \tag{3.42}$$

$$u_{n'} \rightharpoonup u \text{ weakly star in } L^\infty(\mathbb{R}_+; V) \text{ as } n' \to \infty, \tag{3.43}$$

$$\partial_t \Pi_T u_{n'} \rightharpoonup \partial_t \Pi_T u \text{ weakly in } L^2(0, T; H) \text{ as } n' \to \infty. \tag{3.44}$$

Obviously, $\partial_t u \in L^2_{loc}(\mathbb{R}_+; H)$. By Lemma 2.3 we have $\Pi_T u \in \mathcal{C}([0, T]; H^{1-\eta})$ since the embedding $V \hookrightarrow H^{1-\eta}$ is compact. From (3.41) and the uniqueness of limit we have $u = u^*$. Next we verify that $\Pi_T u^*$ is a strong solution of (2.4) on the interval $[0, T]$. Since we have the following convergent relations

$$A\Pi_T u_{n'} \rightharpoonup A\Pi_T u^* \text{ weakly in } L^2(0, T; H) \text{ as } n' \to \infty; \tag{3.45}$$

$$B_N(\Pi_T u_{n'}, \Pi_T u_{n'}) \rightharpoonup B_N(\Pi_T u^*, \Pi_T u^*) \text{ weakly in } L^2(0, T; H) \text{ as } n' \to \infty, \tag{3.46}$$

which were proved in [3], we can pass the limit in equation (2.4) as $n' \to \infty$ and conclude that $\Pi_T u^*$ is a strong solution of equation (2.4) on $[0, T]$.

Step Two We prove that the trajectory absorbing set Λ_V is compact in $\mathcal{C}_{loc}(\mathbb{R}_+; H^{1-\eta})$. From (3.33) we see that $\Pi_T \Lambda_H$ is bounded in $W_{\infty,2}(0, T; V, H)$. Thus $\Pi_T \Lambda_V$ is precompact in $\mathcal{C}([0, T]; H^{1-\eta})$ due to Lemma 2.3. Hence, it suffices to show that $\Pi_T \Lambda_V$ is closed in $\mathcal{C}([0, T]; H^{1-\eta})$ for any $T > 0$. Assume that $\{u_n\} \subset \Lambda_V$ and $\Pi_T u_n \longrightarrow \Pi_T u$ in the metric of $\mathcal{C}([0, T]; H^{1-\eta})$ as $n \to \infty$. Then $u \in \mathcal{T}_V^+$. Moreover, $\partial_t \Pi_T u_n \rightharpoonup \partial_t \Pi_T u$ weakly in $L^2(0, T; H)$ and $u_n \rightharpoonup u$ weakly star in $L^\infty(\mathbb{R}_+; V)$ as $n \to \infty$. Thus

$$\|u\|_{L^\infty(t, t+1; V)} + \|\partial_t u\|_{L^2(t, t+1; H)}$$
$$\leqslant \liminf_{n \to \infty} \|u_n\|_{L^\infty(t, t+1; V)} + \liminf_{n \to \infty} \|\partial_t u_n\|_{L^2(t, t+1; H)} \leqslant 2R_V, \ \forall\, t \geqslant 0.$$

Therefore $u \in \Lambda_V$ and $\Pi_T u \in \mathcal{C}([0, T]; H^{1-\eta})$. The proof is completed. $\qquad\square$

Remember that we have introduced the notation $\mathcal{K}_X$ with $X = H$ or $X = V$ in the Introduction. The main result of this section reads as follows.

Theorem 3.1. *Suppose $g \in H$.*

(1) *The natural translation semigroup $\{S(t)\}_{t \geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_H^{\mathrm{tr}}$ in $\mathcal{T}_H^+$ satisfying Definition 3.2(1)(i)-(iii) and possessing the following structure*

$$\mathcal{A}_H^{\mathrm{tr}} = \Pi_+ \mathcal{K}_H = \{\Pi_+ u(\cdot) \,\big|\, u \in \mathcal{K}_H\} \subset \mathcal{T}_H^+. \tag{3.47}$$

(2) *The natural translation semigroup $\{S(t)\}_{t \geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_V^{\mathrm{tr}}$ in $\mathcal{T}_V^+$ satisfying Definition 3.2(2)(i)-(iii) and possessing the following structure*

$$\mathcal{A}_V^{\mathrm{tr}} = \Pi_+ \mathcal{K}_V = \{\Pi_+ u(\cdot) \,\big|\, u \in \mathcal{K}_V\} \subset \mathcal{T}_V^+. \tag{3.48}$$

