

Applying the Random Variable Transformation method to solve a class of random linear differential equation with discrete delay

Tomás Caraballo^a, J.-C. Cortés^b, A. Navarro-Quiles^{c,d,*}

^a*Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla,*

c/ Tarfia s/n, 41012, Sevilla, Spain

^b*Instituto Universitario de Matemática Multidisciplinar,
Universitat Politècnica de València,*

c/ Camino de Vera s/n, 46022, Valencia, Spain

^c*DeustoTech, University of Deusto, 48007 Bilbao,
Basque Country, Spain*

^d*Facultad de Ingeniería, Universidad de Deusto,
Avda. Universidades, 24, 48007, Bilbao, Basque Country, Spain.*

Abstract

We randomize the following class of linear differential equations with delay, $x'_t(t) = ax_t(t) + bx_{t-\tau}(t - \tau)$, $t > 0$, and initial condition, $x_t(t) = g(t)$, $-\tau \leq t \leq 0$, by assuming that coefficients a and b are random variables and the initial condition $g(t)$ is a stochastic process. We consider two cases, depending on the functional form of the stochastic process $g(t)$, and then we solve, from a probabilistic point of view, both random initial value problems by determining explicit expressions to the first probability density function, $f(x, t; \tau)$, of the corresponding solution stochastic processes. Afterwards, we establish sufficient conditions on the involved random input parameters in order to guarantee that $f(x, t; \tau)$ converges, as $\tau \rightarrow 0^+$, to the first probability density function, say $f(x, t)$, of the corresponding associated random linear problem without delay ($\tau = 0$). The paper concludes with several numerical experiments illustrating our theoretical findings.

Keywords: Random linear differential equation with delay, Probability density function, Random Variable Transformation technique.

1. Introduction and motivation

Ordinary differential equations are useful mathematical tools to model phenomena in areas like Physics, Engineering, Epidemiology, Economics, etc. In many applications, differential equations are formulated using the principle of causality based upon the fact that future state of a system under study is independent of its past state and is solely determined by the current state. Although many phenomena can be properly described using this tenet, there are other situations

*Corresponding author

Email addresses: caraball@us.es (Tomás Caraballo), jccortes@imm.upv.es (J.-C. Cortés), annaqui@doctor.upv.es (A. Navarro-Quiles)

7 where it may be more realistic to model the current state of a physical system (understood in
8 a wide sense) in terms of past information. In that case, it is more suitable to describe the
9 dynamics of the physical system by means of differential equations incorporating the past history
10 of the system under analysis. These kinds of differential equations are usually referred to as
11 delay differential equations (DDEs). One commonly distinguishes two main classes of DDEs
12 depending on the type of delay considered therein. If only a part of the history has a relevant
13 influence on the current state, then discrete DDEs are formulated, while continuous DDEs are
14 those whose delay is unbounded or infinite. In this latter case, the whole past history is taken into
15 account to describe the phenomenon under study. In this paper, we will deal with the following
16 class of linear discrete DDEs with initial condition

$$\begin{cases} x'_\tau(t) = ax_\tau(t) + bx_\tau(t - \tau), & t > 0, \quad \tau > 0, \\ x_\tau(t) = g(t), & -\tau \leq t \leq 0, \end{cases} \quad (1)$$

17 where $\tau > 0$ denotes a prefixed finite delay, a is the coefficient of the non-delay term, $x_\tau(t)$, b
18 is the coefficient of the delay term, $x_\tau(t - \tau)$, and $g(t)$ is an arbitrary function (initial condition)
19 defined on the interval $[-\tau, 0]$. To avoid confusion with the notation introduced for the initial
20 condition and the solution of the initial value problem (IVP) formulated in (1), hereinafter the
21 solution will be denoted by $x_\tau(\cdot)$. This notation is necessary to explicitly indicate the dependence
22 of the solution on the τ parameter since later we will study the convergence of the solution of the
23 IVP (1) as $\tau \rightarrow 0^+$ to the solution of the corresponding IVP without delay. Now, if we assume
24 that $g(t)$ is continuous in $[-\tau, 0]$ and differentiable in $]-\tau, 0[$ then, according to Theorem 1 below,
25 there exists an exact expression of the solution of IVP (1).

26 **Theorem 1 ([1, 2]).** *Let us consider IVP (1) and assume that $g(t)$ is continuous in $[-\tau, 0]$ and*
27 *differentiable in $]-\tau, 0[$, i.e., $g(\cdot) \in C^1([-\tau, 0])$. Then, IVP (1) has a unique solution $x_\tau(\cdot) \in$*
28 *$C^0([-\tau, \infty]) \cap C^1([-\tau, 0]) \cap C^1([0, \infty])$ given by*

$$x_\tau(t) = e^{a(t+\tau)} e_\tau^{b_1, t} g(-\tau) + \int_{-\tau}^0 e^{a(t-s)} e_\tau^{b_1, t-\tau-s} (g'(s) - ag(s)) ds, \quad (2)$$

29 where $b_1 = e^{-a\tau} b$, and $e_\tau^{b_1, t}$ and $e_\tau^{b_1, t-\tau-s}$ denote the delayed exponential function, $e_\tau^{c, t}$, evaluated
30 at $(c, t) = (b_1, t)$ and $(c, t) = (b_1, t - \tau - s)$, respectively. This function appears in a natural way
31 in dealing with the linear discrete DDE (1) since its solution is constructed segment by segment
32 (see for example [2]). For the sake of completeness, below we recall its definition.

33 **Definition 1.** ([2]) *Let c be a real number and $\tau > 0$, then the function*

$$e_\tau^{c, t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + c \frac{t}{1!}, & 0 \leq t < \tau, \\ 1 + c \frac{t}{1!} + c^2 \frac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ \sum_{k=0}^n c^k \frac{(t-(k-1)\tau)^k}{k!}, & (n-1)\tau \leq t < n\tau, \end{cases} \quad (3)$$

34 *is called the delayed exponential function, where $n = \lfloor t/\tau \rfloor + 1$, being $\lfloor x \rfloor$ the greatest integer less*
35 *than or equal to x .*

36 So far we have revised the main definitions and results involving the solution of deterministic
 37 IVP (1) for the linear discrete DDE. When this class of equations is applied to model the dynam-
 38 ics of real phenomena, its input parameters, i.e. the coefficients a and b , and the initial condition,
 39 $g(t)$, must be fixed from experimental data which often involve uncertainties because they are
 40 obtained after measurements and sampling. This fact allows us to treat the input coefficients,
 41 a and b , as random variables (RVs), and the initial condition, $g(t)$, as a stochastic process (SP)
 42 rather than deterministic constants and a classical function, respectively. This leads to the full
 43 randomization of IVP (1)

$$\begin{cases} x'_\tau(t; \omega) &= a(\omega)x_\tau(t; \omega) + b(\omega)x_\tau(t - \tau; \omega), & t > 0, \quad \tau > 0, \\ x_\tau(t; \omega) &= g(t; \omega), & -\tau \leq t \leq 0, \end{cases} \quad (4)$$

44 where $a(\omega)$ and $b(\omega)$ are assumed to be absolutely continuous RVs and $g(t; \omega)$ is a SP, being
 45 all of them defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In accordance with
 46 Theorem 1, and using the so-called ‘‘sample random calculus’’ for SPs [3, App. I], hereinafter it
 47 will be assumed that the initial condition SP, $g(t; \omega)$, satisfies the following condition,

$$g(\cdot; \omega) \in C^1([-\tau, 0] \times \Omega), \quad \text{a.e.}, \quad (5)$$

48 where, as usual, a.e. stands for ‘almost everywhere’, so that the existence of a unique sample
 49 solution SP, $x_\tau(t; \omega)$, to random IVP (4) can be guaranteed.

50 The study of differential equations with delay involving uncertainties has been studied from
 51 different approaches. In [4], authors study a class of stochastic impulsive differential equations
 52 involving Bernoulli distribution where trial lengths vary randomly. In [5], the complete controlla-
 53 bility property of a class of nonlinear stochastic differential equation with delay, in the fractional
 54 sense, is investigated assuming that delays are described by Poisson jumps. Stochastic differen-
 55 tial equations with delay have been proposed to model interesting real problems. For example, in
 56 [6] the nonlinear delay differential neoclassical growth model is analysed assuming that stochas-
 57 tic perturbations of the white noise type. In [7], authors provide sufficient conditions for stability
 58 in probability of the equilibrium point of a social obesity epidemic model with distributed delay
 59 and stochastic perturbations. In dealing with delay differential equations it is usual to study the
 60 behaviour of the solution when the delay tends to zero, i.e., to investigate conditions under which
 61 there is convergence of the solution of the IVP with delay to the corresponding solution of the
 62 IVP when the delay vanishes. In [8] it is shown that the solution of a mixed stochastic delay
 63 differential equation depends continuously on the coefficients and the initial data. In addition,
 64 authors prove the convergence of the solutions to equations with vanishing delay to the solution
 65 of corresponding equations without delay. In [9] one deals with the mean square convergence
 66 and mean square exponential stability of an Euler scheme for a linear impulsive stochastic delay
 67 differential equation.

68 Solving a random (ordinary/partial/fractional/delay/etc.) differential equation means not just
 69 to obtain an exact/approximate solution SP, but also its main statistical properties, like the mean
 70 and the variance functions. These equations are said to be solved, from a probabilistic standpoint,
 71 when the first probability density function (1-PDF) of the solution SP is exact/approximately
 72 obtained since from this deterministic function one can completely characterize the probabilistic
 73 behaviour of the solution SP at every time instant. As a consequence, mean, variance, skewness,
 74 etc., as well as any one-dimensional moment of the solution SP can be derived from the 1-PDF,
 75 **provided these moments exist**. To be specific, if $f(x, t; \tau)$ denotes the 1-PDF of the solution SP
 76 $x_\tau(t; \omega)$ to random IVP (1), then the mean, $\mathbb{E}[x_\tau(t; \omega)]$, and the variance, $\mathbb{V}[x_\tau(t; \omega)]$, functions

77 can be obtained by

$$\mathbb{E} \left[(x_\tau(t; \omega))^k \right] = \int_{\mathbb{R}} x^k f(x, t; \tau) dx, \quad k = 1, 2, \dots$$

78 as

$$\mathbb{E}[x_\tau(t; \omega)] = \int_{\mathbb{R}} x f(x, t; \tau) dx, \quad \mathbb{V}[x_\tau(t; \omega)] = \int_{\mathbb{R}} x^2 f(x, t; \tau) dx - (\mathbb{E}[x_\tau(t; \omega)])^2, \quad (6)$$

79 respectively. Fixed a time instant, say \hat{t} , the computation of the probability that the solution,
80 $x_\tau(\hat{t}; \omega)$, lies within an interval of specific interest can also be computed just by integrating the
81 1-PDF

$$\mathbb{P}[\omega \in \Omega : a \leq x_\tau(\hat{t}; \omega) \leq b] = \int_a^b f(x, \hat{t}; \tau) dx.$$

82 At this point is important to stress that, to the best of our knowledge, this approach has already
83 been dealt with some classes of random fractional, ordinary and partial differential equations and
84 of random difference equations as well (see for instance, [10], [11, 12, 13, 14, 15], [16, 17, 18,
85 19] and [20, 21], respectively), but the corresponding analysis for random DDEs has not been
86 addressed yet.

87 In the spirit of these previous contributions, the main objective of this paper is solving, from
88 a probabilistic point of view, random IVP (4) by obtaining the 1-PDF, $f(x, t; \tau)$, of its solution
89 SP, $x_\tau(t; \omega)$, which, according to the deterministic solution formulated in (2), for each $t \in [(n -$
90 $1)\tau, n\tau[$, with $\tau > 0$ fixed and $n = 1, 2, \dots$, is given by

$$x_\tau(t; \omega) = e^{a(\omega)(t+\tau)} e_{\tau}^{b_1(\omega), t} g(-\tau; \omega) + \int_{-\tau}^0 e^{a(\omega)(t-s)} e_{\tau}^{b_1(\omega), t-\tau-s} (g'(s; \omega) - a(\omega)g(s; \omega)) ds, \quad (7)$$

91 where $b_1(\omega) = e^{-a(\omega)\tau} b(\omega)$. The key tool that will be applied to achieve this goal is the Random
92 Variable Transformation (RVT) method. This technique allows us to obtain the PDF of a random
93 vector, which results from mapping another random vector whose PDF is known. The following
94 result provides the RVT technique in its multidimensional version.

