SUBSPACES OF FREQUENTLY HYPERCYCLIC FUNCTIONS FOR SEQUENCES OF COMPOSITION OPERATORS

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ABSTRACT. In this paper, a criterion for a sequence of composition operators defined on the space of holomorphic functions in a complex domain to be frequently hypercyclic is provided. Such a criterion improves some already known special cases and, in addition, it is also valid to provide dense vector subspaces as well as large closed ones consisting entirely, except for zero, of functions that are frequently hypercyclic.

1. INTRODUCTION, NOTATION AND BACKGROUND

The phenomenon of hypercyclicity has become a trend in the last three decades. Roughly speaking, it means density of some orbit of a vector under the action of an operator or a sequence of operators. When this density is quantified in some optimal sense, the property of frequent hypercyclicity – a concept coined by Bayart and Grivaux [4] – arises naturally. In this paper we are concerned with the study of frequent hypercyclicity of sequences of composition operators acting on holomorphic functions defined in a domain of the complex plane. But prior to going on, let us fix some notation and definitions, mostly standard.

By \mathbb{D} , $\overline{\mathbb{D}}$ and \mathbb{T} we denote, respectively, the open unit disc $\{|z| < 1\}$, the closed unit disc $\{|z| \le 1\}$ and the unit circle $\{|z| = 1\}$ in the complex plane \mathbb{C} . A domain is a nonempty connected open subset $G \subset \mathbb{C}$. Its one-point compactification $G \cup \{\infty\}$ will be represented by G_{∞} . A domain $G \subset \mathbb{C}$ is called *simply connected* provided that $\mathbb{C}_{\infty} \setminus G$ is connected. The symbol H(G) will stand for the space of holomorphic functions on G. It becomes an F-space –that is, a complete metrizable topological vector space– under the topology of uniform convergence on compact subsets of G.

Next, we recall the notion of hypercyclicity. For a good account of concepts, results and history concerning this topic, the reader is referred to the books [5,24]. Let X and Y be two (Hausdorff) topological vector spaces. A sequence $T_n: X \to Y \ (n \in \mathbb{N})$ of continuous linear mappings is said to be *hypercyclic* provided that there is a vector $x_0 \in X$ (called hypercyclic for $(T_n)_n$) such that the

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orbit $\{T_n x_0 : n \in \mathbb{N}\}$ is dense in Y. An operator T on X (i.e., a continuous linear selfmapping $X \to X$) is said to be *hypercyclic* if the sequence $(T^n)_n$ of its iterates is hypercyclic. The corresponding sets of hypercyclic vectors will be respectively denoted by $HC((T_n)_n)$ and HC(T). As a stronger notion of hypercyclicity, the property of frequent hypercyclicity was introduced in [4] for operators, and then extended in [18] to sequences of linear mappings. Given a subset $A \subset \mathbb{N}$, the lower density of A is defined as $\underline{dens}(A) = \liminf_{n\to\infty} \frac{\operatorname{card}(A \cap \{1,\ldots,n\})}{n}$. Assume that X and Y are topological vector spaces. Then a sequence $T_n : X \to Y$ $(n \in \mathbb{N})$ of continuous linear mappings is called *frequently hypercyclic* provided that there is a vector $x_0 \in X$ (called frequently hypercyclic for $(T_n)_n$) such that $\underline{dens}(\{n \in \mathbb{N} : T_n x_0 \in V\}) > 0$ for every nonempty open set $V \subset Y$. An operator T on X is said to be *frequently hypercyclic* if the sequence $(T^n)_n$ of its iterates is frequently hypercyclic. The corresponding sets of frequently hypercyclic vectors will be respectively denoted by $FHC((T_n)_n)$ and FHC(T). Each set HC(T), FHC(T)is dense as soon as it is nonempty.

Our main concern enters the realm of composition operators. If $G \subset \mathbb{C}$ is a domain, then every holomorphic selfmap $\varphi: G \to G$ defines a composition operator on H(G) given by $C_{\varphi}: f \in H(G) \mapsto f \circ \varphi \in H(G)$. By $H(G,G), H_{1-1}(G)$ and $\operatorname{Aut}(G)$ we denote, respectively, the family of such selfmaps, the subfamily of all univalent (i.e., one-to-one) members of H(G,G), and the subfamily of all automorphisms of G (that is, the bijective holomorphic functions $G \to G$). Hence $\operatorname{Aut}(G) \subset H_{1-1}(G) \subset H(G,G)$. According to [10], a sequence $(\varphi_n)_n \subset H(G,G)$ is said to be runaway whenever it satisfies the following property: for every compact set $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \cap \varphi_N(K) = \emptyset$. A function $\varphi \in H(G,G)$ is said to be runaway if the sequence (φ^n) of its compositional iterates is runaway.

The dynamics of a sequence $(C_{\varphi_n})_n$ has been studied by several authors, so as to obtain, among others, the following assertions, in which G is assumed to be a simply connected domain and $\varphi \in H(G, G)$:

- ([10]) Let $(\varphi_n)_n := (z \mapsto a_n z + b_n)_n \subset \operatorname{Aut}(\mathbb{C}) (a_n, b_n \in \mathbb{C}; a_n \neq 0)$ and $(\psi_n)_n := (z \mapsto k_n \frac{z-a_n}{1-a_n z})_n \subset \operatorname{Aut}(\mathbb{D}) (|a_n| < 1 = |k_n|)$. Then $(C_{\varphi_n})_n$ is hypercyclic if and only if $\limsup_{n\to\infty} \min\{|b_n|, |b_n/a_n|\} = +\infty$ (Birkhoff's theorem [17] is the special case $a_n = 0, b_n = an$, with $a \neq 0$), and $(C_{\psi_n})_n$ is hypercyclic if and only if $\limsup_{n\to\infty} |a_n| = 1$.
- The operator C_{φ} is hypercyclic if and only if $\varphi \in H_{1-1}(G)$ and has no fixed point in G ([32, 33]), and if and only if $\varphi \in H_{1-1}(G)$ and is runaway ([23]). Moreover, if $(\varphi_n)_n \subset H_{1-1}(G)$, then $(C_{\varphi_n})_n$ is hypercyclic if and only if $(\varphi_n)_n$ is runaway ([23]).
- ([3,4]) Each translation $\tau_a : f \in H(\mathbb{C}) \mapsto f(\cdot + a) \in H(\mathbb{C}) \ (a \in \mathbb{C} \setminus \{0\})$ is frequently hypercyclic. If $\varphi \in \operatorname{Aut}(\mathbb{D})$ has no fixed point in \mathbb{D} , then $C_{\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ is also frequently hypercyclic.