(3) *The following regularity of the trajectory attractor holds*

$$\mathcal{A}_H^{\mathrm{tr}} = \Pi_+ \mathcal{K}_H = \mathcal{A}_V^{\mathrm{tr}} = \Pi_+ \mathcal{K}_V. \tag{3.49}$$

Proof. The results in items (1) and (2) are obtained directly by employing Lemma 3.1, Lemmas 3.4-3.6 and the abstract theory [31, Theorem 4.1].

We next prove (3.49). To this end, we first establish that any trajectory $u(\cdot)$ issuing from any bounded subset of H will becomes eventually bounded in $L^\infty(\mathbb{R}_+; V)$. Indeed, consider any bounded subset $\mathcal{B}_H$ of H and any $u_0 \in \mathcal{B}_H$. Let $u(\cdot) = S(\cdot)u_0$ be a trajectory issuing from u_0. Then by Lemma 2.2(1), there exists some $t_0 \in (0,1)$ such that $u(t_0) \in V$. Then the function $v(t) = u(t + t_0)$ is a weak solution of equation (2.4) with initial value $v(0) = u(t_0) \in V$. By [3, (4.41)] we have

$$\begin{aligned}
\|u(t + t_0)\|_V^2 = \|v(t)\|_V^2 &\lesssim (\|v(0)\|_V^2 + t\|v(0)\|^2)e^{-\sigma t} + \|g\| \\
&= (\|u(t_0)\|_V^2 + t\|u(t_0)\|^2)e^{-\sigma t} + \|g\|. \tag{3.50}
\end{aligned}$$

Set

$$\Psi = \{u(\cdot) \in \mathcal{T}_H^+ \,\big|\, u(\cdot) \text{ issues from } u_0 \in \mathcal{B}_H\}.$$

Then for above t_0 we see that both $\sup_{u(\cdot) \in \Psi} \|u(t_0)\|_V$ and $\sup_{u(\cdot) \in \Psi} \|u(t_0)\|$ depend only on $\mathcal{B}_H$. We then deduce from (3.50) that there exists some time $t^*(\mathcal{B}_H) > t_0$ such that

$$\|u(t)\|_V^2 \leqslant c, \quad \forall\, t > t^*(\mathcal{B}_H), \tag{3.51}$$

where the positive constant c depends only on $\|g\|$ and $\mathcal{B}_H$. Now, (3.51) and the invariance property of the trajectory attractor tell us that there is some time $t^*(\mathcal{A}_H^{\mathrm{tr}})$ such that $\mathcal{A}_H^{\mathrm{tr}} = S(t^*(\mathcal{A}_H^{\mathrm{tr}}))\mathcal{A}_H^{\mathrm{tr}}$ is bounded in $L^\infty(\mathbb{R}_+; V)$. Also by the invariance property and attracting property of the trajectory attractor, we obtain for any $T > 0$ that

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}(\Pi_T \mathcal{A}_H^{\mathrm{tr}}, \Pi_T \mathcal{A}_V^{\mathrm{tr}}) \\
=&\mathrm{dist}_{\mathcal{C}([0,T];H^{-\eta})}(\Pi_T S(t)\mathcal{A}_H^{\mathrm{tr}}, \Pi_T \mathcal{A}_V^{\mathrm{tr}}) \quad (\forall\, t \in \mathbb{R}_+) \\
\leqslant&\mathrm{dist}_{\mathcal{C}([0,T];H^{1-\eta})}(\Pi_T S(t)\mathcal{A}_H^{\mathrm{tr}}, \Pi_T \mathcal{A}_V^{\mathrm{tr}}) \quad (\forall\, t \geqslant t^*(\mathcal{A}_H^{\mathrm{tr}})) \\
=&\lim_{t \to +\infty} \mathrm{dist}_{\mathcal{C}([0,T];H^{1-\eta})}(\Pi_T S(t)\mathcal{A}_H^{\mathrm{tr}}, \Pi_T \mathcal{A}_V^{\mathrm{tr}}) \\
=&0,
\end{aligned}$$