Theorem 2 (Multidimensional RVT method, [3]). *Let $\mathbf{x}(\omega) = (x_1(\omega), \dots, x_m(\omega))$ and $\mathbf{y}(\omega) = (y_1(\omega), \dots, y_m(\omega))$ be two m -dimensional absolutely continuous random vectors defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbf{r} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a one-to-one deterministic transformation of $\mathbf{x}(\omega)$ into $\mathbf{y}(\omega)$, i.e., $\mathbf{y}(\omega) = \mathbf{r}(\mathbf{x}(\omega))$. Assume that \mathbf{r} is continuous in \mathbf{x} and has continuous partial derivatives with respect to \mathbf{x} . Then, if $f_{\mathbf{x}}(\mathbf{x})$ denotes the joint probability density function of the random vector $\mathbf{x}(\omega)$, and $\mathbf{s} = \mathbf{r}^{-1} = (s_1(y_1, \dots, y_m), \dots, s_m(y_1, \dots, y_m))$ represents the inverse mapping of $\mathbf{r} = (r_1(x_1, \dots, x_m), \dots, r_m(x_1, \dots, x_m))$, the joint probability density function of the random vector $\mathbf{y}(\omega)$ is given by*

$$g_{\mathbf{y}}(\mathbf{y}) = f_{\mathbf{x}}(\mathbf{s}(\mathbf{y})) |J_m|,$$

where $|J_m|$, which is assumed to be different from zero, denotes the absolute value of the Jacobian defined by the determinant

$$J_m = \det \begin{pmatrix} \frac{\partial s_1(y_1, \dots, y_m)}{\partial y_1} & \dots & \frac{\partial s_m(y_1, \dots, y_m)}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(y_1, \dots, y_m)}{\partial y_m} & \dots & \frac{\partial s_m(y_1, \dots, y_m)}{\partial y_m} \end{pmatrix}.$$

4

95 Once we have obtained the 1-PDF, $f(x, t; \tau)$, of the solution SP to random IVP (4), the next
 96 objective of this paper is to study the relationship between $f(x, t; \tau)$ and the 1-PDF, say $f(x, t)$,
 97 of the corresponding random IVP without delay, i.e.,

$$\begin{cases} x'(t; \omega) &= (a(\omega) + b(\omega))x(t; \omega), \quad t > 0, \\ x(0; \omega) &= g(0; \omega) = g_0(\omega), \end{cases} \quad (8)$$

98 where $g_0(\omega)$ is an absolutely continuous RV. To be specific, we will establish conditions in order
 99 to guarantee that

$$\lim_{\tau \rightarrow 0^+} f(x, t; \tau) = f(x, t), \quad \text{for each } (x, t) \in \mathcal{D}(x_\tau(t; \omega)) \cap \mathcal{D}(x(t; \omega)) \times [(n-1)\tau, n\tau[\text{ fixed, } (9)$$

100 being $n = 1, 2, \dots$ and where $\mathcal{D}(x_\tau(t; \omega))$ and $\mathcal{D}(x(t; \omega))$ denote the codomains of SPs $x_\tau(t; \omega)$
 101 and $x(t; \omega)$, respectively. This analysis will focus on the following choices of the initial condition
 102 $g(t; \omega)$ for random IVP (4)

103 • Case I: $g(t; \omega) = e^{a(\omega)t+c(\omega)}$.

104 • Case II: $g(t; \omega) = \sum_{j=0}^m c_j(\omega) t^j, m \geq 0$.

105 The choice in Case I has been made because, as it will be seen later, it allows us to deal with
 106 a scenario that illustrates adequately the main ideas of our approach. While Case II involves
 107 the general case in which the initial condition is a random polynomial of degree m , which has
 108 interest by itself since, similarly to what happens in the deterministic context, many important
 109 random SPs can be approximated by appropriate random polynomials.

110 The rest of this paper is organized as follows. Section 2 is split into two subsections, in
 111 the first one, we determine an explicit expression to the 1-PDF, $f(x, t; \tau)$, of the solution SP to
 112 random IVP (1) in Case I, i.e., when $g(t; \omega) = e^{a(\omega)t+c(\omega)}$. Subsection 2.1 is addressed to establish
 113 sufficient conditions upon the involved random input parameters $a(\omega)$, $b(\omega)$ and $c(\omega)$, in order
 114 to guarantee that $f(x, t; \tau)$ converges, as $\tau \rightarrow 0^+$, to the 1-PDF, $f(x, t)$, of the corresponding
 115 associated random linear problem without delay ($\tau = 0$). For the sake of clarity, Section 3 is
 116 organized analogously as Section 2 to conduct the corresponding study in Case II, i.e., when
 117 the initial condition to random IVP (1) is a random polynomial $g(t; \omega) = \sum_{j=0}^m c_j(\omega) t^j, m \geq$
 118 0. Several numerical examples, corresponding to Cases I and II are exhibited in Section 4.
 119 Conclusions are drawn in Section 5.

120 2. Case I: Computing the 1-PDF of the solution SP and study of the convergence

121 Let us consider random IVP (4) where the initial condition is given by the SP $g(t; \omega) =$
 122 $e^{a(\omega)t+c(\omega)}$. Observe that this situation corresponds to the case where $g(t; \omega)$ is the solution of the
 123 following random IVP

$$\begin{cases} g'(t; \omega) &= a(\omega)g(t; \omega), \quad t \geq -\tau, \quad \tau > 0, \\ g(-\tau; \omega) &= e^{-a(\omega)\tau+c(\omega)}. \end{cases} \quad (10)$$

124 In other words, we are then implicitly considering a stochastic control problem defined by (4)
 125 and (10). In agreement with (7), the solution SP of this control problem is given by

$$x_\tau(t; \omega) = e^{a(\omega)t+c(\omega)} e_\tau^{b_1(\omega), t}, \quad t \in [(n-1)\tau, n\tau[, \quad n \in \mathbb{N} \text{ fixed,} \quad (11)$$

126 where $b_1(\omega) = e^{-a(\omega)\tau} b(\omega)$. Henceforth, we will assume that the random inputs $a(\omega)$, $b(\omega)$ and
 127 $c(\omega)$ are absolutely continuous dependent RVs with a joint PDF denoted by $f_{c,a,b}(c, a, b)$, which
 128 is assumed to be known. Observe that $g(t; \omega)$ satisfies condition (5).

129 2.1. Computing the PDF

130 First at all, notice that if \hat{t} is such that $x_\tau(\hat{t}; \omega) = 0$, then clearly the 1-PDF is given by
 131 $f(x, \hat{t}; \tau) = \delta(x)$, $-\infty < x < \infty$, being $\delta(\cdot)$ the Dirac delta function. Thus, in order to determine the
 132 1-PDF of the solution SP, $x_\tau(t; \omega)$, given by (11), we will only consider time instants t such that
 133 $x_\tau(t; \omega) \neq 0$ a.e. As a consequence, as $x_\tau(t; \omega) = e^{a(\omega)t+c(\omega)} e_\tau^{b_1(\omega), t}$, one derives that $e_\tau^{b_1(\omega), t} \neq 0$
 134 a.e., at every time instant t where the 1-PDF $f(x, t; \tau)$ is going to be determined.

135 Let $t \in [(n-1)\tau, n\tau[$ be fixed, next we will apply the RVT method (see Theorem 2), in order to
 136 obtain the PDF, $f(x, t; \tau)$, of the solution SP, $x_\tau(t; \omega)$, given by (11). This PDF will be expressed
 137 in terms of the joint PDF $f_{c,a,b}(c, a, b)$. To this end, consider the following deterministic mapping
 138 $\mathbf{r} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{aligned} x_1 &= r_1(c, a, b) = e^{at+c} e_\tau^{b_1, t}, \\ x_2 &= r_2(c, a, b) = a, \\ x_3 &= r_3(c, a, b) = b, \end{aligned}$$

139 where $b_1 = e^{-a\tau} b$. The inverse mapping, $\mathbf{s} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, of \mathbf{r} is given by

$$\begin{aligned} c &= s_1(x_1, x_2, x_3) = \ln\left(\frac{x_1 e^{-x_2 t}}{e_\tau^{b_1, t}}\right), \\ a &= s_2(x_1, x_2, x_3) = x_2, \\ b &= s_3(x_1, x_2, x_3) = x_3, \end{aligned}$$

140 where $b_1 = e^{-x_2\tau} x_3$. Observe that the Jacobian of \mathbf{s} is $|J_3| = 1/|x_1| \neq 0$. Moreover, $|J_3|$ is well-
 141 defined since $x_\tau(t; \omega) = x_1 \neq 0$ a.s. for each instant time t . Therefore, applying the RVT method
 142 one obtains the PDF of the random vector $(x_1(\omega), x_2(\omega), x_3(\omega))$,

$$f(x_1, x_2, x_3) = f_{c,a,b}\left(\ln\left(\frac{x_1 e^{-x_2 t}}{e_\tau^{b_1, t}}\right), x_2, x_3\right) \frac{1}{|x_1|}, \text{ where } b_1 = e^{-x_2\tau} x_3.$$

143 Then, marginalizing with respect to the random vector $(x_2(\omega), x_3(\omega)) = (a(\omega), b(\omega))$ and taking
 144 $t \in [(n-1)\tau, n\tau[$ arbitrary, one gets the 1-PDF of the solution SP, $x_\tau(t; \omega)$,

$$f(x, t; \tau) = \int_{\mathbb{R}^2} f_{c,a,b}\left(\ln\left(\frac{x e^{-at}}{e_\tau^{b_1, t}}\right), a, b\right) \frac{1}{|x|} da db, \text{ where } b_1 = e^{-a\tau} b. \quad (12)$$

145 2.2. Convergence

146 As it has been indicated previously, this subsection is devoted to investigate the relationship
 147 between the 1-PDF, $f(x, t; \tau)$, given in (12), as $\tau \rightarrow 0^+$, and the 1-PDF, $f(x, t)$, of the solution
 148 SP to random IPV (8). In order for the corresponding analysis makes sense (put $\tau \rightarrow 0^+$ in the
 149 initial condition of (10)), we will take as initial condition in this latter IVP $g_0(\omega) = e^{c(\omega)}$, so the
 150 solution SP to random IPV (8) is given by

$$x(t; \omega) = e^{c(\omega)} e^{(a(\omega)+b(\omega))t}.$$

151 By applying the RVT method, it can be checked that the 1-PDF of $x(t; \omega)$ is given by

$$f(x, t) = \int_{\mathbb{R}^2} f_{c,a,b}(\ln(x e^{-(a+b)t}), a, b) \frac{1}{|x|} da db. \quad (13)$$

152 In order to find out sufficient conditions that guarantee the convergence given in (9), where
 153 $f(x, t; \tau)$ and $f(x, t)$ are given by (12) and (13), respectively, we first establish the following result
 154 that permits relating the delayed exponential function to the (classical) exponential function as
 155 the delay tends to zero. The proof of this result is based upon the ideas exhibited in [22, Theorem
 156 A.3.].

157 **Theorem 3 (Convergence of the delayed exponential function).** *Let $c \in \mathbb{R}$, $\tau_0 > 0$, $\alpha = 1 +$
 158 $e^{\tau_0|c|} > 1$. Then, for any $\tau \in]0, \tau_0[$,*

$$|e^{ct} - e_{\tau}^{c,t}| \leq \tau|c| \left(e^{\alpha|c|T} + e^{c|T-\tau|} \right), \quad t \in [0, T].$$

159 If $\tau = 0$ then $e^{ct} = e_{\tau}^{c,t}$.

160 **Proof** Let $\tau \in]0, \tau_0[$. We will apply mathematical induction in order to prove that for any $n \in \mathbb{N}$

$$|e^{ct} - e_{\tau}^{c,t}| \leq \tau|c| \left(e^{\alpha|c|n\tau} + e^{c|(n-1)\tau} \right), \quad \text{for } t \in [(n-1)\tau, n\tau]. \quad (14)$$

161 • If $n = 1$, then $t \in [0, \tau]$ and, by the definition of the delayed exponential function given in
 162 (3), $e_{\tau}^{c,t} = 1 + ct$. Therefore, we must prove

$$|e^{ct} - (1 + ct)| \leq \tau|c| \left(e^{\alpha|c|\tau} + 1 \right).$$

163 Applying the Fundamental Calculus Theorem (FCT) and the Mean Value Theorem (MVT)
 164 for integration, one derives

$$\begin{aligned} |e^{ct} - (1 + ct)| &\leq |e^{ct} - 1| + |ct| \stackrel{\text{FCT}}{=} \left| \int_0^t \left[\frac{d}{dx} (e^{cx}) \right] dx \right| + |ct| \leq \int_0^t \left| \frac{d}{dx} (e^{cx}) \right| dx + |ct| \\ &= |c| \int_0^t e^{cx} dx + |ct| \stackrel{t \leq \tau}{\leq} |c| \int_0^{\tau} e^{cx} dx + |c|\tau \stackrel{\text{MVT}}{\leq} |c|\tau e^{c\delta} + |c|\tau \stackrel{\text{(I)}}{\leq} |c|\tau \left(e^{\alpha|c|\tau} + 1 \right). \end{aligned}$$

165 Now, we justify Step (I) previously applied.

166 **Step (I):** By the MVT, $\delta \in [0, \tau]$. Let $\alpha > 1$, then

$$c\delta \leq |c|\delta \leq |c|\tau \leq \alpha|c|\tau,$$

167 and, as the exponential is an increasing function, $e^{c\delta} \leq e^{\alpha|c|\tau}$.