- ([15]) The operator C_{φ} is frequently hypercyclic if and only if it is hypercyclic, so if and only if $\varphi \in H_{1-1}(G)$ and has no fixed point in G.
- ([9]) Let $(\varphi_n)_n := (z \mapsto a_n z + b_n)_n \subset \operatorname{Aut}(\mathbb{C}) \ (a_n, b_n \in \mathbb{C}; a_n \neq 0)$. Assume that there is an unbounded nondecreasing sequence $(\omega_n)_n \subset (0, +\infty)$ satisfying $\lim_{n\to\infty} (|b_n| - \omega_n |a_n|) = +\infty$ and $|b_m - b_n| \ge \omega_{m-n}(|a_m| + |a_n|)$ for all $m, n \in \mathbb{N}$ with m > n. Then $(C_{\varphi_n})_n$ is frequently hypercyclic on $H(\mathbb{C})$. Moreover, if $(b_n) \subset \mathbb{C}$ is a sequence with $\lim_{k\to\infty} \inf_{n\in\mathbb{N}} |b_{n+k} - b_n| = +\infty$, then the sequence of translations (τ_{b_n}) is frequently hypercyclic on $H(\mathbb{C})$.

Recently, a new terminology has been coined with the aim of suggesting largeness in an algebraic sense. This terminology enters the new subject of *lineability*, for whose background we refer the reader to the survey [13] and the book [2]. Assume that X is a vector space and that $A \subset X$. The set A is said to be *lineable* in X provided that there is an infinite dimensional vector space M such that $M \subset A \cup \{0\}$. If X is, in addition, a topological vector space, then A is called: *dense-lineable* in X if there exists a dense vector subspace $M \subset X$ such that $M \subset$ $A \cup \{0\}$; maximal dense-lineable in X if, further, dim $(M) = \dim(X)$; spaceable in X if there exists a closed infinite dimensional vector subspace $M \subset X$ such that $M \subset A \cup \{0\}$.

Let X, Y be topological vector spaces, T an operator on X and $T_n: X \to Y$ $(n \in \mathbb{N})$ be a sequence of continuous linear mappings. By using the preceding language, it is known (see [14, 20, 26, 35]) that if T is hypercyclic, then HC(T) is always dense-lineable in X, even maximal dense-lineable if X is a Banach space (in both cases, the existing dense subspace M is T-invariant). Sufficient conditions for $HC((T_n)_n)$ to be lineable or dense-lineable were furnished in [7, Theorems 1-2] (for maximal dense lineability of $HC((T_n)_n)$, see [12]). In contrast to the property of dense-lineability, not every set HC(T) is spaceable if T is hypercyclic, as Montes has shown in [28]. Sufficient criteria for HC(T) or $HC((T_n)_n)$ to be spaceable can be found in [24, Chapter 10], [2, Section 4.5] and the references contained in them. In particular, if $G \subset \mathbb{C}$ is a domain that is not conformally equivalent to $\mathbb{C} \setminus \{0\}$ and $(\varphi_n)_n \subset \operatorname{Aut}(G)$ is a runaway sequence, then $HC((C_{\varphi_n})_n)$ is spaceable in H(G)[11]; consequently, $HC(C_{\varphi})$ is spaceable if $\varphi \in Aut(G)$ is runaway. Assuming that G is simply connected and $(\varphi_n)_n \subset H_{1-1}(G)$ is a runaway sequence, it can be extracted from [23, Theorem 3.2 and its proof] together with [7, Theorem 2] that $HC((C_{\varphi_n})_n)$ is dense-lineable in H(G).

Concerning frequent hypercyclicity, Bayart and Grivaux [4] proved that if Xis a separable F-space and T is a frequently hypercyclic operator on X, then FHC(T) is dense-lineable (again with T-invariance of the corresponding dense subspace). For conditions for $FHC((T_n)_n)$ to be dense-lineable, see [12]. As for the existence of large closed subspaces, in [19, Theorem 3] a sufficient criterion for the spaceability of FHC(T) is given – from which it is derived as an example that $FHC(\tau_a)$ is spaceable in $H(\mathbb{C})$ for each $a \in \mathbb{C} \setminus \{0\}$ – while Menet [27, Theorem 2.12] provided a criterion to discover spaceability of $FHC((T_n)_n)$. Bès [15] proved in 2013 the spaceability in H(G) of $FHC(C_{\varphi})$ if it is assumed that $G \subset \mathbb{C}$ is a simply connected domain, $\varphi \in H_{1-1}(G)$ and φ has no fixed point in G.

As far as we know, there is not any general result about lineability of the family of frequently hypercyclic functions with respect to *sequences* of composition operators. The aim of this paper is to contribute to fill in this gap. Specifically, we furnish sufficient conditions for a *sequence of composition operators* defined on a given planar simply connected domain to be *frequently hypercyclic* and to enjoy the property that the family of its corresponding frequently hypercyclic functions is not only nonempty but also it *contains vector spaces* that are, in several senses, large. This will be performed in Sections 2 and 3. In the final Section 4 a number of examples will be provided.

2. Frequently hypercyclic sequences of composition operators

A sequence $(K_n)_n$ of compact subsets of a domain $G \subset \mathbb{C}$ is said to be *exhaustive* if $G = \bigcup_{n=1}^{\infty} K_n$ and $K_n \subset K_{n+1}^{\circ}$ for all $n \in \mathbb{N}$, where A° denotes the interior of A. In particular, it satisfies that, given a compact set $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \subset K_N$. A compact set $K \subset \mathbb{C}$ is said to be *Mergelyan* whenever it lacks holes, that is, whenever $\mathbb{C} \setminus K$ is connected. The symbol $\mathcal{M}(G)$ will stand for the family of Mergelyan compact subsets of G. If G is simply connected, an exhaustive sequence of compact subsets of G satisfying $(K_n)_n \subset \mathcal{M}(G)$ always exists: see, e.g., [31, Chap. 13].

Given a compact subset $K \subset G$, r > 0 and a function $f \in H(G)$, by $||f||_K$ we mean the maximum of |f(z)| over K, while $B_K(f,r)$ will denote the set of all functions $h \in H(G)$ such that $||h - f||_K < r$. The sets $B_K(f,r)$ form a base for the open sets of H(G).

We first give an easy necessary condition for a sequence $(\varphi_n)_n \subset H(G,G)$ to generate a frequently hypercyclic sequence $(C_{\varphi_n})_n$ of composition operators.

Proposition 2.1. If $(C_{\varphi_n})_n$ is frequently hypercyclic on H(G), then $(\varphi_n)_n$ is weakly frequently runaway, that is, for every compact set $K \subset G$, one has

$$\underline{\operatorname{dens}}(\{n \in \mathbb{N} : K \cap \varphi_n(K) = \varnothing\}) > 0.$$

Proof. Fix any compact set $K \subset G$. Let $n \in \mathbb{N}$ be a natural number such that $K \cap \varphi_n(K) \neq \emptyset$ and pick $z_n \in K$ such that $\varphi_n(z_n) \in K$. Let $f \in FHC((C_{\varphi_n})_n)$ and define g as the constant function $g(z) := 1 + ||f||_K$. Then

$$||g - C_{\varphi_n} f||_K \ge |g(z_n) - f(\varphi_n(z_n))| \ge 1 + ||f||_K - |f(\varphi_n(z_n))| \ge 1,$$

where the facts $z_n \in K$ and $\varphi_n(z_n) \in K$ have been used. Hence

$$\{n \in \mathbb{N} : C_{\varphi_n} f \in B_K(g, 1)\} \subset \{n \in \mathbb{N} : K \cap \varphi_n(K) = \emptyset\}$$

But the frequent hypercyclicity of f implies that the smaller set has positive lower density, so the bigger one too.