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which implies $\mathcal{A}_H^{\mathrm{tr}} \subseteq \mathcal{A}_V^{\mathrm{tr}}$. At the same time, since all strong solutions are weak solutions, (3.47) and (3.48) give $\mathcal{A}_V^{\mathrm{tr}} \subseteq \mathcal{A}_H^{\mathrm{tr}}$. The proof is completed. $\qquad\square$

4 Invariant measure and trajectory statistical solution for the 3D globally modified Navier-Stokes equations

In this section we will investigate the existence and uniqueness of invariant Borel probability measure and trajectory statistical solution for the 3D globally modified Navier-Stokes equations. Our idea originates from [6]. Here we extend the approach of [6] and use the trajectory attractor to construct the invariant measure and trajectory statistical solution.

According to [2, Definition 3.1], we now give the concept of the trajectory statistical solution for the addressed 3D globally modified Navier-Stokes equations. For brevity, we denote

$$\mathcal{F}^+ = \mathcal{C}([0, +\infty); H^{-\eta}),$$

which has been endowed with the metric (see (3.1))

$$\mathrm{d}_{\mathcal{F}^+}(u, v) = \mathrm{d}_{\mathcal{C}([0,+\infty);H^{-\eta})}(u, v) = \sum_{T \in \mathbb{N}} \frac{1}{2^T} \frac{\sup\{\mathrm{d}_{H^{-\eta}}(u(t), v(t)) : 0 \leqslant t \leqslant T\}}{1 + \sup\{\mathrm{d}_{H^{-\eta}}(w(t), \phi(t)) : 0 \leqslant t \leqslant T\}}.$$

Definition 4.1. *We say a Borel probability measure ρ on $\mathcal{F}^+$ is a $\mathcal{T}_H^+$-trajectory statistical solution over $[0, +\infty)$ (or simply a trajectory statistical solution) for equations (1.1)-(1.2) if*

(1) *ρ is tight for any $\mathcal{B} \in \mathcal{B}(\mathcal{F}^+)$ (the collection of Borel sets of $\mathcal{F}^+$) in the sense that*

$$\rho(\mathcal{B}) = \sup\left\{\rho(E) \big| E \in \mathcal{B}(\mathcal{F}^+) \ \text{and} \ E \subset \mathcal{B}\right\};$$

(2) *ρ is supported by a Borel subset of $\mathcal{F}^+$ included in $\mathcal{T}_H^+$.*

To construct the invariant Borel probability measure and trajectory statistical solution, we first establish two lemmas. Let $\mathcal{C}(\mathcal{F}^+)$ be the set of all continuous functions from $\mathcal{F}^+$ to $\mathbb{R}$. For any $\epsilon > 0$, denote $K_\epsilon = \{w \in \mathcal{F}^+ \,|\, \inf\limits_{v \in K} \mathrm{d}_{\mathcal{F}^+}(w, v) < \epsilon\}$.

Lemma 4.1. *Let K be some compact subset of $\mathcal{F}^+$. Then for every $\psi \in \mathcal{C}(\mathcal{F}^+)$, there exists some $\epsilon > 0$ such that*

$$\sup_{w \in K_\epsilon} |\psi(w)| < +\infty.$$

Proof. Fix some $\psi \in \mathcal{C}(\mathcal{F}^+)$. For every $w \in K$ we can choose $\delta = \delta_w > 0$ such that for every $v \in \mathcal{B}(w; \delta_w) = \{\Phi \in \mathcal{F}^+ \,|\, \mathrm{d}_{\mathcal{F}^+}(w, \Phi) < \delta_w\}$ there holds $|\psi(w) - \psi(v)| < 1$. Picking numbers $\delta_w > 0$ in this way we can construct an open covering $\Lambda = \{\mathcal{B}(w; \frac{\delta_w}{3}) \,|\, w \in K\}$ for K. Since K is compact in $\mathcal{F}^+$, we may extract from this open covering a finite one

$$\Lambda_m = \left\{\mathcal{B}(w_1; \frac{\delta_{w_1}}{3}), \mathcal{B}(w_2; \frac{\delta_{w_2}}{3}), \cdots, \mathcal{B}(w_m; \frac{\delta_{w_m}}{3})\right\}.$$

Set

$$\epsilon = \frac{\min\{\delta_{w_1}, \delta_{w_2}, \cdots, \delta_{w_m}\}}{3}, \quad C = 1 + \max_{1 \leqslant j \leqslant m} |\psi(w_j)|.$$