168 • Now, assuming that claim (14) is satisfied for $n \geq 1$ (induction hypothesis), we will apply
 169 the FCT to prove inequality (14) for $n + 1$. Let $t \in [n\tau, (n + 1)\tau]$,

$$\begin{aligned}
|e_{\tau}^{c,t} - e^{ct}| &\stackrel{\text{(II)}}{\leq} |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + \int_{n\tau}^{(n+1)\tau} \left| \frac{d}{ds}(e_{\tau}^{c,s} - e^{cs}) \right| ds \\
&= |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + |c| \int_{n\tau}^{(n+1)\tau} |e_{\tau}^{c,s-\tau} - e^{cs}| ds \\
&\leq |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + |c| \int_{n\tau}^{(n+1)\tau} |e_{\tau}^{c,s-\tau} - e^{c(s-\tau)}| ds \\
&\quad + |c| \int_{n\tau}^{(n+1)\tau} |e^{c(s-\tau)} - e^{cs}| ds \\
&\stackrel{\text{(III)}}{\leq} |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + |c| \int_{n\tau}^{(n+1)\tau} |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) ds \\
&\quad + |c| \int_{n\tau}^{(n+1)\tau} \int_{s-\tau}^s \left| \frac{d}{d\sigma}(e^{c\sigma}) \right| d\sigma ds \\
&= |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + (|c|\tau)^2 (e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) \\
&\quad + |c|^2 \int_{n\tau}^{(n+1)\tau} \int_{s-\tau}^s e^{c\sigma} d\sigma ds \\
&\stackrel{\text{(IV)}}{\leq} |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + (|c|\tau)^2 (e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) \\
&\quad + |c|^2 \int_{n\tau}^{(n+1)\tau} \int_{s-\tau}^s e^{\alpha|c|n\tau + |c|\tau} d\sigma ds \\
&= |c|\tau(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + (|c|\tau)^2 (e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau}) + |c|^2 \tau^2 e^{\alpha|c|n\tau + |c|\tau} \\
&= |c|\tau(e^{\alpha|c|n\tau} (1 + |c|\tau + |c|\tau e^{|c|\tau}) + e^{|c|(n-1)\tau} (1 + |c|\tau)) \\
&\stackrel{\text{(V)}}{\leq} |c|\tau(e^{\alpha|c|(n+1)\tau} + e^{|c|n\tau}).
\end{aligned}$$

170

Now, we justify Steps (II)–(V) previously applied.

171

Step (II): By the FCT

$$\int_{n\tau}^t \frac{d}{ds}(e_{\tau}^{c,s} - e^{cs}) ds = e_{\tau}^{c,t} - e^{ct} - e_{\tau}^{c,n\tau} + e^{cn\tau}.$$

172 Then by the induction hypothesis and taking into account that $t \leq (n+1)\tau$, we have

$$\begin{aligned} |e_{\tau}^{c,t} - e^{ct}| &\leq |e_{\tau}^{c,n\tau} - e^{cn\tau}| + \int_{n\tau}^t \left| \frac{d}{ds} (e_{\tau}^{c,s} - e^{cs}) \right| ds \\ &\leq |c|\tau \left(e^{\alpha|c|n\tau} + e^{|c|(n-1)\tau} \right) + \int_{n\tau}^{(n+1)\tau} \left| \frac{d}{ds} (e_{\tau}^{c,s} - e^{cs}) \right| ds. \end{aligned}$$

173 **Step (III):** In this step we apply both the FCT and the hypothesis of induction (14).

174 **Step (IV):** Following the same argument of Step (I), being $\alpha > 1$

$$c\sigma \leq |c|\sigma \leq |c|s \leq |c|(n+1)\tau \leq \alpha|c|n\tau + |c|\tau,$$

175 so, $e^{c\sigma} \leq e^{\alpha|c|n\tau + |c|\tau}$.

176 **Step (V):** Applying that $e^p \geq 1 + p$, $\forall p \geq 0$ and taking $\alpha = 1 + e^{|c|\tau_0}$, as $0 < \tau \leq \tau_0$, then

$$1 + |c|\tau + |c|\tau e^{|c|\tau} = 1 + |c|\tau(1 + e^{|c|\tau}) \leq 1 + |c|\tau(1 + e^{|c|\tau_0}) = 1 + \alpha|c|\tau \leq e^{\alpha|c|\tau}.$$

177 And analogously, $1 + |c|\tau \leq e^{|c|\tau}$.

178 This concludes the proof. \(\square\)

180 Before showing the convergence (9), we need to establish some technical results that will be
181 required in the subsequent analysis.

182 Let $c \in \mathbb{R}$ be arbitrary and let us take $\tau_0 \rightarrow 0^+$ (thus $\tau \rightarrow 0^+$, too) in Theorem 3. Then

$$\lim_{\tau \rightarrow 0^+} e_{\tau}^{c,t} = e^{ct}, \quad t \in [0, T], \quad (15)$$

183 where, according to Definition 3, $t = (n-1)\tau$, $n \rightarrow +\infty$ and $\tau \rightarrow 0^+$. Furthermore, as the delayed
184 exponential function is continuous, one derives that

$$\text{if } \lim_{\tau \rightarrow 0^+} f(\tau) = \hat{f}, \text{ then } \lim_{\tau \rightarrow 0^+} e_{\tau}^{f(\tau),t} = e^{\hat{f}t} \neq 0. \quad (16)$$

185 In particular, if we take $f(\tau) = e^{-a\tau} b$ with $a, b \in \mathbb{R}$ fixed, as $e^{-a\tau} b \xrightarrow{\tau \rightarrow 0^+} b$ in (16), then

$$\lim_{\tau \rightarrow 0^+} e_{\tau}^{e^{-a\tau} b, t} = e^{bt} \neq 0.$$

186 Therefore, there exists $\tau_1 > 0$ such that

$$e_{\tau}^{b_1, t} \neq 0, \quad b_1 = e^{-a\tau} b, \quad \forall \tau \in [0, \tau_1].$$

187 As a consequence of the continuity of the logarithm function and of the delayed exponential
188 function (15), one further obtains

$$\lim_{\tau \rightarrow 0^+} \ln \left(\frac{x e^{-at}}{e_{\tau}^{b_1, t}} \right) = \ln \left(x e^{-(a+b)t} \right).$$

189 Therefore, for all $\epsilon > 0$ there exists $\tau_2 : 0 < \tau_2 \leq \tau_1$ such that

$$\left| \ln \left(\frac{x e^{-at}}{e_\tau^{b_1, t}} \right) - \ln \left(x e^{-(a+b)t} \right) \right| < \epsilon, \quad \forall \tau \in]0, \tau_2[. \quad (17)$$

190 To prove the convergence $f(x, t; \tau) \xrightarrow{\tau \rightarrow 0^+} f(x, t)$ introduced in (9), hereinafter the following
191 hypotheses will be assumed

H1 : The random vector of coefficients $(a(\omega), b(\omega))$ is independent of the RV $c(\omega)$, i.e.,
 $f_{c,a,b}(c, a, b) = f_c(c) f_{a,b}(a, b)$.

192

H2 : $f_c(c)$ is Lipschitz continuous in \mathbb{R} , i.e.,
 $\exists L_{f_c} \geq 0 : |f_c(c_{0,1}) - f_c(c_{0,2})| \leq L_{f_c} |c_{0,1} - c_{0,2}|, \quad \forall c_{0,1}, c_{0,2} \in \mathbb{R}$,

193 Let $\tau \in]0, \tau^*[: 0 < \tau^* < \tau_2$ and $(x, t) \in \mathcal{D}(x_\tau(t; \omega)) \cap \mathcal{D}(x(t; \omega)) \times [(n-1)\tau, n\tau[\subset \mathbb{R} \times [0, T]$
194 being all of them fixed. Then, taking into account expressions (12) and (13), for $\epsilon^* > 0$ arbitrary

$$\begin{aligned} |f(x, t; \tau) - f(x, t)| &= \frac{1}{|x|} \left| \int_{\mathbb{R}^2} \left[f_{c,a,b} \left(\ln \left(\frac{x e^{-at}}{e_\tau^{b_1, t}} \right), a, b \right) - f_{c,a,b} \left(\ln \left(x e^{-(a+b)t} \right), a, b \right) \right] da db \right| \\ &\stackrel{\mathbf{H1}}{=} \frac{1}{|x|} \left| \int_{\mathbb{R}^2} \left[f_c \left(\ln \left(\frac{x e^{-at}}{e_\tau^{b_1, t}} \right) \right) - f_c \left(\ln \left(x e^{-(a+b)t} \right) \right) \right] f_{a,b}(a, b) da db \right| \\ &\leq \frac{1}{|x|} \int_{\mathbb{R}^2} \left| f_c \left(\ln \left(\frac{x e^{-at}}{e_\tau^{b_1, t}} \right) \right) - f_c \left(\ln \left(x e^{-(a+b)t} \right) \right) \right| f_{a,b}(a, b) da db \\ &\stackrel{\mathbf{H2}}{\leq} \frac{L_{f_c}}{|x|} \int_{\mathbb{R}^2} \left| \ln \left(\frac{x e^{-at}}{e_\tau^{b_1, t}} \right) - \ln \left(x e^{-(a+b)t} \right) \right| f_{a,b}(a, b) da db \\ &\stackrel{\text{(VI)}}{<} \frac{L_{f_c}}{|x|} \epsilon^* \frac{|x|}{L_{f_c}} \int_{\mathbb{R}^2} f_{a,b}(a, b) da db = \epsilon^*, \end{aligned}$$

195 where $b_1 = e^{-a\tau} b$. In Step (VI), we have applied (17) with $\epsilon = \epsilon^* \frac{|x|}{L_{f_c}} > 0$, while in the last step
196 we have used that $\int_{\mathbb{R}^2} f_{a,b}(a, b) da db = 1$, since $f_{a,b}(a, b)$ is a PDF.

197 Summarizing, the following result has been established

198 **Theorem 4.** Consider the random discrete delay differential equation (4) with $g(t; \omega) = e^{a(\omega)t+c(\omega)}$
199 and whose solution SP, $x_\tau(t; \omega)$, is given by (11). Assume that $a(\omega)$, $b(\omega)$ and $c(\omega)$ are absolutely
200 continuous RVs being $f_{c,a,b}(c, a, b)$ their joint PDF. Then, the 1-PDF, $f(x, t; \tau)$, of the solution
201 SP $x_\tau(t; \omega)$ is given by (12) at every time instant t where $x_\tau(t; \omega) \neq 0$. Further assume that hy-
202 potheses **H1** and **H2** hold, then $f(x, t; \tau)$ converges to the 1-PDF, $f(x, t)$, of the solution SP to the
203 random linear differential equation (8), according to (9).

204 3. Case II: Computing the PDF of the solution SP and study of the convergence

205 In this section we will analyze Case II following an analogous structure to the one exhibited
206 in Section 2. Thus, we will consider random IVP (4) assuming that the initial condition is given

207 by a random polynomial of degree m

$$g(t, \omega) = \sum_{j=0}^m c_j(\omega) t^j, \quad m \geq 0, \quad (18)$$

208 where $c_j(\omega)$, $j = 0, 1, \dots, m$ are absolutely continuous RVs defined on a common complete
 209 probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of generality in the subsequent study, we will assume no
 210 independence among the involved random inputs, i.e., henceforth, we will assume that $f_{\mathbf{w}}(\mathbf{w})$ is
 211 the PDF of the random vector $\mathbf{w}(\omega) = (c_0(\omega), c_1(\omega), \dots, c_m(\omega), a(\omega), b(\omega))$ made of polynomial
 212 coefficients, $c_j(\omega)$, $0 \leq j \leq m$ of the initial condition given by (18), and of coefficients $a(\omega)$ and
 213 $b(\omega)$ of the random delayed differential equation (4).

214 3.1. Computing the PDF

215 In this case, according to expression (7), the solution SP $x_\tau(t; \omega)$ is given by

$$x_\tau(t; \omega) = \varphi_\tau^0(t, \omega) c_0(\omega) + \sum_{j=1}^m \varphi_\tau^j(t, \omega) c_j(\omega), \quad (19)$$

216 being

$$\varphi_\tau^0(t, \omega) = e^{a(\omega)(t+\tau)} e^{b_1(\omega)t} - \int_{-\tau}^0 a(\omega) e^{a(\omega)(t-s)} e^{b_1(\omega), t-\tau-s} ds, \quad (20)$$

217 and

$$\varphi_\tau^j(t, \omega) = e^{a(\omega)(t+\tau)} e^{b_1(\omega)t} (-\tau)^j + \int_{-\tau}^0 e^{a(\omega)(t-s)} e^{b_1(\omega), t-\tau-s} (j - a(\omega)s) s^{j-1} ds, \quad j = 1, 2, \dots, m, \quad (21)$$

218 where $b_1(\omega) = e^{-a(\omega)\tau} b(\omega)$ and $\tau > 0$ is a fixed delay.

219 **Remark 1.** Notice that, in order to have a non-trivial solution, at least one coefficient $c_j(\omega)$, in
 220 the initial condition $\sum_{j=0}^m c_j(\omega) t^j$ must be, in the probabilistic sense, different from zero. This
 221 fact is guaranteed since $c_j(\omega)$, $j = 0, 1, \dots, m$, are absolutely continuous RVs.