The last proposition can suggest conditions guaranteeing that $FHC((C_{\varphi_n})_n) \neq \emptyset$. We restrict ourselves to the rather illustrative case of simply connected domains. Neither the results nor the approaches in [15, 19, 27] will be used in our proofs.

Prior to establish our main result in this section (Theorem 2.6, whose proof is inspired by the one of Theorem 4.1 in [9]), two auxiliary results are needed.

Lemma 2.2. Any sequence of natural numbers with positive lower density can always be split into infinitely many disjoint subsequences each of which has also positive lower density.

Proof. Let $A = \{n_1 < n_2 < \cdots < n_k < \cdots\}$ be a sequence in \mathbb{N} with $\underline{\operatorname{dens}}(A) > 0$. This means that there is C > 0 with $n_k \leq Ck$ for all $k \in \mathbb{N}$. If we define $A_1 = \{m_k\}_{k\geq 1} := \{n_1 < n_3 < n_5 < \cdots\}$, then $m_k \leq C(2k-1) \leq 2Ck$ for all $k \in \mathbb{N}$, so that $\underline{\operatorname{dens}}(A_1) > 0$. Now, we split $A \setminus A_1$ into $A_2 = \{p_k\}_{k\geq 1} := \{n_2 < n_6 < n_{10} < n_{14} < \cdots\}$ and $A \setminus (A_1 \cup A_2) = \{n_{4k}\}_{k\geq 1}$. Then $p_k \leq C(4k-2) \leq 4Ck$ for all $k \geq 1$, which yields $\underline{\operatorname{dens}}(A_2) > 0$. Then we divide $\{n_{4k}\}_{k\geq 1}$ into two sequences with positive lower density. It is clear that this procedure gives pairwise disjoint sets $A_n \subset \mathbb{N}$ with $\underline{\operatorname{dens}}(A_n) > 0$ for all $n \in \mathbb{N}$.

Our second auxiliary assertion (Theorem 2.4 below) is known as *Nersesjan's* approximation theorem and can be found in [22, Chapter 4] (see also [29]).

Definition 2.3. Let $G \subset \mathbb{C}$ be a domain. A relatively closed set $F \subset G$ is said to be a *Carleman set of* G provided that the following conditions hold:

- (i) $G_{\infty} \setminus F$ is connected,
- (ii) $G_{\infty} \setminus F$ is locally connected at ∞ ,
- (iii) F "lacks long islands", that is, for every compact set $K \subset G$ there is a neighborhood V of ∞ in G_{∞} such that no component of F° meets both K and V.

Note that (ii) in the last definition is equivalent to the following (see [22, p. 143]): for each neighborhood U of ∞ there exists a neighborhood $V \subset U$ of ∞ such that each point $z_0 \in V \setminus (F \cup \{\infty\})$ can be connected in $U \setminus F$ with a point that is arbitrarily close to ∞ .

Theorem 2.4. A relatively closed set $F \subset G$ is a Carleman set of G if and only if for every function f continuous on F and holomorphic on F° and every positive and continuous function $\varepsilon : F \to [0, +\infty)$, there is a function $g \in H(G)$ such that

$$|f(z) - g(z)| < \varepsilon(z)$$
 for all $z \in F$.

We say that a family \mathcal{F} of subsets of a given set X is *pairwise disjoint* if $A \cap B = \emptyset$ for every pair of distinct $A, B \in \mathcal{F}$. If $A \subset \mathbb{C}$ then ∂A and $\partial_{\infty} A$ will stand for the boundary of A in \mathbb{C} and in \mathbb{C}_{∞} , respectively, so that $\partial_{\infty} A$ equals either ∂A or $\{\infty\} \cup \partial A$, depending on whether A is bounded or not.

In order to formulate appropriately our criterion for frequent hypercyclicity, let us introduce the next concept.

Definition 2.5. Let $G \subset \mathbb{C}$ be a domain and $(\varphi_n) \subset H(G, G)$. We say that (φ_n) is strongly frequently runaway provided that there exists an exhaustive sequence $(K_{\nu})_{\nu} \subset \mathcal{M}(G)$ as well as a family $\{A(\nu) : \nu \in \mathbb{N}\}$ of subsets of \mathbb{N} satisfying the following conditions:

- (P1) dens $(A(\nu)) > 0$ for all $\nu \in \mathbb{N}$.
- (P2) Each of the families $\{A(\nu); \nu \in \mathbb{N}\}$ and $\mathcal{K} := \{\varphi_n(K_\nu) : n \in A(\nu), \nu \in \mathbb{N}\}$ is pairwise disjoint.
- (P3) Given a compact subset $K \subset G$ there are only finitely many $L \in \mathcal{K}$ with $K \cap L \neq \emptyset$.

Note that thanks to (P3) the notion introduced in the preceding definition is stronger than the one of a weakly frequently runaway sequence given in Proposition 2.1.

Theorem 2.6. Let $G \subset \mathbb{C}$ be a simply connected domain and $(\varphi_n)_n$ be a strongly frequently runaway sequence in $H_{1-1}(G)$. Then the sequence $(C_{\varphi_n})_n$ is frequently hypercyclic. In other words, the set $FHC((C_{\varphi_n})_n)$ is not empty.

Proof. Let $(P_l)_l$ be a dense sequence of H(G), for instance the sequence of polynomials whose coefficients have rational real and imaginary parts. Divide each set $A(\nu)$ into infinitely many disjoint subsets $A(\nu, l)$ $(l \in \mathbb{N})$, each of them with positive lower density. This is possible thanks to (P1) and Lemma 2.2.

Now, we use the fact that if K is a compact set with N holes and φ is an injective holomorphic mapping on a neighborhood of K, then $\varphi(K)$ also has N holes (see [30, p. 276]). In particular, if $K \in \mathcal{M}(G)$, then $\varphi(K)$ lack holes. Define the set

$$F := \bigcup_{\nu \in \mathbb{N}} \bigcup_{n \in A(\nu)} \varphi_n(K_{\nu}).$$

From (P2), the sets $\varphi_n(K_\nu)$ are pairwise disjoint compact subsets of G having no holes. From (P3), it follows that they escape to the boundary of G. Therefore Fis closed in G, and $G_\infty \setminus F$ is connected as well as locally connected at ∞ . The components of F° are the sets $(\varphi_n(K_\nu))^\circ$. Given a compact subset $K \subset G$ there are by (P3) only finitely many $L_1, \ldots, L_N \in \mathcal{K}$ with $K \cap L_j \neq \emptyset$ $(j = 1, \ldots, N)$. Then $V := \{\infty\} \cup (G \setminus (K \cup L_1 \cup \cdots \cup L_N))$ is a neighborhood of ∞ in G_∞ satisfying that no component of F° meets both K and V. Hence F is a Carleman set of G. Now define $g: F \to \mathbb{C}$ as follows:

$$g(z) := P_l(\varphi_n^{-1}(z))$$
 if $z \in \varphi_n(K_\nu), n \in A(\nu, l)$ and $l, \nu \in \mathbb{N}$.