Given any $w \in K_\epsilon$ we can choose $v \in K$ so that $\mathrm{d}_{\mathcal{F}^+}(w, v) < 2\epsilon$. Since Λ_m covers K we can choose w_j such that $\mathrm{d}_{\mathcal{F}^+}(v, w_j) < \frac{\delta_{w_j}}{3}$. Hence we obtain

$$\mathrm{d}_{\mathcal{F}^+}(w, w_j) < \mathrm{d}_{\mathcal{F}^+}(w, v) + \mathrm{d}_{\mathcal{F}^+}(v, w_j) < 2\epsilon + \frac{\delta_{w_j}}{3} \leqslant \delta_{w_j}$$

and conclude that $|\psi(w)| \leqslant C$. By the arbitrariness of $w \in K_\epsilon$, we end the proof. $\square$

Lemma 4.2. *Let K be some compact subset of $\mathcal{F}^+$ and let $\psi,\ \phi \in \mathcal{C}(\mathcal{F}^+)$ satisfying $\psi(w) = \phi(w)$ for every $w \in K$. Then for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup\limits_{w \in K_\delta} |\psi(w) - \phi(w)| < \epsilon$.*

Proof. Consider given $\epsilon > 0$. For every $w \in K$ we pick $\gamma_w > 0$ so that $|\phi(w) - \phi(v)| + |\psi(w) - \psi(v)| < \epsilon$ whenever $v \in \mathcal{B}(w; \gamma_w)$. Due to the compactness of K in $\mathcal{F}^+$, we can cover K with a finite collection

$$\Lambda_k = \left\{ \mathcal{B}(w_1; \frac{\gamma_{w_1}}{3}), \mathcal{B}(w_2; \frac{\gamma_{w_2}}{3}), \cdots, \mathcal{B}(w_k; \frac{\gamma_{w_k}}{3}) \right\} \text{ with } w_j \in K, j = 1, 2, \cdots, k.$$

Set $3\delta = \min\{\gamma_{w_1}, \gamma_{w_2}, \cdots, \gamma_{w_k}\}$, then for every $v \in K_\delta$ we can choose $w \in K$ so that $\mathrm{d}_{\mathcal{F}^+}(v, w) < 2\delta$. Notice that K is covered by Λ_k, we may take w_j such that $\mathrm{d}_{\mathcal{F}^+}(w, w_j) < \frac{\gamma_{w_j}}{3}$. Thus

$$\mathrm{d}_{\mathcal{F}^+}(v, w_j) \leqslant \mathrm{d}_{\mathcal{F}^+}(v, w) + \mathrm{d}_{\mathcal{F}^+}(w, w_j) \leqslant 2\delta + \frac{\gamma_{w_j}}{3} \leqslant \gamma_{w_j}.$$

Therefore for arbitrary $v \in K_\delta$, there exists some j such that $v \in \mathcal{B}(w_j; \gamma_{w_j})$. Keeping in mind that $\psi(w_j) = \phi(w_j)$, we have

$$|\psi(v) - \phi(v)| \leqslant |\psi(v) - \psi(w_j)| + |\phi(w_j) - \phi(v)| < \epsilon.$$

The proof is completed. $\square$

We next recall the definition of generalized Banach limit and a useful property.

Definition 4.2. ([12, 22]) *A generalized Banach limit is any linear functional, which we denote by* $\mathrm{LIM}_{t \to +\infty}$, *defined on the space of all bounded real-valued functions on* $[0, +\infty)$ *that satisfies*

(1) $\mathrm{LIM}_{t \to +\infty} h(t) \geqslant 0$ *for nonnegative functions* $h(\cdot)$ *on* $[0, +\infty)$;

(2) $\mathrm{LIM}_{t \to +\infty} h(t) = \lim\limits_{t \to +\infty} h(t)$ *if the usual limit* $\lim\limits_{t \to +\infty} h(t)$ *exists.*

Let B_+ be the collection of all bounded real-valued functions on $[0, +\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t \to +\infty}$, the following useful property

$$|\mathrm{LIM}_{t \to +\infty} h(t)| \leqslant \limsup_{t \to +\infty} |h(t)|, \quad \forall h(\cdot) \in B_+, \tag{4.1}$$

is presented in [12, (1.38)] and in [6, (2.3)].