222 Let $t \in [(n-1)\tau, n\tau[$ be fixed, and let us apply the RVT method (Theorem 2) to obtain the
 223 PDF of the solution SP, $x_\tau(t; \omega)$, in terms of the PDF, $f_{\mathbf{w}}(\mathbf{w})$, of the random vector $\mathbf{w}(\omega) =$
 224 $(c_0(\omega), c_1(\omega), \dots, c_m(\omega), a(\omega), b(\omega))$. To this end, we will define the following mapping $\mathbf{r} :$
 225 $\mathbb{R}^{m+3} \rightarrow \mathbb{R}^{m+3}$ whose components are defined by

$$\begin{aligned} x_1 &= r_1(c_0, c_1, \dots, c_m, a, b) = \varphi_\tau^0(t, a, b) c_0 + \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j, \\ x_2 &= r_2(c_0, c_1, \dots, c_m, a, b) = c_1, \\ x_3 &= r_3(c_0, c_1, \dots, c_m, a, b) = c_2, \\ &\vdots \\ x_{m+1} &= r_{m+1}(c_0, c_1, \dots, c_m, a, b) = c_m, \\ x_{m+2} &= r_{m+2}(c_0, c_1, \dots, c_m, a, b) = a, \\ x_{m+3} &= r_{m+3}(c_0, c_1, \dots, c_m, a, b) = b. \end{aligned}$$

226 Observe that, for the sake of clarity, in the previous expression we have emphasized that φ_τ^j
 227 depends on a and b . The inverse mapping of \mathbf{r} is $\mathbf{s} : \mathbb{R}^{m+3} \rightarrow \mathbb{R}^{m+3}$ whose components are

$$\begin{aligned} c_0 &= s_1(x_1, x_2, \dots, x_{m+3}) = \frac{x_1 - \sum_{j=1}^m \varphi_\tau^j(t, x_{m+2}, x_{m+3})x_{j+1}}{\varphi_\tau^0(t, x_{m+2}, x_{m+3})}, \\ c_1 &= s_2(x_1, x_2, \dots, x_{m+3}) = x_2, \\ c_2 &= s_3(x_1, x_2, \dots, x_{m+3}) = x_3, \\ &\vdots \\ c_m &= s_{m+1}(x_1, x_2, \dots, x_{m+3}) = x_{m+1}, \\ a &= s_{m+2}(x_1, x_2, \dots, x_{m+3}) = x_{m+2}, \\ b &= s_{m+3}(x_1, x_2, \dots, x_{m+3}) = x_{m+3}. \end{aligned}$$

228 The Jacobian of mapping \mathbf{s} is given by

$$|J_{m+3}| = \frac{1}{|\varphi_\tau^0(t, x_{m+2}, x_{m+3})|} \neq 0.$$

229 Notice that, the absolute value of the Jacobian is well-defined since $a(\omega)$ and $b(\omega)$ are abso-
 230 lutely continuous RVs, thus by (20) $\varphi_\tau^0(t, x_{m+2}, x_{m+3})$ is different from zero with probability one.
 231 Therefore, the PDF of the random vector $\mathbf{x}(\omega) = (x_1(\omega), x_2(\omega), \dots, x_{m+3}(\omega))$ in terms of the PDF
 232 of $\mathbf{w}(\omega) = (c_0(\omega), c_1(\omega), \dots, c_m(\omega), a(\omega), b(\omega))$ is given by

$$f_{\mathbf{x}}(\mathbf{x}) = f_{\mathbf{w}} \left(\frac{x_1 - \sum_{j=1}^m \varphi_\tau^j(t, x_{m+2}, x_{m+3})x_{j+1}}{\varphi_\tau^0(t, x_{m+2}, x_{m+3})}, x_2, \dots, x_{m+3} \right) \frac{1}{|\varphi_\tau^0(t, x_{m+2}, x_{m+3})|}.$$

233 Then, marginalizing with respect to the random vector $(x_2(\omega), x_3(\omega), \dots, x_{m+3}(\omega)) = (c_1(\omega),$
 234 $\dots, c_m(\omega), a(\omega), b(\omega))$, given $\tau > 0$ and taking $t \in [(n-1)\tau, n\tau[$ arbitrary, the 1-PDF of the
 235 solution SP $x_\tau(t; \omega)$ becomes

$$f(x, t; \tau) = \int_{\mathbb{R}^{m+2}} f_{\mathbf{w}} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)}, c_1, \dots, c_m, a, b \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_m \dots dc_1. \quad (22)$$

236 3.2. Convergence

237 This subsection is addressed to study conditions in order to the 1-PDF, $f(x, t; \tau)$, given by
 238 (22), that corresponds to random IVP (4) with delay, converges to the 1-PDF, $f(x, t)$, of the
 239 solution SP of the corresponding non-delayed random IVP

$$\begin{cases} x'(t; \omega) &= (a(\omega) + b(\omega))x(t; \omega), \quad t \geq 0, \\ x(0; \omega) &= c_0(\omega). \end{cases} \quad (23)$$

240 To compute the 1-PDF, $f(x, t)$, notice that the solution SP of random IVP (23) is given by

$$x(t; \omega) = c_0(\omega) e^{(a(\omega)+b(\omega))t}.$$

241 Then applying the RVT method (see Theorem 2), it is straightforward to check that

$$f(x, t) = \int_{\mathbb{R}^2} f_{c_0, a, b} \left(x e^{-(a+b)t}, a, b \right) e^{-(a+b)t} db da. \quad (24)$$

242 Here, $f_{c_0,a,b}(c_0, a, b)$ stands for the joint PDF of the RVs $c_0(\omega)$, $a(\omega)$ and $b(\omega)$, which is obtained
 243 by marginalizing the PDF, $f_{c_0,c_1,\dots,c_m,a,b}(c_0, c_1, \dots, c_m, a, b)$, with respect to the RVs $c_1(\omega), \dots, c_m(\omega)$
 244 (notice that by hypothesis this PDF is known),

$$f_{c_0,a,b}(c_0, a, b) = \int_{\mathbb{R}^m} f_{c_0,c_1,\dots,c_m,a,b}(c_0, c_1, \dots, c_m, a, b) dc_m \cdots dc_1.$$

245 To prove the convergence $f(x, t; \tau) \xrightarrow{\tau \rightarrow 0^+} f(x, t)$ introduced in (9), hereinafter the following
 246 hypotheses will be assumed

$$\hat{\mathbf{H}}1 : \quad c_0(\omega), c_1(\omega), \dots, c_m(\omega), a(\omega), b(\omega) \text{ are independent RVs, i.e.,} \\ f_{c_0,c_1,\dots,c_m,a,b}(c_0, c_1, \dots, c_m, a, b) = f_{c_0}(c_0)f_{c_1}(c_1) \cdots f_{c_m}(c_m)f_a(a)f_b(b).$$

$$\hat{\mathbf{H}}2 : \quad f_{c_0}(c_0) \text{ is Lipschitz continuous in } \mathbb{R}, \text{ i.e.,} \\ \exists L_{f_{c_0}} : |f_{c_0}(c_{0,1}) - f_{c_0}(c_{0,2})| \leq L_{f_{c_0}}|c_{0,1} - c_{0,2}|, \quad \forall c_{0,1}, c_{0,2} \in \mathbb{R}.$$

247 Let $\tau \in]0, \tau^*[$, where τ^* will be specified later and $(x, t) \in \mathcal{D}(x_\tau(t; \omega)) \cap \mathcal{D}(x(t; \omega)) \times [(n -$
 248 $1)\tau, n\tau[\subset \mathbb{R} \times [0, T]$ fixed, then taking into account (22) and (24)

$$\begin{aligned} |f(x, t; \tau) - f(x, t)| &= \left| \int_{\mathbb{R}^{m+2}} f_{c_0,c_1,\dots,c_m,a,b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)}, c_1, \dots, c_m, a, b \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_m \cdots dc_1 \right. \\ &\quad \left. - \int_{\mathbb{R}^2} f_{c_0,a,b} \left(x e^{-(a+b)t}, a, b \right) e^{-(a+b)t} db da \right| \\ &\stackrel{\text{(VII)}}{\leq} \int_{\mathbb{R}^{m+2}} \left| f_{c_0,c_1,\dots,c_m,a,b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)}, c_1, \dots, c_m, a, b \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} \right. \\ &\quad \left. - f_{c_0,c_1,\dots,c_m,a,b} \left(x e^{-(a+b)t}, c_1, \dots, c_m, a, b \right) e^{-(a+b)t} \right| db da dc_m \cdots dc_1 \\ &\stackrel{\hat{\mathbf{H}}1}{=} \int_{\mathbb{R}^{m+2}} \left| f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} \right. \\ &\quad \left. - f_{c_0} \left(x e^{-(a+b)t} \right) e^{-(a+b)t} \right| f_{c_1}(c_1) \cdots f_{c_m}(c_m) f_a(a) f_b(b) db da dc_m \cdots dc_1 \\ &\leq \int_{\mathbb{R}^{m+2}} \left(\left| f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} - f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)} \right) e^{-(a+b)t} \right| \right. \\ &\quad \left. + \left| f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b)c_j}{\varphi_\tau^0(t, a, b)} \right) e^{-(a+b)t} - f_{c_0} \left(x e^{-(a+b)t} \right) e^{-(a+b)t} \right| \right) \\ &\quad \times f_{c_1}(c_1) \cdots f_{c_m}(c_m) f_a(a) f_b(b) db da dc_m \cdots dc_1 \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^{m+2}} \left(\underbrace{f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right)}_{(A)} \underbrace{\left| \frac{1}{|\varphi_\tau^0(t, a, b)|} - e^{-(a+b)t} \right|}_{(B)} \right. \\
&\quad \left. + \underbrace{\left| f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) - f_{c_0} (x e^{-(a+b)t}) \right|}_{(C)} \underbrace{e^{-(a+b)t}}_{(D)} \right) \\
&\quad \times f_{c_1}(c_1) \cdots f_{c_m}(c_m) f_a(a) f_b(b) db da dc_m \cdots dc_1 \\
&\stackrel{(VIII)}{<} \int_{\mathbb{R}^{m+2}} \left[\left(L_{f_{c_0}} (\epsilon_0 + e^{|a|T} e^{|b|T}) \left(|x| + \sum_{j=1}^m \epsilon_j |c_j| \right) + F_0 \right) \epsilon_0 \right. \\
&\quad \left. + L_{f_{c_0}} \left(|x| \epsilon_0 + (\epsilon_0 + e^{|a|T} e^{|b|T}) \sum_{j=1}^m \epsilon_j |c_j| \right) e^{|a|T} e^{|b|T} \right) \\
&\quad \times f_{c_1}(c_1) \cdots f_{c_m}(c_m) f_a(a) f_b(b) db da dc_1 \cdots dc_m \\
&= \epsilon_0 \mathbb{E} \left[L_{f_{c_0}} (\epsilon_0 + e^{|a(\omega)|T} e^{|b(\omega)|T}) \left(|x| + \sum_{j=1}^m \epsilon_j |c_j(\omega)| \right) + F_0 \right] \\
&\quad + L_{f_{c_0}} \mathbb{E} \left[\left(|x| \epsilon_0 + (\epsilon_0 + e^{|a(\omega)|T} e^{|b(\omega)|T}) \sum_{j=1}^m \epsilon_j |c_j(\omega)| \right) e^{|a(\omega)|T} e^{|b(\omega)|T} \right] \\
&= \epsilon_0 \mathbb{E} \left[\left(L_{f_{c_0}} (\epsilon_0 + e^{|a(\omega)|T} e^{|b(\omega)|T}) \left(|x| + \sum_{j=1}^m \epsilon_j |c_j(\omega)| \right) + F_0 \right) \right] \\
&\quad + L_{f_{c_0}} \mathbb{E} \left[\left(|x| \epsilon_0 e^{|a(\omega)|T} e^{|b(\omega)|T} + (\epsilon_0 e^{|a(\omega)|T} e^{|b(\omega)|T} + e^{2|a(\omega)|T} e^{2|b(\omega)|T}) \sum_{j=1}^m \epsilon_j |c_j(\omega)| \right) \right] \\
&= \epsilon_0 \left(L_{f_{c_0}} (\epsilon_0 + \mathbb{E} [e^{|a(\omega)|T}] \mathbb{E} [e^{|b(\omega)|T}]) \left(|x| + \sum_{j=1}^m \epsilon_j \mathbb{E} [|c_j(\omega)|] \right) + F_0 \right) \\
&\quad + L_{f_{c_0}} (|x| \epsilon_0 \mathbb{E} [e^{|a(\omega)|T}] \mathbb{E} [e^{|b(\omega)|T}] + (\epsilon_0 \mathbb{E} [e^{|a(\omega)|T}] \mathbb{E} [e^{|b(\omega)|T}] \\
&\quad + \mathbb{E} [e^{2|a(\omega)|T}] \mathbb{E} [e^{2|b(\omega)|T}]) \sum_{j=1}^m \epsilon_j \mathbb{E} [|c_j(\omega)|]).
\end{aligned}$$

250 Let us justify the steps made throughout the previous expression.

251

252 **Step (VII):** Let $N > M$, if the joint PDF, $g_{\mathbf{x}_N}(x_1, \dots, x_N)$, of a random vector, say,

$$\mathbf{x}_N(\omega) = (x_1(\omega), \dots, x_M(\omega), x_{M+1}(\omega), \dots, x_N(\omega))$$

253 is marginalized with respect to RVs $x_{M+1}(\omega), \dots, x_N(\omega)$, the joint PDF of the random vector

254 $\mathbf{x}_M(\omega) = (x_1(\omega), \dots, x_M(\omega))$ is obtained via

$$g_{\mathbf{x}_M}(x_1, \dots, x_M) = \int_{\mathbb{R}^{N-M}} g_{\mathbf{x}_N}(x_1, \dots, x_M, x_{M+1}, \dots, x_N) dx_N \cdots dx_{M+1}.$$

255 Using the notation of previous development

$$f_{c_0, a, b}(c_0, a, b) = \int_{\mathbb{R}^m} f_{c_0, c_1, \dots, c_m, a, b}(c_0, c_1, \dots, c_m, a, b) dc_m \cdots dc_1.$$