It is obvious that g is continuous on F and holomorphic in F° . Hence, by Nersesjan's Theorem (Theorem 2.4), there exists $f \in H(G)$ such that $|f(z) - g(z)| < \varepsilon(z)$ for $z \in F$, where $\varepsilon : G \to [0, +\infty)$ is a function that goes to zero as z approaches the boundary of G (for instance, take $\varepsilon(z) := \chi(z, \partial_{\infty}G)$, where χ denotes the chordal distance on \mathbb{C}_{∞}).

We claim that f is frequently hypercyclic for $(C_{\varphi_n})_n$. Indeed, fix $l, \nu \in \mathbb{N}$ and $z \in K_{\nu}$. Then, for $n \in A(l, \nu)$ we have

$$|C_{\varphi_n}(f)(z) - P_l(z)| = |f(\varphi_n(z)) - g(\varphi_n(z))| < \varepsilon(\varphi_n(z)).$$
(1)

But (P3) implies that $\varphi_n(z)$ tends to the boundary as $n \to \infty$ uniformly on K_{ν} , so $\sup_{z \in K_{\nu}} \varepsilon(\varphi_n(z)) \to 0$. Hence, given $\delta > 0$, there is $n_0 = n_0(\nu) \in \mathbb{N}$ such that $\sup_{z \in K_{\nu}} \varepsilon(\varphi_n(z)) < \delta$ for all $n \ge n_0$. This, together with (1), gives us that for any $l, \nu \in \mathbb{N}, z \in K_{\nu}$ and $n \in A(l, \nu) \setminus \{1, \ldots, n_0\}$, we have

$$|C_{\varphi_n}(f)(z) - P_l(z)| < \delta.$$

Since $\underline{\operatorname{dens}}(A(l,\nu)\setminus\{1,\ldots,n_0\}) = \underline{\operatorname{dens}}(A(l,\nu)) > 0$ and the sets $B_{K_{\nu}}(P_l,\delta)$ form a base for the open sets of H(G), the frequent hypercyclicity of f is proved. \Box

The assumptions of Theorem 2.6 imply the frequent hypercylicity of the sequence of composition operators. In the next section we will establish that, under the same conditions, the set $FHC((C_{\varphi_n})_n)$ enjoys a large algebraic size.

In view of Proposition 2.1 and Theorem 2.6, we want to pose here the natural problem of characterization of the frequent hypercyclicity of the sequence (C_{φ_n}) in terms of properties of (φ_n) . Note that, contrarily to the case of the iterates of one composition operator (see [15] and Section 1), hypercyclicity and frequent hypercyclicity are not equivalent for a general sequence (C_{φ_n}) . Indeed, consider a simply connected domain $G \subset \mathbb{C}$ and take a $\varphi \in H_{1-1}(G)$ without fixed points, and then define (φ_n) as $\varphi_{2^k} = \varphi^k$ (the compositional kth-iterate of φ) for $k \in \mathbb{N}$, and $\varphi_k(z) = z$ for $k \in \mathbb{N} \setminus \{2^k : k \in \mathbb{N}\}$. It is clear that (C_{φ_n}) is hypercyclic but not frequently hypercyclic. Other –still unsolved– questions concerning the frequent hypercyclicity of sequences of composition operators on $H(\mathbb{C})$ defined by similarities were posed in [9, Remark 4.3.2].

3. Vector subspaces of frequently hypercyclic sequences of composition operators

As we have mentioned earlier, under the condition of being strongly frequently runaway we get a high degree of algebraic genericity. This will be shown in the next two theorems.

We first consider spaceability. The following technical result will be needed in the proof. Recall that two basic sequences $(x_n)_n$, $(y_n)_n$ in a Banach space $(X, \|\cdot\|)$ are said to be *equivalent* if, for every sequence $(a_n)_n$ of scalars, the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n y_n$ converges. This happens (see [6]) if and only if there exist two constants $m, M \in (0, +\infty)$ such that

$$m \left\| \sum_{j=1}^{J} a_j x_j \right\| \le \left\| \sum_{j=1}^{J} a_j y_j \right\| \le M \left\| \sum_{j=1}^{J} a_j x_j \right\|$$

for all scalars a_1, \ldots, a_J and all $J \in \mathbb{N}$. By using the first inequality, we are easily led to the next lemma, whose proof can be found in [8, Lemma 2.1]. By $L^2(\mathbb{T})$ we denote the Hilbert space of all Lebesgue-measurable functions $f : \mathbb{T} \to \mathbb{C}$ with finite quadratic norm $||f||_2 = (\int_0^{2\pi} |f(e^{i\theta})| \frac{d\theta}{2\pi})^{1/2}$. It is well known that the sequence $(z^n)_n$ is a basic sequence in $L^2(\mathbb{T})$.

Lemma 3.1. Assume that G is a domain with $\overline{\mathbb{D}} \subset G$ and that $(f_j)_j \subset H(G)$ is a sequence such that it is a basic sequence in $L^2(\mathbb{T})$ that is equivalent to $(z^j)_j$. If $\left(h_l := \sum_{j=1}^{J(l)} c_{j,l} f_j\right)_l$ is a sequence in span $\{f_j : j \ge 1\}$ converging in H(G), then we have

$$\sup_{l \in \mathbb{N}} \sum_{j=1}^{J(l)} |c_{j,l}|^2 < +\infty.$$

Theorem 3.2. Let $G \subset \mathbb{C}$ be a simply connected domain and $(\varphi_n)_n$ be a strongly frequently runaway sequence in $H_{1-1}(G)$. Then the set $FHC((C_{\varphi_n})_n)$ is spaceable in H(G).

Proof. Let $(K_{\nu})_{\nu} \subset \mathcal{M}(G)$ and $\{A(\nu) : \nu \in \mathbb{N}\}$ be, respectively, an exhaustive sequence of compact sets and a countable family of sets of positive integers given by the strongly frequent runawayness of $(\varphi_n)_n$.

Without loss of generality, we can assume that $\overline{\mathbb{D}} \subset K_1 \subset G$, because frequent hypercyclicity is stable under translations and dilations. We are going to modify the proof of Theorem 2.6 appropriately to get spaceability.

Let $(P_l)_l$ be a dense sequence of H(G) and divide each set $A(\nu)$ into infinitely many disjoint subsets $A(\nu, l, p)$ $(l, p \in \mathbb{N})$, each of them having positive lower density. Observe that condition (P3) of Definition 2.5 implies the existence of $k_1 \in \mathbb{N}$ such that $K_1 \cap \varphi_n(K_\nu) = \emptyset$ for all $\nu \ge k_1$ and all $n \in A(\nu)$. Similarly to the proof of Theorem 2.6, it follows from conditions (P2) and (P3) that the set

$$F := K_1 \cup \bigcup_{\nu \ge k_1} \bigcup_{n \in A(\nu)} \varphi_n(K_\nu)$$

is a Carleman set of G.