The main result of this section reads as follows.

Theorem 4.1. *Let* $\mathrm{LIM}_{t\to+\infty}$ *be a given generalized Banach limit. Then for any* $u \in \mathcal{T}_H^+$, *there exists a unique Borel probability measure* ρ_u *on* $\mathcal{F}^+$ *such that*

$$\int_{\mathcal{F}^+} \psi(v)\mathrm{d}\rho_u(v) = \mathrm{LIM}_{t\to+\infty} \frac{1}{t}\int_0^t \psi(S(s)u)\mathrm{d}s, \quad \forall \psi \in \mathcal{C}(\mathcal{F}^+). \qquad (4.2)$$

Moreover, this Borel probability measure ρ_u *is a trajectory statistical solution for the 3D globally modified Navier-Stokes equations* (1.1)-(1.2) *and* ρ_u *satisfies the following invariant property and regularity*

$$\int_{\mathcal{F}^+} \psi(S(\tau)v)\mathrm{d}\rho_u(v) = \int_{\mathcal{F}^+} \psi(v)\mathrm{d}\rho_u(v) = \int_{\mathcal{A}_H^{\mathrm{tr}}} \psi(v)\mathrm{d}\rho_u(v)$$

$$= \int_{\mathcal{A}_V^{\mathrm{tr}}} \psi(v)\mathrm{d}\rho_u(v), \quad \forall \tau \geqslant 0, \ \forall \psi \in \mathcal{C}(\mathcal{F}^+). \qquad (4.3)$$

Proof. Let $\mathrm{LIM}_{t\to+\infty}$ be a given generalized Banach limit. Consider given $u \in \mathcal{T}_H^+$ and $\psi \in \mathcal{C}(\mathcal{F}^+)$. Since the translation semigroup $\{S(s)\}_{s\geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_H^{\mathrm{tr}} \subset \mathcal{T}_H^+ \subset \mathcal{F}^+$, we see from the attracting property of the trajectory attractor that for every $\epsilon > 0$, there exists a time $t_\epsilon \geqslant 0$ such that

$$S(s)u \in \mathcal{A}_{H,\epsilon}^{\mathrm{tr}} = \{\Phi \in \mathcal{F}^+ \mid \inf_{v\in\mathcal{A}_H^{\mathrm{tr}}} \mathrm{d}_{\mathcal{F}^+}(\Phi, v) < \epsilon\}, \text{ for every } s \geqslant t_\epsilon. \qquad (4.4)$$

By Lemma 4.1 we can pick $\epsilon > 0$ such that

$$C_1 = \sup_{v\in\mathcal{A}_{H,\epsilon}^{\mathrm{tr}}} |\psi(v)| < +\infty. \qquad (4.5)$$

Notice that the interval $[0, t_\epsilon]$ is compact and that $s \longmapsto |\psi(S(s)u)|$ is continuous, we can take t_ϵ as required in (4.4) for the picked ϵ, and see that

$$C_2 = \sup_{s\in[0,t_\epsilon]} |\psi(S(s)u)| < +\infty. \qquad (4.6)$$

Hence, we conclude for every $t > 0$ that

$$\frac{1}{t}\int_0^t \psi(S(s)u)\mathrm{d}s \leqslant C_1 + C_2 < +\infty,$$

which implies that the map defined by $t \longmapsto \dfrac{1}{t}\displaystyle\int_0^t \psi(S(s)u)\mathrm{d}s$ is bounded over $[0, +\infty)$. Therefore, we claim that

$$\mathcal{L}_u(\psi) = \mathrm{LIM}_{t\to+\infty} \frac{1}{t}\int_0^t \psi(S(s)u)\mathrm{d}s \qquad (4.7)$$

is well defined as a positive linear functional on $\mathcal{C}(\mathcal{F}^+)$. We next establish that the positive linear functional $\mathcal{L}_u(\psi)$ depends only on the values of ψ on the trajectory attractor $\mathcal{A}_H^{\mathrm{tr}}$. In other words, we will prove that if $\psi(v) = \widetilde{\psi}(v)$ for every $v \in \mathcal{A}_H^{\mathrm{tr}}$