256 Therefore, substituting this expression in the left-hand side of (VII), this term can be expressed

257 as

$$\begin{aligned} & \left| \int_{\mathbb{R}^{m+2}} f_{c_0, c_1, \dots, c_m, a, b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_m \cdots dc_1 \right. \\ & \quad \left. - \int_{\mathbb{R}^2} f_{c_0, a, b} \left(x e^{-(a+b)t}, a, b \right) e^{-(a+b)t} db da \right| \\ &= \left| \int_{\mathbb{R}^{m+2}} f_{c_0, c_1, \dots, c_m, a, b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_m \cdots dc_1 \right. \\ & \quad \left. - \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^m} f_{c_0, c_1, \dots, c_m, a, b} \left(x e^{-(a+b)t}, c_1, \dots, c_m, a, b \right) dc_1 \cdots dc_m \right) e^{-(a+b)t} db da \right| \\ &= \left| \int_{\mathbb{R}^{m+2}} f_{c_0, c_1, \dots, c_m, a, b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_m \cdots dc_1 \right. \\ & \quad \left. - \int_{\mathbb{R}^{m+2}} f_{c_0, c_1, \dots, c_m, a, b} \left(x e^{-(a+b)t}, c_1, \dots, c_m, a, b \right) e^{-(a+b)t} db da dc_1 \cdots dc_m \right| \\ &= \left| \int_{\mathbb{R}^{m+2}} \left(f_{c_0, c_1, \dots, c_m, a, b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} \right. \right. \\ & \quad \left. \left. - f_{c_0, c_1, \dots, c_m, a, b} \left(x e^{-(a+b)t}, c_1, \dots, c_m, a, b \right) e^{-(a+b)t} \right) db da dc_m \cdots dc_1 \right| \\ &\leq \int_{\mathbb{R}^{m+2}} \left| f_{c_0, c_1, \dots, c_m, a, b} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) \frac{1}{|\varphi_\tau^0(t, a, b)|} \right. \\ & \quad \left. - f_{c_0, c_1, \dots, c_m, a, b} \left(x e^{-(a+b)t}, c_1, \dots, c_m, a, b \right) e^{-(a+b)t} \right| db da dc_m \cdots dc_1, \end{aligned}$$

258 which is just the right-hand side of (VII).

259 **Step (VIII):** In this step we will prove that (A) and (D) are bounded, and (B) and (C) tend to zero
 260 as $\tau \rightarrow 0^+$.

261 • Let us see that expression (B) tends to zero as $\tau \rightarrow 0^+$.

262 According to (20), for each $\omega \in \Omega$, given $a(\omega) = a$ and $b(\omega) = b$, it is known that
 263 $\varphi_\tau^0(t, a, b) = e^{a(t+\tau)} e_\tau^{b_1, t} - a \int_{-\tau}^0 e^{a(t-s)} e_\tau^{b_1, t-\tau-s} ds$, then by the MVT and the convergence of
 264 the delayed exponential function to the exponential one, Theorem 3, we have

$$\lim_{\tau \rightarrow 0^+} \varphi_\tau^0(t, a, b) = e^{(a+b)t}.$$

265 By continuity, it is clear that $1/\varphi_\tau^0(t, a, b) \xrightarrow{\tau \rightarrow 0^+} 1/e^{(a+b)t}$, then for all $\epsilon_0 > 0$ there exists τ_1
 266 such that for every $\tau \in]0, \tau_1[\cap]0, \tau_0[$ (being τ_0 defined in Theorem 3),

$$\left| \frac{1}{\varphi_\tau^0(t, a, b)} - \frac{1}{e^{(a+b)t}} \right| < \epsilon_0. \quad (25)$$

267 • Let us see that expression (A) is bounded.

268 Let $F_0 = f_{c_0}(0) > 0$, then by the Lipschitz condition, hypothesis $\hat{\mathbf{H}}1$,

$$\begin{aligned} f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) &\leq L_{f_{c_0}} \frac{1}{|\varphi_0(t, a, b; \tau)|} \left[|x| + \sum_{j=1}^m |\varphi_\tau^j(t, a, b)| |c_j| \right] + F_0 \\ &\leq L_{f_{c_0}} (\epsilon_0 + e^{|a|T} e^{|b|T}) \left(|x| + \sum_{j=1}^m \epsilon_j |c_j| \right) + F_0, \end{aligned}$$

269 where last inequality is justified by formula (25),

$$\left| \frac{1}{\varphi_\tau^0(t, a, b)} \right| \leq \left| \frac{1}{\varphi_\tau^0(t, a, b)} - \frac{1}{e^{(a+b)t}} \right| + \frac{1}{e^{(a+b)t}} < \epsilon_0 + e^{-(a+b)t} \leq \epsilon_0 + e^{|a|T} e^{|b|T}, \quad t \in [0, T].$$

270 On the other hand, by (21), $\varphi_\tau^j(t, a, b) = e^{a(t+\tau)} e_\tau^{b_1, t} (-\tau)^j + \int_{-\tau}^0 e^{a(t-s)} e_\tau^{b_1, t-\tau-s} (j-as) s^{j-1} ds$,
 271 $1 \leq j \leq m$, then by the MVT, the convergence of the delayed exponential function to the
 272 exponential function and, Theorem 3, we have

$$\lim_{\tau \rightarrow 0^+} \varphi_\tau^j(t, a, b) = 0, \quad t \in [0, T].$$

273 Thus, for each $j = 1, \dots, m$ and $\epsilon_j > 0$ arbitrary, there exists $\tau_{j+1} > 0$ such that for every
 274 $\tau \in]0, \tau_{j+1}[\cap]0, \tau_0[$

$$\left| \varphi_\tau^j(t, a, b) \right| < \epsilon_j. \quad (26)$$

275 • Let us see that expression (C) tends to zero as $\tau \rightarrow 0^+$.

276

By the Lipschitz condition, hypothesis $\hat{\mathbf{H}}1$, and expressions (25)–(26), one derives

$$\begin{aligned} & \left| f_{c_0} \left(\frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} \right) - f_{c_0} \left(x e^{-(a+b)t} \right) \right| \leq L_{f_{c_0}} \left| \frac{x - \sum_{j=1}^m \varphi_\tau^j(t, a, b) c_j}{\varphi_\tau^0(t, a, b)} - x e^{-(a+b)t} \right| \\ & \leq L_{f_{c_0}} \left(|x| \left| \frac{1}{\varphi_\tau^0(t, a, b)} - e^{-(a+b)t} \right| + \frac{1}{|\varphi_\tau^0(t, a, b; \tau)|} \sum_{j=1}^m |\varphi_\tau^j(t, a, b)| |c_j| \right) \\ & < L_{f_{c_0}} \left(|x| \epsilon_0 + \left(\epsilon_0 + e^{|a|T} e^{|b|T} \right) \sum_{j=1}^m \epsilon_j |c_j| \right), \end{aligned}$$

277

with $\tau \in \cap_{j=0}^{m+1}]0, \tau_j[$.

278

- Let us see that expression (D) is bounded. Indeed, it is clear that

$$e^{-(a+b)t} \leq e^{|a|T} e^{|b|T}, \quad t \in [0, T].$$

279

Then, the right hand-side of the inequality of Step (VIII) is obtained.

280

Finally, we use the definition of the expectation and the independence between the RVs $a(\omega)$, $b(\omega)$ and $c_j(\omega)$, $j = 1, 2, \dots, m$, obtaining the last expression.

281

Now, we assume the following hypothesis in order to prove the convergence

$$\hat{\mathbf{H}}3 : \mathbb{E} \left[\left(e^{a(\omega)T} \right)^2 \right] = k_a < +\infty, \quad \mathbb{E} \left[\left(e^{b(\omega)T} \right)^2 \right] = k_b < +\infty, \quad \mathbb{E} [|c_j|] = k_j < +\infty.$$

283

Notice that if $\mathbb{E} [(y(\omega))^2] = k_y < \infty$, then by the Cauchy-Schwarz inequality $\mathbb{E} [y(\omega)] \leq \mathbb{E} [(y(\omega))^2]^{1/2} = k_y^{1/2} < \infty$.

284

285

286

Therefore, for $\tau \in]0, \tau^*[$, $\tau^* = \min\{\tau_j : 0 \leq j \leq m\}$, $(x, t) \in \mathcal{D}(x_\tau(t; \omega)) \cap \mathcal{D}(x(t; \omega)) \times [(n-1)\tau, n\tau[\subset \mathbb{R} \times [0, T]$ all of them being fixed

$$\begin{aligned} |f(x, t; \tau) - f(x, t)| & \leq \epsilon_0 \left(L_{f_{c_0}} \left(\epsilon_0 + \mathbb{E} [e^{a(\omega)T}] \mathbb{E} [e^{b(\omega)T}] \right) \left(|x| + \sum_{j=1}^m \epsilon_j \mathbb{E} [|c_j(\omega)|] \right) + F_0 \right) \\ & \quad + L_{f_{c_0}} |x| \epsilon_0 \mathbb{E} [e^{a(\omega)T}] \mathbb{E} [e^{b(\omega)T}] \\ & \quad + L_{f_{c_0}} \left(\epsilon_0 \mathbb{E} [e^{a(\omega)T}] \mathbb{E} [e^{b(\omega)T}] + \mathbb{E} [e^{2|a(\omega)T}] \mathbb{E} [e^{2|b(\omega)T}] \right) \sum_{j=1}^m \epsilon_j \mathbb{E} [|c_j(\omega)|] \\ & \leq \epsilon_0 \left(L_{f_{c_0}} \left(\epsilon_0 + k_a^{1/2} k_b^{1/2} \right) \left(|x| + \sum_{j=1}^m \epsilon_j k_j \right) + F_0 \right) \\ & \quad + L_{f_{c_0}} \left(|x| \epsilon_0 k_a^{1/2} k_b^{1/2} + \left(\epsilon_0 k_a^{1/2} k_b^{1/2} + k_a k_b \right) \sum_{j=1}^m \epsilon_j k_j \right). \end{aligned}$$

287 Summarizing, the following result has been established.

288 **Theorem 5.** Consider the random discrete delay differential equation (4) with $g(t; \omega)$ given
 289 by (18) and whose solution SP, $x_\tau(t; \omega)$, is given by (19)–(21). Let us assume that $\mathbf{w}(\omega) =$
 290 $(c_0(\omega), c_1(\omega), \dots, c_m(\omega), a(\omega), b(\omega))$, is an absolutely continuous random vector being $f_{\mathbf{w}}(\mathbf{w})$
 291 their joint PDF. Then, the 1-PDF, $f(x, t; \tau)$, of the solution SP $x_\tau(t; \omega)$ is given by (22). Fur-
 292 ther assume that hypotheses $\hat{\mathbf{H}}1$ – $\hat{\mathbf{H}}3$ hold, then $f(x, t; \tau)$ converges to the 1-PDF, $f(x, t)$, of the
 293 solution SP to the random linear differential equation (23), according to (9).

294 4. Numerical examples

295 This section is devoted to illustrate our theoretical findings by means of several numerical
 296 experiments where the 1-PDF of the solution SP to random IVP (4) is computed in the two cases
 297 previously investigated. For the sake the clarity in the presentation, the explicit expressions
 298 of the 1-PDFs, in each one of the numerical examples, are reported in Appendix B since their
 299 mathematical representation are somewhat cumbersome. It is important to point out that we have
 300 chosen a wide variety of probability distributions for the input parameters $a(\omega)$, $b(\omega)$ and $c(\omega)$.

301 **Example 1.** Let us consider random IVP (4) whose initial condition has the form $g(t; \omega) =$
 302 $e^{a(\omega)t+c(\omega)}$, which corresponds to Case I. We will assume that $a(\omega)$, $b(\omega)$ and $c(\omega)$ are independent
 303 continuous absolutely RVs (hence hypothesis $\mathbf{H}1$ is fulfilled) with the following distributions:

- 304 • $a(\omega)$ is a Gaussian RV with zero mean and standard deviation 0.1, i.e., $a(\omega) \sim N(0; 0.1)$.
- 305 • $b(\omega)$ is a Beta RV with parameters 2 and 3, i.e., $b(\omega) \sim Be(2; 3)$.
- 306 • $c(\omega)$ is an Exponential RV with mean 1/20, i.e., $c(\omega) \sim Exp(20)$.

307 Since the first derivative of the PDF of RV $c(\omega)$ is bounded, hypothesis $\mathbf{H}2$ also holds. Therefore,
 308 assumptions of Theorem 4 are fulfilled. Now, we check numerically that the 1-PDF, $f(x, t; \tau)$,
 309 of the solution SP of random IVP (4) with $g(t; \omega) = e^{a(\omega)t+c(\omega)}$ converges to the 1-PDF, $f(x, t)$,
 310 of the solution SP of random IVP (8). To this end, in Figure 1 we have plotted $f(x, t)$ together
 311 with $f(x, t; \tau)$ with different delays $\tau \in \{0.01, 0.05, 0.1, 0.5, 2\}$ at different time instants $t = 0.1$,
 312 $t = 0.5$ and $t = 1$. From this graphical representation we can observe that $f(x, t; \tau)$ converges
 313 rapidly to $f(x, t)$ as $\tau \rightarrow 0^+$. To numerically assess this convergence, in Table 1 we show the
 314 error between $f(x, t)$ and $f(x, t; \tau)$ for the values of the delays and the time instants previously
 315 indicated, according to the following error measure

$$e_\tau^{PDF}(t) = \int_{\mathbb{R}} |f(x, t; \tau) - f(x, t)| dx. \quad (27)$$

316 We observe that for t fixed, the error $e_\tau^{PDF}(t)$ decreases as $\tau \rightarrow 0^+$, as expected. We also observe
 317 that the velocity of the convergence decreases as t increases.