Now, for each $\mu \in \mathbb{N}$, let us define the function $g_{\mu}: F \to \mathbb{C}$ by

$$g_{\mu}(z) := \begin{cases} z^{\mu} & \text{if } z \in K_{1} \\ P_{l}(\varphi_{n}^{-1}(z)) & \text{if } z \in \varphi_{n}(K_{\nu}), \, \nu \geq k_{1}, \, n \in A(\nu, l, \mu), \, l \in \mathbb{N} \\ 0 & \text{if } z \in \varphi_{n}(K_{\nu}), \, \nu \geq k_{1}, \, n \in A(\nu, l, p), \, p \neq \mu, \, l \in \mathbb{N}. \end{cases}$$

It is plain that each g_{μ} is continuous on F and holomorphic in F° . Consequently, by Nersesjan's Theorem (Theorem 2.4), there exists a function $f_{\mu} \in H(G)$ such that

$$|f_{\mu}(z) - g_{\mu}(z)| < \frac{1}{3^{\mu}} \cdot \min\{1, \varepsilon(z)\} \text{ for all } z \in F,$$
(2)

where, again, $\varepsilon(z) = \chi(z, \partial_{\infty}G)$ as in the proof of Theorem 2.6.

We claim that the closed linear space generated by $(f_{\mu})_{\mu}$ in H(G), namely,

$$M := \overline{\operatorname{span}} \{ f_{\mu} : \, \mu \in \mathbb{N} \},\$$

is an infinite dimensional closed vector space consisting, except for zero, of frequently hypercyclic functions for $(C_{\varphi_n})_n$.

With this aim, observe first that, since $\overline{\mathbb{D}} \subset K_1$, we have $|f_{\mu}(z) - z^{\mu}| < \frac{1}{3^{\mu}}$ for all $z \in \mathbb{T}$. Let $(e^*_{\mu})_{\mu}$ be the sequence of coefficient functionals corresponding to the basic sequence $(z^{\mu})_{\mu}$ in $L^2(\mathbb{T})$. Since $||e^*_{\mu}||_2 = 1$ ($\mu \in \mathbb{N}$), one obtains

$$\sum_{\mu=1}^{\infty} \|e_{\mu}^{*}\|_{2} \cdot \|f_{\mu} - z^{\mu}\|_{2} < \sum_{\mu=1}^{\infty} \frac{1}{3^{\mu}} = \frac{1}{2} < 1.$$
(3)

From (3) and the basis perturbation theorem [21, p. 50] it follows that $(f_{\mu})_{\mu}$ is also a basic sequence in $L^2(\mathbb{T})$ that is equivalent to $(z^{\mu})_{\mu}$. In particular, the functions f_{μ} ($\mu \in \mathbb{N}$) are linearly independent and M is an infinite dimensional closed vector space.

Now, fix any $f \in M \setminus \{0\}$ and let $f = \sum_{\mu \in \mathbb{N}} \alpha_{\mu} f_{\mu}$ be its representation in $L^{2}(\mathbb{T})$. As $f \neq 0$ there is some nonzero coefficient α_{μ} , which without loss of generality may supposed to be α_{1} (indeed, if α_{m} , and not α_{1} , is the first nonzero coefficient, the reasonings below would involve sums $\sum_{\mu=m}^{N_{j}}, \sum_{\mu=m+1}^{N_{j}}, \sum_{\mu=m}^{\infty}$ and the inequality $|f_{m}(\varphi_{n}(z)) - P_{l}(z)| < \varepsilon(\varphi_{n}(z))$, which produce the same effects as if m = 1). Furthermore, by the invariance under scalar multiplication of frequent hypercyclicity, we can assume that $\alpha_{1} = 1$. Thanks to the definition of M, there is a sequence $\left(h_{j} := \sum_{\mu=1}^{N_{j}} \alpha_{j,\mu} f_{\mu}\right)_{j}$ converging to f in H(G). Hence

$$h_j \longrightarrow f$$
 in $L^2(\mathbb{T})$,

because the topology in H(G) is finer than the one in $L^2(\mathbb{T})$. By the continuity of each projection, we obtain that $\alpha_{j,1} \to 1$ $(j \to \infty)$, and without loss of generality we may assume that $\alpha_{j,1} = 1$ for all $j \in \mathbb{N}$ (if not, just take $H_j := h_j + (1 - \alpha_{1,j})f_1$). Finally, by Lemma 3.1, there is a constant H > 0 such that

$$\sum_{\mu=1}^{N_j} |\alpha_{j,\mu}|^2 \le H \quad \text{for all} \quad j \in \mathbb{N}.$$
(4)

Next, fix $l, \nu \in \mathbb{N}$ with $\nu \geq k_1$ and $n \in A(\nu, l, 1)$. On the one hand, the set $\varphi_n(K_\nu)$ is compact. Then, given $\delta > 0$, there exists $j = j(n) \in \mathbb{N}$ such that

$$|f(\varphi_n(z)) - h_j(\varphi_n(z))| < \frac{\delta}{2} \quad \text{for all} \quad z \in K_\nu.$$
(5)

But on the other hand, applying (2) and (4) together with the Cauchy–Schwarz inequality guarantees that, for all $z \in K_{\nu}$,

$$\begin{aligned} |h_{j}(\varphi_{n}(z)) - P_{l}(z)| &\leq |f_{1}(\varphi_{n}(z)) - P_{l}(z)| + \sum_{\mu=2}^{N_{j}} |\alpha_{j,\mu}| |f_{\mu}(\varphi_{n}(z))| \\ &\leq \varepsilon(\varphi_{n}(z)) + \left(\sum_{\mu=2}^{N_{j}} |\alpha_{j,\mu}|^{2}\right)^{1/2} \left(\sum_{\mu=2}^{N_{j}} |f_{\mu}(\varphi_{n}(z))|^{2}\right)^{1/2} \\ &< \varepsilon(\varphi_{n}(z)) + \sqrt{H} \sum_{\mu=2}^{\infty} \frac{1}{3^{\mu}} \cdot \varepsilon(\varphi_{n}(z)) \\ &< (1 + \sqrt{H}) \cdot \varepsilon(\varphi_{n}(z)). \end{aligned}$$

$$(6)$$

Given $\nu \in \mathbb{N}$, there is $n_0 = n_0(\nu) \in \mathbb{N}$ such that $\sup_{z \in K_\nu} \varepsilon(\varphi_n(z)) < \delta/(2 + 2\sqrt{H})$ for $n \ge n_0$. Finally, from (5), (6) and the triangle inequality (which, incidentally, neutralizes the effect of the dependence j = j(n)), an argument similar to that of the end of the proof of Theorem 2.6 yields frequent $(C_{\varphi_n})_n$ -hypercyclicity for f. This concludes the proof.

A slight modification of the arguments in the proof of the last result allows us to prove the maximal dense lineability of $FHC((C_{\varphi_n})_n)$ as well.

Theorem 3.3. Let $G \subset \mathbb{C}$ be a simply connected domain and $(\varphi_n)_n$ be a strongly frequently runaway sequence in $H_{1-1}(G)$. Then the set $FHC((C_{\varphi_n})_n)$ is maximal dense-lineable.

Proof. Again, let $(K_{\nu})_{\nu}$ and $\{A(\nu) : \nu \in \mathbb{N}\}$ be an exhaustive sequence of compact sets and a countable family of subsets of \mathbb{N} given by the strongly frequent runawayness of $(\varphi_n)_n$. Without loss of generality we may assume that $\overline{\mathbb{D}} \subset K_1$.