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then $\mathcal{L}_u(\psi) = \mathcal{L}_u(\widetilde{\psi})$. In fact, for any given $\epsilon > 0$, we can choose, by Lemma 4.2 a corresponding $\delta > 0$ such that

$$\sup_{v \in \mathcal{A}_{H,\delta}^{\mathrm{tr}}} |\psi(v) - \widetilde{\psi}(v)| < \epsilon, \tag{4.8}$$

where $\mathcal{A}_{H,\delta}^{\mathrm{tr}} = \{\Phi \in \mathcal{F}^+ \mid \inf_{v \in \mathcal{A}_H^{\mathrm{tr}}} \mathrm{d}_{\mathcal{F}^+}(\Phi, v) < \delta\}$. We now pick $t_\delta > 0$ such that $S(s)u \in \mathcal{A}_{H,\delta}^{\mathrm{tr}}$ for every $s \geqslant t_\delta$. Set

$$C_\delta = \sup_{s \in [0, t_\delta]} (|\psi(S(s)u)| + |\widetilde{\psi}(S(s)u)|).$$

Similar to (4.6), we find that $C_\delta < +\infty$. Applying (4.1), (4.6) and (4.8), we have

$$\begin{aligned}
|\mathcal{L}_u(\psi - \widetilde{\psi})| &= \left| \mathrm{LIM}_{t \to +\infty} \frac{1}{t} \int_0^t (\psi(S(s)u) - \widetilde{\psi}(S(s)u)) \mathrm{d}s \right| \\
&\leqslant \limsup_{t \to +\infty} \frac{1}{t} \left| \int_0^t (\psi(S(s)u) - \widetilde{\psi}(S(s)u)) \mathrm{d}s \right| \\
&\leqslant \limsup_{t \to +\infty} \frac{1}{t} \int_0^{t_\delta} \left| \psi(S(s)u) - \widetilde{\psi}(S(s)u) \right| \mathrm{d}s \\
&\quad + \limsup_{t \to +\infty} \frac{1}{t} \int_{t_\delta}^t \left| \psi(S(s)u) - \widetilde{\psi}(S(s)u) \right| \mathrm{d}s \\
&\leqslant \limsup_{t \to +\infty} \frac{t_\delta C_\delta}{t} + \limsup_{t \to +\infty} \frac{(t - t_\delta)\epsilon}{t} \leqslant \epsilon.
\end{aligned}$$

By the arbitrariness of $\epsilon > 0$, we obtain the desired assertion that the positive linear functional $\mathcal{L}_u(\psi)$ depends only on the values of ψ on the trajectory attractor $\mathcal{A}_H^{\mathrm{tr}}$.

We now define $G(\psi) = \mathcal{L}_u(\ell(\psi))$ for $\psi \in \mathcal{C}(\mathcal{A}_H^{\mathrm{tr}})$, where $\ell(\psi)$ is an extension of ψ from $\mathcal{C}(\mathcal{A}_H^{\mathrm{tr}})$ to $\mathcal{C}(\mathcal{F}^+)$ given by the Tietze theorem (see [12, Theorem A.7]). By Definition 4.2(1) and (4.7), we can prove that $G(\cdot)$ is a positive linear functional on $\mathcal{C}(\mathcal{A}_H^{\mathrm{tr}})$. Since $\mathcal{A}_H^{\mathrm{tr}} \subset \mathcal{T}_H^+$, and $\mathcal{T}_H^+$ is a locally compact topological space, we see, by the Kakutani-Riesz Representation Theorem (see [12, Theorem A.1]), that there exists a unique positive, finite, Borel measure ρ_u on $\mathcal{A}_H^{\mathrm{tr}}$ such that

$$G(\psi) = \int_{\mathcal{A}_H^{\mathrm{tr}}} \psi(v) \mathrm{d}\rho_u(v). \tag{4.9}$$

We extend ρ_u (by zero) to a Borel measure on $\mathcal{F}^+$ by taking $\rho_u(E) = \rho_u(\mathcal{A}_H^{\mathrm{tr}} \cap E)$ for $E \in \mathcal{B}(\mathcal{F}^+)$. So for every $\psi \in \mathcal{C}(\mathcal{F}^+)$,

$$G(\psi) = \mathcal{L}_u(\psi) = \mathrm{LIM}_{t \to +\infty} \frac{1}{t} \int_0^t \psi(S(s)u) \mathrm{d}s = \int_{\mathcal{A}_H^{\mathrm{tr}}} \psi(v) \mathrm{d}\rho_u(v) = \int_{\mathcal{F}^+} \psi(v) \mathrm{d}\rho_u(v),$$

and clearly $\rho_u(\mathcal{F}^+ \backslash \mathcal{A}_H^{\mathrm{tr}}) = 0$. At the same time, since $\mathcal{F}^+$ is a metrizable space (cf. [13, Proposition 2.1(2)]), every finite Borel measure is tight in the sense of Definition 4.1(1) (see [1, Theorem 12.5]. Therefore, ρ_u is a trajectory statistical solution for the 3D globally modified Navier-Stokes equations.