318 **Example 2.** In this second example we consider that the initial condition in random IVP (4) is
 319 a polynomial of degree m , $g(t; \omega) = \sum_{j=0}^m c_j(\omega)t^j$, $m \geq 0$ which corresponds with Case II studied
 320 before. We will consider two problems, when the initial condition is a constant RV (i.e., random
 321 polynomial of degree $m = 0$) and when is a random polynomial of degree $m = 1$.

322 Problem 1: $g(t; \omega) = c_0(\omega)$.

323
 324 Let $T = 0.5$ and assume that $a(\omega)$, $b(\omega)$ and $c_0(\omega)$ are independent RVs (hence hypothesis
 325 $\hat{\mathbf{H}}1$ is fulfilled) with the following distributions:

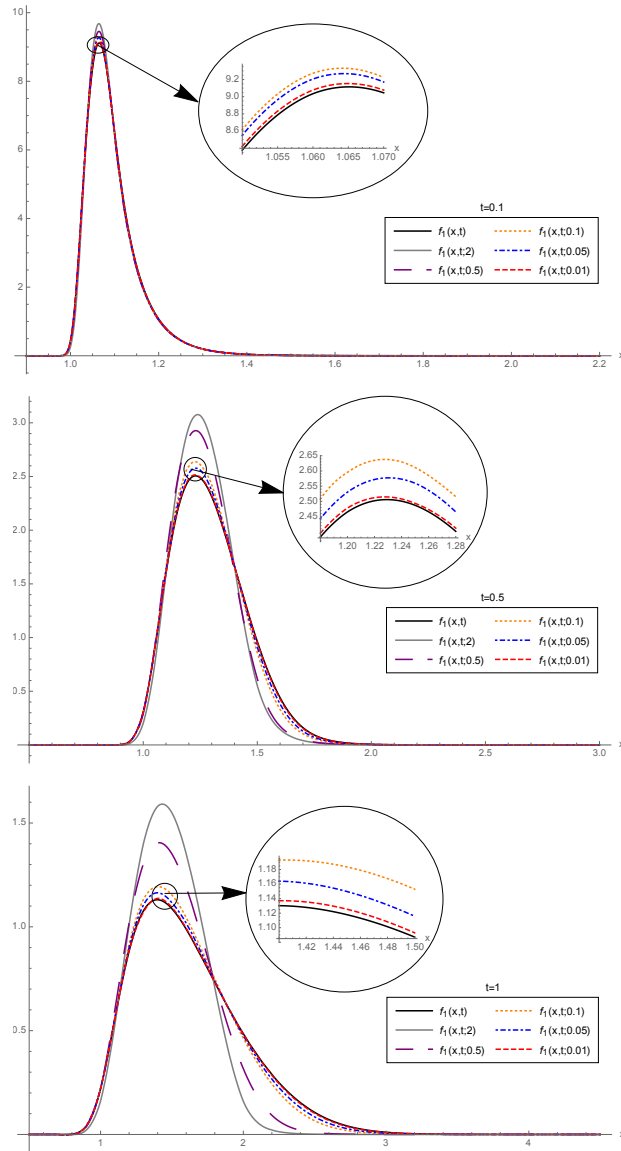


Figure 1: Graphical representation of the PDFs, $f(x, t)$ and $f(x, t; \tau)$, with different delays $\tau \in \{0.01, 0.05, 0.1, 0.5, 2\}$, at different time instants: $t = 0.1$ (top), $t = 0.5$ (center) and $t = 1$ (bottom) in the context of Example 1.

$e_{\tau}^{\text{PDF}}(t)$	$\tau = 2$	$\tau = 0.5$	$\tau = 0.1$	$\tau = 0.05$	$\tau = 0.01$
$t = 0.10$	0.040793	0.027599	0.021251	0.015753	0.003949
$t = 0.50$	0.215938	0.172538	0.061891	0.032553	0.006153
$t = 1.00$	0.382194	0.252025	0.069224	0.035766	0.007786

Table 1: Error measure $e_{\tau}^{\text{PDF}}(t)$, defined by (27), with different delays $\tau \in \{0.01, 0.05, 0.1, 0.5, 2\}$, at different time instants, $t \in \{0.1, 0.5, 1\}$, in the context of Example 1.

- 326 • $a(\omega)$ follows a truncated Gaussian distribution on the interval $I = [-1, 1]$ with zero mean
327 and standard deviation 0.1, i.e., $a(\omega) \sim N_I(0; 0.1)$.
- 328 • $b(\omega)$ is an Exponential RV with mean 1/50, i.e., $b(\omega) \sim \text{Exp}(50)$.
- 329 • $c_0(\omega)$ is a Beta RV with parameters 2 and 3, i.e., $c_0(\omega) \sim \text{Be}(2; 3)$.

330 Since $f'_{c_0}(c_0)$ is bounded, hypothesis $\hat{\mathbf{H}}2$ also holds. Finally, we compute the three values
331 given in hypothesis $\hat{\mathbf{H}}3$ with $T = 0.5$, obtaining

$$k_a = 1.08507, \quad k_b = 1.02041, \quad k_0 = 0.4,$$

332 which are all finite. Therefore, assumptions of Theorem 5 hold. Now, as in the previous example,
333 in order to see the convergence of $f(x, t; \tau)$ to $f(x, t)$, in Figure 2 we have plotted $f(x, t)$ together
334 with $f(x, t; \tau)$ with different delays $\tau \in \{0.1, 0.5, 1, 2, 3\}$ at the time instants $t = 0.1$ and $t = 0.5$.
335 In addition, in Table 2 we have calculated the error given in formula (27). From both we can see
336 graphical and *numerical* the convergence as $\tau \rightarrow 0^+$.

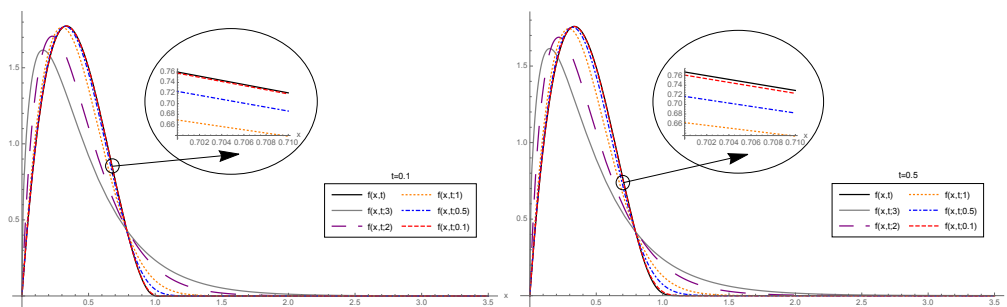


Figure 2: Graphical representation of the PDFs $f(x, t)$ and $f(x, t; \tau)$, with different delays $\tau \in \{0.1, 0.5, 1, 2, 3\}$, at different time instants $t = 0.1$ (left) and $t = 0.5$ (right) in the context of Problem 1 in Example 2.

337 In addition, in Figure 3 the mean and the variance of $x(t; \omega)$ and $x_{\tau}(t; \omega)$ for different $\tau \in$
338 $\{0.1, 0.5, 1, 2, 3\}$ in the time interval $[0, 0.5]$ have been represented. To calculate the mean and
339 the variance of $x_{\tau}(t; \omega)$ we have used the expressions of $\mathbb{E}[x_{\tau}(t; \omega)]$ and $\mathbb{V}[x_{\tau}(t; \omega)]$, respectively,
340 given in (6), where $f(x, t; \tau)$ is defined by (22). In a similar way we have computed the mean and
341 the variance of $x(t; \omega)$ but using (24). We can observe the convergence as $\tau \rightarrow 0^+$. For sake of
342 clarity, the error defined by (28) has been calculated in Table 3.

$e_{\tau}^{\text{PDF}}(t)$	$\tau = 3$	$\tau = 2$	$\tau = 1$	$\tau = 0.5$	$\tau = 0.1$
$t = 0.10$	0.425626	0.261867	0.098525	0.028427	0.001875
$t = 0.50$	0.449224	0.287860	0.115015	0.043369	0.005402

Table 2: Error measure $e_{\tau}^{\text{PDF}}(t)$, defined by (27), with different delays $\tau \in \{0.1, 0.5, 1, 2, 3\}$, at different time instants, $t \in \{0.1, 0.5\}$, in the context of Problem 1 of Example 2.

$$e_{\tau}^{\mathbb{E}} = \int_0^T |\mathbb{E}[x_{\tau}(t; \omega)] - \mathbb{E}[x(t; \omega)]| dt, \quad e_{\tau}^{\mathbb{V}} = \int_0^T |\mathbb{V}[x_{\tau}(t; \omega)] - \mathbb{V}[x(t; \omega)]| dt. \quad (28)$$

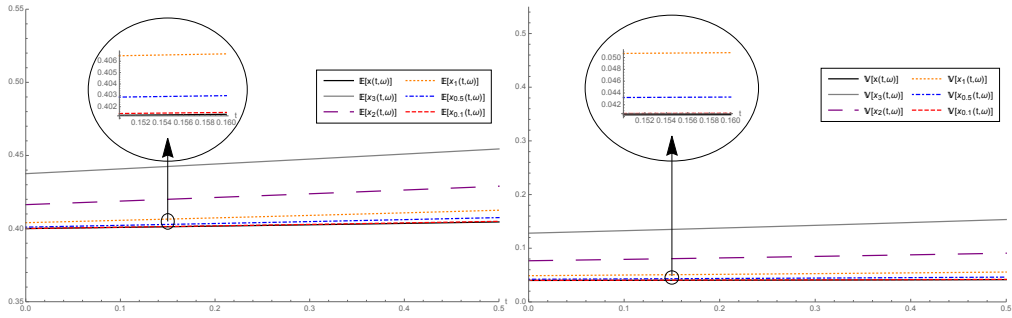


Figure 3: Graphical representation of the mean (left) and the variance (right) of the solutions SP $x_t(t; \omega)$ and $x_t(t; \omega)$, with different delays $\tau \in \{0.01, 0.05, 0.1, 0.5, 2\}$, at the time interval $[0, 1]$ in the context of Problem 1 in the Example 2.

	$\tau = 3$	$\tau = 2$	$\tau = 1$	$\tau = 0.5$	$\tau = 0.1$
$e_{\tau}^{\mathbb{E}}$	0.0218545	0.0101997	0.00301312	0.000995992	0.00011704
$e_{\tau}^{\mathbb{V}}$	0.049849	0.0214272	0.00574011	0.00172316	0.000165323

Table 3: Error measures $e_{\tau}^{\mathbb{E}}$ and $e_{\tau}^{\mathbb{V}}$, defined by (28), with different delays $\tau \in \{0.1, 0.5, 1, 2, 3\}$, in the context of Problem 1 in Example 2.

343 Problem 2: $g(t; \omega) = c_0(\omega) + c_1(\omega)t$.

344

345 We take $T = 0.5$ and we will assume that $a(\omega)$, $b(\omega)$, $c_0(\omega)$ and $c_1(\omega)$ are independent RVs
346 (hence hypothesis $\hat{\mathbf{H}}1$ is fulfilled) with the following distributions:

347

- $a(\omega)$ is a Beta RV with parameters 2 and 3, i.e., $a(\omega) \sim \text{Be}(2; 3)$.

348

- $b(\omega)$ follows a Gaussian distribution with mean -1 and standard deviation 0.2 , i.e., $b(\omega) \sim N(-1; 0.2)$.

349

350

- $c_0(\omega)$ is a Uniform RV in the interval $[-1, 1]$, i.e., $c_0(\omega) \sim U([-1, 1])$.

351 • $c_1(\omega)$ follows an Exponential distribution with mean $1/20$, i.e., $c_1(\omega) \sim \text{Exp}(20)$.

352 Since $c_0(\omega)$ follows a Uniform distribution, then its PDF is a constant and therefore the
 353 derivative of the PDF is 0, then hypothesis $\hat{\mathbf{H}}2$ also holds. Finally, we compute the three values
 354 given in hypothesis $\hat{\mathbf{H}}3$ with $T = 0.5$, obtaining

$$k_a = 1.52247, \quad k_b = 2.77319, \quad k_0 = 0.5, \quad k_1 = 0.05,$$

355 which are all finite. Therefore, assumptions of Theorem 5 are satisfied. Now, as in the previous
 356 example, in order to see the convergence of $f(x, t; \tau)$ to $f(x, t)$, in Figure 4 we have plotted $f(x, t)$
 357 together with $f(x, t; \tau)$ with different delays $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$ at time instants $t =$
 358 0.1 and $t = 0.5$. In addition, in Table 4 we have calculated the error given in formula (27). From
 359 both we can see graphical and *numerical* the convergence as $\tau \rightarrow 0^+$.