Let $(P_l)_l$ be a dense sequence in H(G) and divide each set $A(\nu)$ into infinitely many mutually disjoint subsets $A(\nu, l, p)$ $(l, p \in \mathbb{N})$, each of them with positive lower density. For $\mu \in \mathbb{N}$ fixed, by (P3) there exists $k_{\mu} \in \mathbb{N}$ such that for all $\nu \geq k_{\mu}$ and all $n \in A(\nu)$, we have $K_{\mu} \cap \varphi_n(K_{\nu}) = \emptyset$. Note that the sequence $(k_{\mu})_{\mu} \subset \mathbb{N}$ can be chosen to be strictly increasing. Similarly to the proofs of Theorems 2.6 and 3.2, one derives from (P2) and (P3) that every set

$$F_{\mu} := K_{\mu} \cup \bigcup_{\nu \ge k_{\mu}} \bigcup_{n \in A(\nu)} \varphi_n(K_{\nu})$$

is a Carleman set.

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For each $\mu \in \mathbb{N}$, let us define the function $g_{\mu}: F_{\mu+1} \to \mathbb{C}$ as follows:

$$g_{\mu}(z) := \begin{cases} P_{\mu}(z) & \text{if } z \in K_{\mu+1} \\ P_{l}(\varphi_{n}^{-1}(z)) & \text{if } z \in \varphi_{n}(K_{\nu}), \ \nu \geq k_{\mu+1}, \ n \in A(\nu, l, \mu+1), \ l \in \mathbb{N} \\ 0 & \text{if } z \in \varphi_{n}(K_{\nu}), \ \nu \geq k_{\mu+1}, \ n \in A(\nu, l, p), \ p \neq \mu+1, \ l \in \mathbb{N}. \end{cases}$$

Observe that $g_{\mu}(z) = 0$ for all $z \in \varphi_n(K_{\nu})$ where $n \in A(\nu, l, 1), \nu \geq k_{\mu+1}$ and $l \in \mathbb{N}$. We are going to use these compact sets later.

As in the proof of Theorem 3.2, each g_{μ} is continuous on $F_{\mu+1}$ and holomorphic on $F_{\mu+1}^{\circ}$. Hence, for each $\mu \in \mathbb{N}$, Theorem 2.4 guarantees the existence of a function $f_{\mu} \in H(G)$ such that

$$|f_{\mu}(z) - g_{\mu}(z)| < \frac{1}{\mu} \cdot \min\{1, \varepsilon(z)\} \text{ for all } z \in F_{\mu+1}, \tag{7}$$

where $\varepsilon(z) = \chi(z, \partial_{\infty}G)$. From (7), we get, in particular, that $|f_{\mu}(z) - P_{\mu}(z)| < \frac{1}{\mu}$ for all $z \in K_{\mu+1}$. So

$$\lim_{\mu \to \infty} \|f_{\mu} - P_{\mu}\|_{K_{\mu+1}} = 0$$

and, since the sequence $(P_{\mu})_{\mu}$ is dense in H(G) and $(K_{\mu})_{\mu}$ is an exhaustive sequence of compact sets, we derive that $(f_{\mu})_{\mu}$ is dense in H(G). Now, we define the set

$$M_d := \operatorname{span}\{f_\mu : \mu \in \mathbb{N}\},\$$

the linear span of the latter sequence. It is plain that M_d is a dense vector subspace of H(G). We claim that $M_d \setminus \{0\} \subset FHC((C_{\varphi_n})_n)$.

To this end, fix $H = \sum_{\mu=1}^{N} \lambda_{\mu} f_{\mu} \in M_d \setminus \{0\}$. Without loss of generality, we may assume that $\lambda_N = 1$ (because of the stability of frequent hypercyclicity under scaling). Fix $l, \nu \in \mathbb{N}$ with $\nu \geq k_{N+1}$, $n \in A(\nu, l, N+1)$ and $z \in K_{\nu}$, so that $\varphi_n(z) \in \varphi_n(K_{\nu})$. It follows from (7) that

$$|H(\varphi_n(z)) - P_l(z)| \leq |f_N(\varphi_n(z)) - P_l(z)| + \sum_{\mu=1}^{N-1} |\lambda_\mu| |f_\mu(\varphi_n(z))|$$

$$< \alpha \cdot \varepsilon(\varphi_n(z)),$$

where $\alpha := 1 + \sum_{\mu=1}^{N-1} |\lambda_{\mu}|$. Continuing as in the final part of the proof of Theorem 2.6 – but in a somewhat easier way – we get the frequent hypercyclicity of the function H, so proving the claim.

Observe that, from (7) and the definition of g_{μ} , we also obtain the next inequality, that will be used later:

$$|H(\varphi_n(z))| < \alpha \cdot \varepsilon(\varphi_n(z)) \text{ for all } z \in K_{\nu}, n \in A(\nu, l, 1), \nu \ge k_{l+1}, l \in \mathbb{N}.$$
(8)

Now, we go back to the sets $A(\nu, l, 1)$. For fixed $\nu, l \in \mathbb{N}$, let us divide $A(\nu, l, 1)$ into infinitely many sequences $A(\nu, l, 1, \mu)$ ($\mu \in \mathbb{N}$), each of them with positive

lower density. Let $F := F_1 = K_1 \cup \bigcup_{\nu \ge k_1} \bigcup_{n \in A(\nu)} \varphi_n(K_\nu)$ and proceed as in the proof of Theorem 3.2, but defining $h_\mu : F \to \mathbb{C}$ ($\mu \in \mathbb{N}$) in the next way:

$$h_{\mu}(z) := \begin{cases} z^{\mu} & \text{if } z \in K_1 \\ P_l(\varphi_n^{-1}(z)) & \text{if } z \in \varphi_n(K_{\nu}), \ \nu \ge k_1, \ n \in A(\nu, l, 1, \mu), \ l \in \mathbb{N} \\ 0 & \text{elsewhere.} \end{cases}$$

Observe that, for $p \ge 2$ and $l \in \mathbb{N}$, we have $h_{\mu}(z) = 0$ if $z \in \varphi_n(K_{\nu})$ with $\nu \ge k_1$ and $n \in A(\nu, l, p)$.

Next, by applying Theorem 2.4 as in the proof of Theorem 3.2, we get functions $\Phi_{\mu} \in H(G) \ (\mu \in \mathbb{N})$ satisfying

$$|\Phi_{\mu}(z) - h_{\mu}(z)| < \frac{1}{3^{\mu}} \cdot \min\{1, \varepsilon(z)\}$$
 for all $z \in F$.

Proceeding as in the last mentioned theorem – except for the fact that only the sequences $A(\nu, l, 1, \mu)$ ($\mu \in \mathbb{N}$) are handled instead of all sequences $A(\nu, l, \mu)$ – it is obtained that the functions Φ_{μ} ($\mu \in \mathbb{N}$) are linearly independent and that $M_s \setminus \{0\} \subset FHC((C_{\varphi_n})_n)$, where $M_s := \overline{\operatorname{span}}\{\Phi_{\mu} : \mu \in \mathbb{N}\}$. Since M_s is a closed infinite dimensional vector subspace of the separable F-space H(G), an application of Baire's category theorem yields dim $(M_s) = \mathfrak{c} = \dim(H(G))$, where \mathfrak{c} denotes the cardinality of continuum. Specifically, given $\Phi \in M_s \setminus \{0\}$, $l, \nu \in \mathbb{N}$ with ν large enough and $\delta > 0$, there is some $\mu_0 \in \mathbb{N}$ such that

$$|\Phi(\varphi_n(z)) - P_l(z)| < \delta/2 \tag{9}$$

for all $n \in A(\nu, l, 1, \mu_0)$ large enough and all $z \in K_{\nu}$.