We next prove that ρ_u is invariant under the action of $\{S(t)\}_{t\geqslant 0}$. To this end, we fix any $t^* \geqslant 0$ and any $\psi \in \mathcal{C}(\mathcal{F}^+)$. Since the interval $[0, t^*]$ is compact and $s \mapsto |\psi(S(s)u)|$ is continuous, we have

$$\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t^*}\psi(S(s)u)\mathrm{d}s = 0. \tag{4.10}$$

At the same time, we use (4.1) and (4.5) to get

$$\left|\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_t^{t+t^*}\psi(S(s)u)\mathrm{d}s\right| \leqslant \limsup_{t\to+\infty}\frac{t^*}{t}C_1 = 0. \tag{4.11}$$

Therefore, (4.10) and (4.11) give

$$\begin{aligned}
\int_{\mathcal{F}^+}\psi(S(t^*)v)\mathrm{d}\rho_u(v) &= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\psi(S(t^*)S(s)u)\mathrm{d}s\\
&= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\psi(S(t^*+s)u)\mathrm{d}s\\
&= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_{t^*}^{t+t*}\psi(S(s)u)\mathrm{d}s\\
&= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_t^{t+t^*}\psi(S(s)u)\mathrm{d}s - \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t^*}\psi(S(s)u)\mathrm{d}s\\
&\quad + \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\psi(S(s)u)\mathrm{d}s\\
&= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\psi(S(s)u)\mathrm{d}s = \int_{\mathcal{F}^+}\psi(v)\mathrm{d}\rho_u(v),
\end{aligned}$$

which is the desired invariant property. Finally, remember that the regularity of the trajectory statistical solution means that it is supported by a set in the trajectory space in which all weak solutions are in fact strong solutions. This is a directly conclusion of the regularity of the trajectory attractor which has been proved in Theorem 3.1(3). The proof is completed. □

5 Summary and Remark

In this article, we first prove the existence and regularity of the trajectory attractor for a 3D system of globally modified Navier-Stokes equations. The regularity of the trajectory attractor reveals the trajectory asymptotic smoothing effect of the addressed equations in the sense that the trajectories become eventually more smoother than the initial values. We then use the natural translation semigroup and trajectory attractor to construct trajectory statistical solution for the 3D globally modified Navier-Stokes equations. The constructed trajectory statistical solution is a tight Borel probability measure, which is supported by the trajectory attractor and is invariant under the action of the translation semigroup. Moreover, the constructed trajectory statistical solution is asymptotic regular in the sense that it is supported by a set in the trajectory space in which all weak solutions are in fact strong solutions.

We believe that the approach using trajectory attractor to construct trajectory statistical solution here can be raised to an abstract level for a broad class of evolutionary equations displaying the property of global existence of weak solutions without a known result of global uniqueness, such as the 2D dissipative Euler equations [9], the 3D MHD equations [7, 14, 32, 38] and the 3D convective Brinkman-Forchheimer equations [36]. The fundamental hypotheses on the concrete evolutionary equations is that the trajectory space is a metrizable normal topological space and the natural translation semigroup possesses a compact trajectory attractor.

We end this article with a remark. If we set $\mu_t = S(t)\rho_u$ (where ρ_u is the Borel probability measure obtained in Theorem 4.1) and rewrite equation (2.4) as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u)$$

for $\mathcal{G}(u) = g - Au + B_N(u, u)$, then we can prove the following Liouville-type equation in Statistical Mechanics

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}^+} \Phi(v)\mathrm{d}\mu_t(v) = \int_{\mathcal{F}^+} (\mathcal{G}(v), \Phi'(v))\mathrm{d}\mu_t(v),$$

for all "test" functions Φ (cf. [12, P$_{178}$, Definition 1.2]). We will investigate this issue in another paper.

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