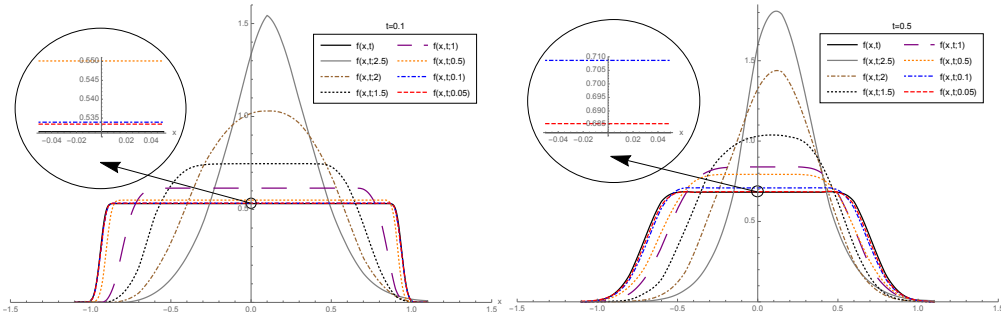


Figure 4: Graphical representation of the PDFs $f(x, t)$ and $f(x, t; \tau)$, with different delays $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$, at different time instants $t = 0.1$ (left) and $t = 0.5$ (right) in the context of Problem 2 in Example 2.

$e_{\tau}^{\text{PDF}}(t)$	$\tau = 2.5$	$\tau = 2$	$\tau = 1.5$	$\tau = 1$
$t = 0.10$	0.853492	0.666295	0.454429	0.236062
$t = 0.50$	0.808335	0.639961	0.440874	0.248945

$e_{\tau}^{\text{PDF}}(t)$	$\tau = 0.5$	$\tau = 0.1$	$\tau = 0.05$
$t = 0.10$	0.0631106	0.00712427	0.00581714
$t = 0.50$	0.182441	0.0520957	0.0178104

Table 4: Error measure $e_{\tau}^{\text{PDF}}(t)$, defined by (27), with different delays $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$, at different time instants: $t \in \{0.10, 0.50\}$, in the context of Problem 2 in Example 2.

360 In Figure 5 the mean and the variance of $f(x, t)$ and $f_1(x, t; \tau)$ for different $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$
 361 in the time interval $[0, 0.5]$ have been represented. We can observe the convergence as $\tau \rightarrow 0^+$.
 362 As in the previous Problem 1, in Table 5 we show the error for the mean and the variance defined
 363 by (28). We observe that the difference of these errors becomes smaller as τ goes to zero.

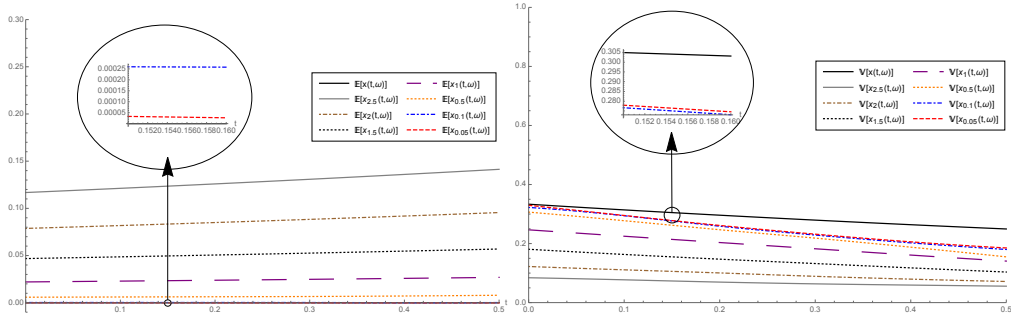


Figure 5: Graphical representation of the mean (left) and the variance (right) of the solutions SP $x(t; \omega)$ and $x_\tau(t; \omega)$, with different delays $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$, at the time interval $[0, 0.5]$ in the context of Problem 2 of Example 2.

	$\tau = 2.5$	$\tau = 2$	$\tau = 1.5$	$\tau = 1$
$e_\tau^{\mathbb{E}}$	0.0642834	0.0434024	0.0258283	0.0121818
$e_\tau^{\mathbb{V}}$	0.11049	0.0967728	0.0742367	0.0479234
	$\tau = 0.5$	$\tau = 0.1$	$\tau = 0.05$	
$e_\tau^{\mathbb{E}}$	0.0033024	0.000101439	0.000011333	
$e_\tau^{\mathbb{V}}$	0.0282562	0.0208731	0.0194024	

Table 5: Error measures $e_\tau^{\mathbb{E}}$ and $e_\tau^{\mathbb{V}}$, defined by (28), with different delays $\tau \in \{0.05, 0.1, 0.5, 1, 1.5, 2, 2.5\}$, in the context of Problem 2 in Example 2.

364 5. Conclusions

365 The main goal of the paper has been to compute the first probability density function (1-PDF)
366 of the solution stochastic process of an important class of random linear differential equations
367 with discrete delay. The study has been split into two cases depending on the functional form of
368 the initial condition. **It is important to point out that the success of our approach relies upon the**
369 **knowledge of an explicit expression of the solution. Otherwise, RVT method would need to be**
370 **combined with numerical strategies.** In the examples, we have shown how to compute the mean
371 and the variance of the solution stochastic process from the 1-PDF. It is important to point out
372 that the analysis performed in this contribution is also useful to deal with other types of initial
373 conditions different from the ones presented here. To the best of our knowledge, this is the first
374 time that this class of random differential equations with delay are studied following the proposed
375 approach. In a future contribution, we plan to go a step forward by studying **the** case which the
376 coefficients of the differential equation are stochastic processes rather than random variables.

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387 Conflict of Interest Statement

388 The authors declare that there is no conflict of interests regarding the publication of this
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391 Appendix A. Proof of Theorem 1: Existence and uniqueness of the solution

392 First, we will show how to construct the solution given in expression (2). To this end, first it
393 is convenient to construct the solution of the following IVP

$$\begin{cases} x'_\tau(t) = bx_\tau(t - \tau), & t > 0, \quad \tau > 0, \\ x_\tau(t) = 1, & -\tau \leq t \leq 0, \end{cases} \quad (\text{A.1})$$

394 which corresponds to the IVP (1) with $a = 0$ and $g(t) = 1$. We will see that its solution is the
395 delayed exponential function, $x_\tau(t) = e_\tau^{b,t}$. This fact will check in the subintervals $[-\tau, 0]$, $[0, \tau]$,
396 $[\tau, 2\tau]$, ...:

- 397 • If $t \in [-\tau, 0]$, it is clear, taken into account the definition of the delayed exponential func-
398 tion (see (3)), that $x_\tau(t) = 1 = e_\tau^{b,t}$, $-\tau \leq t \leq 0$, satisfies the IVP (A.1).

- 399 • If $t \in [0, \tau]$, we integrate the DDE in (A.1) on the interval $[0, t]$

$$\int_0^t x'_\tau(s) ds = \int_0^t b x_\tau(s - \tau) ds.$$

400 As $s \in [0, t]$ and $t \in [0, \tau]$, $s - \tau \in [-\tau, 0]$, then by the definition of the delayed exponential
401 function $x_\tau(s - \tau) = 1$. Therefore, applying the FCT

$$x_\tau(t) - x_\tau(0) = \int_0^t b ds \stackrel{x_\tau(0)=1}{\implies} x_\tau(t) = 1 + bt = e_\tau^{b,t}, \quad 0 \leq t \leq \tau.$$

- 402 • If $t \in [\tau, 2\tau]$, we integrate the DDE in (A.1) on the interval $[\tau, t]$

$$\int_\tau^t x'_\tau(s) ds = \int_\tau^t b x_\tau(s - \tau) ds.$$

403 Analogously, $s \in [\tau, t]$ and $t \in [\tau, 2\tau]$ implies $s - \tau \in [0, \tau]$, then, by the previous step,
404 $x_\tau(s - \tau) = 1 + b(s - \tau)$. Therefore, applying the FCT and integrating one gets

$$x_\tau(t) - x_\tau(\tau) = \int_\tau^t b(1 + b(s - \tau)) ds \stackrel{x_\tau(\tau)=1+b\tau}{\implies} x_\tau(t) = 1 + bt + b^2 \frac{(t - \tau)^2}{2} = e_\tau^{b,t}, \quad \tau \leq t \leq 2\tau.$$

405 Following this reasoning, it is straightforward we conclude by induction that the solution of IVP
406 (A.1) is given by $x_\tau(t) = e_\tau^{b,t}$, for each $t \in [(n - 1)\tau, n\tau]$, $n = 0, 1, 2, \dots$

407 On the other hand, suppose that the initial condition in (A.1) is a time-dependent function $x_\tau(t) =$
408 $g(t)$, with $-\tau \leq t \leq 0$. By the method of variation of constants [], the solution can be written as

$$x_\tau(t) = x_\tau^*(t)c + \int_{-\tau}^0 x_\tau^*(t - \tau - s)\phi'(s) ds, \quad (\text{A.2})$$

409 being $x_\tau^*(t) = e_\tau^{b,t}$ the solution of (A.1), c an unknown constant and $\phi(s)$ is an unknown continu-
410 ously differentiable function which must be determined. If $-\tau \leq t \leq 0$, then $x_\tau(t) = g(t)$. Thus,
411 substituting in (A.2)

$$g(t) = x_\tau^*(t)c + \int_{-\tau}^0 x_\tau^*(t - \tau - s)\phi'(s) ds, \quad -\tau \leq t \leq 0.$$

412 We split this integral into the two following integrals

$$g(t) = x_\tau^*(t)c + \int_{-\tau}^t x_\tau^*(t - \tau - s)\phi'(s) ds + \int_t^0 x_\tau^*(t - \tau - s)\phi'(s) ds.$$

413 Now, by the properties of the delayed exponential function $x_\tau^*(t) = e_\tau^{b,t}$ one gets

- 414 • $-\tau \leq t \leq 0$ implies $x_\tau^*(t) = 1$.
415 • When $s \in [-\tau, t]$, then $t - \tau - s \in [-\tau, 0]$. Therefore, $x_\tau^*(t - \tau - s) = 1$.
416 • When $s \in [t, 0]$, then $t - \tau - s \in [-2\tau, -\tau]$. Therefore, $x_\tau^*(t - \tau - s) = 0$ if $s \neq t$ and
417 $x_\tau^*(t - \tau - s) = 1$ if $s = t$.

418 Thus,

$$g(t) = c + \int_{-\tau}^t \phi'(s) ds \stackrel{\text{FCT}}{=} c + \phi(t) - \phi(-\tau).$$

419 Therefore, we can take in (A.2) $\phi(t) = g(t)$ and $c = g(-\tau)$ so that the previous equation fulfils.
420 Summarizing, the solution is given by

$$x_\tau(t) = e_\tau^{b_1, t} g(-\tau) + \int_{-\tau}^0 e_\tau^{b_1, t-\tau-s} g'(s) ds. \quad (\text{A.3})$$

421 Now, we are ready to build the solution of the IVP (1). To this end, we will follow the same
422 argument as before, so first we will construct the solution of the IVP

$$\begin{cases} x'_\tau(t) = ax_\tau(t) + bx_\tau(t-\tau), & t > 0, \quad \tau > 0, \\ x_\tau(t) = e^{at}, & -\tau \leq t \leq 0, \end{cases} \quad (\text{A.4})$$

423 where the initial condition e^{at} is chosen in order to consider the non-delayed part $ax_\tau(t)$. In this
424 case, and by the result obtained for the IVP (A.1), the solution must be of the form $x_\tau(t) =$
425 $e^{at} e_\tau^{b_1, t}$. We will determine b_1 by imposing that $x_\tau(t) = e^{at} e_\tau^{b_1, t}$ satisfies the DDE displayed in
426 IVP (A.4):

$$\begin{aligned} x'_\tau(t) = \frac{d}{dt} \left(e^{at} e_\tau^{b_1, t} \right) &= a e^{at} e_\tau^{b_1, t} + e^{at} b_1 e_\tau^{b_1, t-\tau} \\ &= ax_\tau(t) + b_1 e^{a\tau} e^{a(t-\tau)} e_\tau^{b_1, t-\tau} \\ &= ax_\tau(t) + b_1 e^{a\tau} x_\tau(t-\tau). \end{aligned} \quad (\text{A.5})$$

427 Notice that we have used the differentiating rule: $\frac{d}{dt} \left(e_\tau^{b_1, t} \right) = b_1 e_\tau^{b_1, t-\tau}$. This derivative can
428 be directly checked by computing the derivative of the delayed exponential in each subinterval
429 from its definition (see (3)). Comparing (A.5) with (A.4) is enough taking $b_1 e^{a\tau} = b$ so that
430 $x_\tau^*(t) = e^{at} e_\tau^{b_1, t}$ with $b_1 = e^{-a\tau} b$ be a solution of the DDE of (A.4). As we are interested in
431 computing the solution of the IVP (1) whose initial condition is a time-dependent function $g(t)$,
432 we will use the method of variation of constants again. So, we seek a solution of the IVP (1) in
433 the form,

$$x_\tau(t) = x_\tau^*(t)c + \int_{-\tau}^0 x_\tau^*(t-\tau-s)\phi(s) ds, \quad (\text{A.6})$$

434 where $x_\tau^*(t) = e^{at} e_\tau^{b_1, t}$ with $b_1 = e^{-a\tau} b$, c is an unknown constant and $\phi(s)$ is an unknown
435 continuously differentiable function that must be calculated. As before, we choose c and $\phi(s)$ so
436 that the initial condition is satisfied, $x_\tau(t) = g(t)$, $-\tau \leq t \leq 0$, i.e.,

$$g(t) = x_\tau^*(t)c + \int_{-\tau}^0 x_\tau^*(t-\tau-s)\phi(s) ds, \quad -\tau \leq t \leq 0. \quad (\text{A.7})$$

437 To obtain c , we set $t = -\tau$. Then, by the definition of the delayed exponential function one gets

- 438 • $x_\tau^*(-\tau) = e^{-a\tau}$.
- 439 • $s \in [-\tau, 0]$, then $t - \tau - s = -2\tau - s \in [-2\tau, -\tau]$. Then, $x_\tau^*(t - \tau - s) = 0$ if $-\tau < s \leq 0$ and
440 $x_\tau^*(t - \tau - s) = 1$ if $s = -\tau$.