Finally, consider the vector subspace of H(G) given by

$$M_{\max} := M_d + M_s.$$

Note that M_{\max} is dense (because it contains M_d) and has dimension $\mathbf{c} = \dim (H(G))$ (because it contains M_s , and $\dim(M_s) = \mathbf{c}$). Fix a function $f \in M_{\max} \setminus \{0\}$. Two cases are possible. If $f \in M_d$, then $f \in FHC((C_{\varphi_n})_n)$. If $f \notin M_d$, then there is a function $\Phi \in M_s \setminus \{0\}$ (as in the preceding paragraph) and another function $H \in M_d$ such that $f = \Phi + H$. Therefore $(C_{\varphi_n}f)(z) = \Phi(\varphi_n(z)) + H(\varphi_n(z))$. Given $l, \nu \in \mathbb{N}$ with ν large enough, we had obtained (see (8)) the existence of $\alpha > 0$ such that $|H(\varphi_n(z))| < \alpha \cdot \chi(\varphi_n(z), \partial_\infty G)$ for all $z \in K_{\nu}$ and all $n \in A(\nu, l, 1)$ (so for all $n \in A(\nu, l, 1, \mu_0)$). Since $\sup_{z \in K_{\nu}} \chi(\varphi_n(z), \partial_\infty G) \to 0$ as $n \to \infty$, we get for $n \in A(\nu, l, 1, \mu_0)$ large enough that $|H(\varphi_n(z))| < \delta/2$ for all $z \in K_{\nu}$. Thanks to (9) and the triangle inequality, we get $|(C_{\varphi_n}f)(z) - P_l(z)| < \delta$ for the same n, z. Then the proof of the fact $f \in FHC((C_{\varphi_n})_n)$ is concluded as soon as we take into account the density of $(P_l)_l$ in H(G) and the property $\underline{dens}(A(\nu, l, 1, \mu_0)) > 0$.

Remarks 3.4. 1. Let $G \subset \mathbb{C}$ be any domain of \mathbb{C} and $(\varphi_n)_n \subset H(G,G)$, and let us consider the family $FHC_{\mathcal{M}}((C_{\varphi_n})_n) := \{f \in H(G) : \underline{dens}(\{n \in \mathbb{N} :$ $f \circ \varphi_n \in B_K(g, \delta)\}) > 0$ for each $K \in \mathcal{M}(G)$, each $\delta > 0$ and each $g \in H(G)\}$. Under the same assumptions as in Theorems 2.6-3.2-3.3 (the strongly frequent runawayness of $(\varphi_n)_n$ in $H_{1-1}(G)$), and with the same proofs, it is obtained that the set $FHC_{\mathcal{M}}((C_{\varphi_n})_n)$ is spaceable and maximal dense-lineable in H(G). Note that, if G is multiply connected, then $FHC_{\mathcal{M}}((C_{\varphi_n})_n)$ is bigger than $FHC((C_{\varphi_n})_n)$. In fact, if G is a finitely connected domain that is not simply connected and $(\varphi_n)_n \subset H_{1-1}(G)$, then the set $HC((C_{\varphi_n})_n)$ (so the set $FHC((C_{\varphi_n})_n)$ as well) is empty: see [23, Theorem 3.15].

2. According to Bès [15] (see Section 1), if G is simply connected and $\varphi \in H(G, G)$, then C_{φ} is hypercyclic if and only if it is frequently hypercyclic, and if and only if $FHC(C_{\varphi})$ is spaceable. Since we do not know whether the property of being strongly frequently runaway for (φ_n) characterizes the frequent hypercyclicity of (C_{φ_n}) , a natural question arises for sequences of composition operators: If $\{\varphi_n\}_{n\geq 1} \subset H_{1-1}(G)$ is such that (C_{φ_n}) is frequently hypercyclic, is $FHC((C_{\varphi_n}))$ spaceable (and even maximal dense-lineable)?

4. Examples and final remarks

1. The first example is motivated by one in [9]. Consider the slit complex plane $G := \mathbb{C} \setminus (-\infty, 0]$. Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\beta \ge 1 + \alpha$. For $N \in \mathbb{N}$, $z^{1/N}$ will represent the principal branch of the Nth root of z in G. Fix $N \in \mathbb{N}$. Then the mappings

$$\varphi_n(z) := n^{\alpha} z^{1/N} + n^{\beta} \quad (n \in \mathbb{N})$$

belong to $H_{1-1}(G) \setminus \text{Aut}(G)$. Let C > 0 be a constant to be specified later, and consider the numbers $R_{\nu} = (C \nu^{\beta-\alpha})^N$ and the compact sets

$$K_{\nu} = \left\{ z = re^{i\theta} : \min\left\{\frac{1}{R_{\nu}}, 1\right\} \le r \le R_{\nu}, \, |\theta| \le \pi \left(1 - \frac{1}{\nu}\right) \right\} \, (\nu \in \mathbb{N}).$$

Note that $K_{\nu} \subset \overline{D}(0, R_{\nu})$, the closed disc with center 0 and radius R_{ν} . Hence $\varphi_n(K_{\nu}) \subset \overline{D}(n^{\beta}, n^{\alpha}R_{\nu}^{1/N})$. Plainly, all K_{ν} 's lack holes and form an exhaustive sequence of compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. By [18, Lemma 2.2], there exist pairwise disjoint subsets $A(l, \nu)$ $(l, \nu \geq 1)$ of \mathbb{N} with $\underline{dens}(A(l, \nu)) > 0$ such that, for any $n \in A(l, \nu)$ and $m \in A(k, \mu)$, we have $n \geq \nu$ and $|n - m| \geq \nu + \mu$ if $n \neq m$. Define $A(\nu) := \bigcup_{l \in \mathbb{N}} A(l, \nu)$. Of course, $\underline{dens}(A(\nu)) > 0$ for every $\nu \in \mathbb{N}$, and the family $\{A(\nu) : \nu \in \mathbb{N}\}$ is pairwise disjoint. Pick two distinct members $A, B \in \mathcal{K} := \{\varphi_n(K_{\nu}) : n \in A(\nu), \nu \in \mathbb{N}\}$. Then there are distinct $m, n \in \mathbb{N}$ (m > n, say) as well as sets $A(l, \nu)$, $A(k, \mu)$ such that $n \in A(l, \nu), m \in A(k, \mu), A = \varphi_m(K_{\mu})$ and $B = \varphi_n(K_{\nu})$. It is easy to see that $\sigma := \inf_{t>1} \frac{t^{\beta-1}}{t^{\alpha}(t-1)^{\beta-\alpha}} > 0$. Take $C := \min\{1/2, \sigma/4\}$. Since m > n, we get $\frac{(m/n)^{\beta}-1}{(m/n)^{\alpha}((m/n)-1)^{\beta-\alpha}} \geq 4C$, that is, $m^{\beta} - n^{\beta} \geq 4C \cdot m^{\alpha}(m-n)^{\beta-\alpha}$. The distance between the centers of the discs