441 Therefore, substituting in (A.7), $g(-\tau) = e^{-a\tau} c$ and isolating c we obtain $c = e^{a\tau} g(-\tau)$. To
442 determine the function $\phi(t)$, we split the integral in expression (A.7) into two integrals

$$g(t) = x_\tau^*(t)c + \int_{-\tau}^t x_\tau^*(t-\tau-s)\phi(s) ds + \int_t^0 x_\tau^*(t-\tau-s)\phi(s) ds. \quad (\text{A.8})$$

443 Following the same reasoning, by the definition of the delayed exponential function one deduces

444 • $x_\tau^*(t) = e^{at}$.

445 • In the first integral, $s \in [-\tau, t]$, then $t-\tau-s \in [-\tau, t] \subset [-\tau, 0]$. Then, $x_\tau^*(t-\tau-s) = e^{a(t-\tau-s)}$.

446 • In the second integral, $s \in [t, 0]$, then $t-\tau-s \in [t-\tau, -\tau] \subset [-2\tau, -\tau]$. Then, $x_\tau^*(t-\tau-s) = 0$
 447 if $t < s \leq 0$ and $x_\tau^*(t-\tau-s) = e^{-a\tau}$ if $s = t$.

448 Therefore, substituting in (A.8) and taking into account that $c = e^{a\tau} g(-\tau)$,

$$g(t) = e^{a(t+\tau)} g(-\tau) + \int_{-\tau}^t e^{a(t-\tau-s)} \phi(s) ds. \quad (\text{A.9})$$

449 Differentiating relation (A.9), by general Leibniz rule, we obtain

$$g'(t) = a e^{a(t+\tau)} g(-\tau) + a \int_{-\tau}^t e^{a(t-\tau-s)} \phi(s) ds + e^{-a\tau} \phi(t). \quad (\text{A.10})$$

450 Solving the system (A.9)–(A.10), we obtain

$$\phi(t) = e^{a\tau} (g'(t) - ag(t)).$$

451 Finally, substituting $x_\tau^*(t) = e^{at} e^{b_1 t}$ with $b_1 = e^{-a\tau} b$, $c = e^{a\tau} g(-\tau)$ and $\phi(t) = e^{a\tau} (g'(t) - ag(t))$
 452 in (A.6), we obtain the solution given in (2) to the IVP (1).

453 Uniqueness follows because the function $f(t, x, y) = ax + by$ defining the right-hand side of the
 454 DDE (1) (with $x \equiv x_\tau$ and $y \equiv x_\tau(t-\tau)$) is continuous with respect to t for every x and y , and it
 455 is locally Lipschitzian with respect to x and y because is linear [23, p.296].

456

457 Appendix B. Explicit expressions for the 1-PDFs in the Examples 1 and 2

458 **Example 1:** $a(\omega)$, $b(\omega)$ and $c(\omega)$ are independent RVs ($f_{c,a,b}(c, a, b) = f_c(c)f_a(a)f_b(b)$) with
 459 the following distributions

460 • $a(\omega) \sim N(\mu = 0; \sigma = 0.1)$

$$f_a(a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(a-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}}, \quad a \in \mathbb{R}.$$

461 • $b(\omega) \sim Be(\alpha = 2; \beta = 3)$

$$f_b(b) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} b^{\alpha-1} (1-b)^{\beta-1} = 12b(1-b)^2, \quad b \in]0, 1[, \quad (0, \text{ in otherwise}).$$

462 • $c(\omega) \sim Exp(\lambda = 20)$

$$f_c(c) = \lambda e^{-\lambda c} = 20 e^{-20c}, \quad c > 0, \quad (0, \text{ in otherwise}).$$

463 We compute the 1-PDF of the solution SP of the IVP (8), substituting the above distributions in
 464 expression (13)

$$\begin{aligned} f(x, t) &= \int_0^1 \int_{-\infty}^{\infty} f_c(\ln(x e^{-(a+b)t})) f_a(a) f_b(b) \frac{1}{|x|} da db \\ &= \int_0^1 \int_{-\infty}^{\infty} f_c(\ln(x e^{-(a+b)t})) \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}} 12b(1-b)^2 \frac{1}{|x|} da db. \end{aligned}$$

465 Since $c = c(\omega)$ has an Exponential distribution, the last integral must be calculated taking into
 466 account the condition $\ln(x e^{-(a+b)t}) > 0$. This is the reason why we have not performed the formal
 467 substitution $f_c(\ln(x e^{-(a+b)t})) = 20(x e^{-(a+b)t})^{-20}$ in the previous expression for $f(x, t)$.
 468 Now, we determine the expression of the 1-PDF of the solution SP to the IVP (4), substituting
 469 the above distributions in expression (12)

$$\begin{aligned} f(x, t; \tau) &= \int_0^1 \int_{-\infty}^{\infty} f_c\left(\ln\left(\frac{x e^{-at}}{e^{b_1 t}}\right)\right) f_a(a) f_b(b) \frac{1}{|x|} da db \\ &= \int_0^1 \int_{-\infty}^{\infty} f_c\left(\ln\left(\frac{x e^{-at}}{e^{b_1 t}}\right)\right) \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}} 12b(1-b)^2 \frac{1}{|x|} da db, \end{aligned}$$

470 where $b_1 = e^{-a\tau} b$. Again, this last double integral must be calculated taking into account the
 471 condition $\ln\left(\frac{x e^{-at}}{e^{b_1 t}}\right) > 0$.

472 **Example 2, Problem 1:** $a(\omega)$, $b(\omega)$ and $c_0(\omega)$ are independent RVs with the following distribu-
 473 tions

- 474 • $a(\omega) \sim N_I(\mu = 0; \sigma = 0.1)$, with $I = [\alpha, \beta] = [-1, 1]$.

$$f_a(a) = \frac{\phi\left(\frac{a-\mu}{\sigma}\right)}{\sigma\left(\Phi\left(\frac{\beta-\mu}{\sigma}\right) - \Phi\left(\frac{\alpha-\mu}{\sigma}\right)\right)}, \quad \text{where } \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{and } \Phi(x) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right),$$

475 being $\operatorname{erf}(x)$ the error function, defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

476 Notice that, in this case, it can be proved that $\Phi\left(\frac{\beta-\mu}{\sigma}\right) - \Phi\left(\frac{\alpha-\mu}{\sigma}\right) = 1$. Then,

$$f_a(a) = \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}}, \quad a \in [-1, 1], \quad (0, \text{ in otherwise}).$$

- 477 • $b(\omega) \sim \operatorname{Exp}(\lambda = 50)$

$$f_b(b) = 50 e^{-50b}, \quad b > 0, \quad (0, \text{ in otherwise}).$$

- 478 • $c_0(\omega) \sim \operatorname{Be}(\alpha = 2; \beta = 3)$

$$f_{c_0}(c_0) = 12c_0(1-c_0)^2, \quad c_0 \in]0, 1[, \quad (0, \text{ in otherwise}).$$

479 We compute the 1-PDF of the solution SP of the IVP (8), substituting the above distributions in
 480 expression (24):

$$\begin{aligned} f(x, t) &= \int_{-1}^1 \int_0^{+\infty} f_{c_0}(x e^{-(a+b)t}) f_a(a) f_b(b) e^{-(a+b)t} db da \\ &= \int_{-1}^1 \int_0^{+\infty} f_{c_0}(x e^{-(a+b)t}) \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}} 50 e^{-50b} e^{-(a+b)t} db da. \end{aligned}$$

481 This integral must be calculated taking into account the restriction $0 < x e^{-(a+b)t} < 1$ since $f_{c_0}(\cdot)$
 482 is the PDF of a Beta random variable.

483 Now, we determine the expression of the 1-PDF of the solution SP of the IVP (4), substituting
 484 the above distributions in expression (22)

$$\begin{aligned} f(x, t; \tau) &= \int_{-1}^1 \int_0^{+\infty} f_{c_0}\left(\frac{x}{\varphi_\tau^0(t, a, b)}\right) f_a(a) f_b(b) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da \\ &= \int_{-1}^1 \int_0^{+\infty} f_{c_0}\left(\frac{x}{\varphi_\tau^0(t, a, b)}\right) \frac{1}{\sqrt{0.02\pi}} e^{-\frac{a^2}{0.02}} 50 e^{-50b} \frac{1}{|\varphi_\tau^0(t, a, b)|} db da, \end{aligned}$$

485 where $\varphi_\tau^0(t, a, b)$ is defined by (20) being $b_1(\omega) = e^{-a(\omega)\tau} b(\omega)$, $\tau > 0$. As $c_0(\omega)$ has a Beta
 486 distribution, this integral must be calculated taking into account the condition $0 < x/\varphi_\tau^0(t, a, b) <$
 487 1.

488 **Example 2, Problem 2:** $a(\omega)$, $b(\omega)$, $c_0(\omega)$ and $c_1(\omega)$ are independent RVs with the following
 489 distributions

- 490 • $a(\omega) \sim Be(\alpha = 2; \beta = 3)$

$$f_a(a) = 12a(1-a)^2, \quad a \in]0, 1[, \quad (0, \text{ in otherwise}).$$

- 491 • $b(\omega) \sim N(\mu = -1; \sigma = 0.2)$

$$f_b(b) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(b-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{0.08\pi}} e^{-\frac{(b+1)^2}{0.08}}, \quad b \in \mathbb{R}.$$

- 492 • $c_0(\omega) \sim U([u_1 = -1, u_2 = 1])$

$$f_{c_0}(c_0) = \frac{1}{u_2 - u_1} = \frac{1}{2}, \quad c_0 \in]-1, 1[, \quad (0, \text{ in otherwise}).$$

- 493 • $c_1(\omega) \sim Exp(\lambda = 20)$

$$f_{c_1}(c_1) = 20 e^{-20c_1}, \quad c_1 > 0, \quad (0, \text{ in otherwise}).$$

494 We compute the 1-PDF of the solution SP of the IVP (8), substituting the above distributions in
 495 expression (24)

$$\begin{aligned} f(x, t) &= \int_0^1 \int_{-\infty}^{+\infty} f_{c_0}(x e^{-(a+b)t}) f_a(a) f_b(b) e^{-(a+b)t} db da \\ &= \int_0^1 \int_{-\infty}^{+\infty} f_{c_0}(x e^{-(a+b)t}) 12a(1-a)^2 \frac{1}{\sqrt{0.08\pi}} e^{-\frac{(b+1)^2}{0.08}} e^{-(a+b)t} db da, \end{aligned}$$

496 where this last double integral must be calculated taking into account the condition $-1 < x e^{-(a+b)\tau} <$
 497 1 since $c_0(\omega) \sim U([-1, 1])$.
 498 Now, we determine the expression of the 1-PDF of the solution SP of the IVP (4), substituting
 499 the above distributions in expression (22)

$$\begin{aligned}
 f(x, t; \tau) &= \int_0^{+\infty} \int_0^1 \int_{-\infty}^{+\infty} f_{c_0} \left(\frac{x - \varphi_\tau^1(t, a, b)c_1}{\varphi_\tau^0(t, a, b)} \right) f_{c_1}(c_1) f_a(a) f_b(b) \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_1 \\
 &= \int_0^{+\infty} \int_0^1 \int_{-\infty}^{+\infty} f_{c_0} \left(\frac{x - \varphi_\tau^1(t, a, b)c_1}{\varphi_\tau^0(t, a, b)} \right) 20 e^{-20c_1} 12a(1-a)^2 \frac{1}{\sqrt{0.08\pi}} e^{-\frac{(b+1)^2}{0.08}} \frac{1}{|\varphi_\tau^0(t, a, b)|} db da dc_1 \\
 &= \int_0^{+\infty} \int_0^1 \int_{-\infty}^{+\infty} f_{c_0} \left(\frac{x - \varphi_\tau^1(t, a, b)c_1}{\varphi_\tau^0(t, a, b)} \right) \frac{240}{\sqrt{0.08\pi}} e^{-\left(\frac{(b+1)^2}{0.08} + 20c_1\right)} \frac{a(1-a)^2}{|\varphi_\tau^0(t, a, b)|} db da dc_1,
 \end{aligned}$$

500 where $\varphi_\tau^0(t, a, b)$ and $\varphi_\tau^1(t, a, b)$ are defined by (20) and (21) (for $j = 1$), being $b_1(\omega) = e^{-a(\omega)\tau} b(\omega)$,
 501 $\tau > 0$. The last triple integral must be calculated taking into account the following condition
 502 $-1 < (x - \varphi_\tau^1(t, a, b)c_1)/\varphi_\tau^0(t, a, b) < 1$ since $c_0(\omega) \sim U([-1, 1])$.

503 **Remark:** Finally, we point out that the integrals defining $f(x, t)$ and $f(x, t; \tau)$, in Example 1 and
 504 in Example 2 (Problems 1 and 2), have been computed numerically by Mathematica[®] package.

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