$$D_{1} := \overline{D}(m^{\beta}, m^{\alpha}R_{\mu}^{1/N}) \text{ and } D_{2} := \overline{D}(n^{\beta}, n^{\alpha}R_{\nu}^{1/N}) \text{ is } m^{\beta} - n^{\beta}. \text{ But observe that}$$
$$m^{\beta} - n^{\beta} > 4C \cdot \frac{m^{\alpha} + n^{\alpha}}{2} \cdot (m - n)^{\beta - \alpha} \ge 4C \cdot \frac{m^{\alpha} + n^{\alpha}}{2} \cdot (\mu + \nu)^{\beta - \alpha}$$
$$\ge 2C \cdot \frac{m^{\alpha} + n^{\alpha}}{2} \cdot (\mu^{\beta - \alpha} + \nu^{\beta - \alpha}) \ge m^{\alpha}C\mu^{\beta - \alpha} + n^{\alpha}C\nu^{\beta - \alpha}$$
$$= \text{radius}(D_{1}) + \text{radius}(D_{2}).$$

Consequently, $D_1 \cap D_2 = \emptyset$ and so $A \cap B = \emptyset$. Then \mathcal{K} is pairwise disjoint. Finally, fix a compact set $K \subset \mathbb{C} \setminus (-\infty, 0]$. There is R > 0 such that $K \subset \overline{D}(0, R)$. If $L \in \mathcal{K}$, there exists $(l, \nu) \in \mathbb{N}^2$ and $n \in A(l, \nu)$ (so $n \geq \nu$) with $L = \varphi_n(K_\nu) \subset \overline{D}(n^\beta, n^\alpha R_\nu^{1/N})$. It follows that each $w \in L$ satisfies $|w - n^\beta| \leq n^\alpha R_\nu^{1/N}$, hence (recall that $C \leq 1/2$) we get $|w| \geq n^\beta - n^\alpha R_\nu^{1/N} = n^\beta - n^\alpha C \nu^{\beta-\alpha} \geq n^\beta - n^\alpha C n^{\beta-\alpha} \geq (1/2)n^\beta > R$ if n is large enough. Therefore $K \cap \varphi_n(K_\nu) = \emptyset$, so $K \cap L = \emptyset$ except for finitely many $L \in \mathcal{K}$, because no n may belong to two different $A(l, \nu)$'s. According with Theorems 2.6-3.2-3.3, $(C_{\varphi_n})_n$ is frequently hypercyclic and the set $FHC((C_{\varphi_n})_n)$ is spaceable and maximal dense-lineable in $H(\mathbb{C} \setminus (-\infty, 0])$.

2. If G and Ω are simply connected domains different from \mathbb{C} , then the Riemann conformal representation theorem (see, e.g., [1]) provides an isomorphism (that is, a bijective biholomorphic mapping) $f: G \to \Omega$. Therefore, if $\varphi \in H(G, G)$, we have $f \circ \varphi \circ f^{-1} \in H(\Omega, \Omega)$ and the mapping $h \in H(\Omega) \mapsto h \circ f \in H(G)$ is a linear homeomorphism. Hence every example of a sequence $(\varphi_n)_n \subset H(G, G)$ satisfying that $FHC((C_{\varphi_n})_n)$ is nonempty (spaceable, maximal dense-lineable, resp.) provides us with an example of a sequence $(\Phi_n)_n \subset H(\Omega, \Omega)$ such that $FHC((C_{\Phi_n})_n)$ is nonempty (spaceable, maximal dense-lineable, resp.). Indeed, take $\Phi_n := f \circ \varphi_n \circ f^{-1}$ $(n \in \mathbb{N})$. For instance, we have in accordance with Example 1 that $FHC((C_{\varphi_n})_n)$ is spaceable and maximal dense-lineable in $H(\mathbb{C} \setminus (-\infty, 0])$, where $\varphi_n(z) = n + z^{1/N}$. Moreover, the mapping $f: z \in \mathbb{C} \setminus (-\infty, 0] \mapsto \frac{z^{1/2}-1}{z^{1/2}+1} \in \mathbb{D}$ is an isomorphism with inverse $f^{-1}(z) = (\frac{1+z}{1-z})^2$. Consequently, the set $FHC((C_{\Phi_n})_n)^{1/2}-1$.

3. Let a > 0 and $\gamma \ge 1$. It is easy to check that every linear fractional function $\Phi_n(z) := 1 + \frac{2(z-1)}{2-ian^{\gamma}(z-1)}$ $(n \in \mathbb{N})$ is an automorphism of \mathbb{D} . Moreover, every Φ_n is parabolic (see [32, pp. 6–7]), that is, it has a unique fixed point, located at \mathbb{T} (namely, at 1). With arguments similar – but easier – to those given in Example 1, we can check that if $\gamma \ge 1$, $\Pi_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is the open right half-plane and $\varphi_n(z) := z + ian^{\gamma}$ for each $n \in \mathbb{N}$ (note that $(\varphi_n)_n \subset \operatorname{Aut}(\Pi_+)$), then $FHC((C_{\varphi_n})_n)$ is spaceable and maximal dense-lineable in $H(\Pi_+)$. Since $f(z) := \frac{1+z}{1-z}$ is an isomorphism between \mathbb{D} and Π_+ with inverse $\frac{z-1}{z+1}$, and since

 $\Phi_n = f^{-1} \circ \varphi_n \circ f$, we obtain as in Example 2 the spaceability as well as the maximal dense-lineability for $FHC((C_{\Phi_n})_n)$ in $H(\mathbb{D})$. The case $\gamma = 1$ gives the same properties (spaceability was already known, see Section 1) for the set $FHC(C_{\varphi})$, where φ is the parabolic automorphism of \mathbb{D} given by $\varphi(z) = 1 + \frac{2(z-1)}{2-ia(z-1)}$.

4. Let $G = \mathbb{C}$. In this case $H_{1-1}(G) = \operatorname{Aut}(\mathbb{C}) = \{ \operatorname{similarities} z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0 \}$. Examples of sequences of similarities $(\varphi_n(z) = a_n z + b_n)_n$ for which $(C_{\varphi_n})_n$ is frequently hypercyclic were provided in [9, Theorem 4.1] (see Section 1). In its proof, it is shown in fact that these sequences (φ_n) are strongly frequently runaway. Then, according to Theorems 2.6-3.2-3.3, we obtain that for each of such sequences the set $FHC((C_{\varphi_n})_n)$ is not only nonempty but also spaceable and maximal dense-lineable in $H(\mathbb{C})$.

5. In this final and rather general example –from which the parabolic automorphism φ defined in the last line of Example 3 above is a particular instance– we will not use Theorems 2.6-3.2-3.3. We say that a domain $G \subset \mathbb{C}$ is a *Jordan domain* whenever $\partial_{\infty}G$ is a homeomorphic image of \mathbb{T} (hence G need not be bounded; for instance, an open half-plane is a Jordan domain). We exhibit the mentioned collection of examples in the following proposition.

Proposition 4.1. The following assertions hold:

- (a) Assume that G is a Jordan domain in \mathbb{C} and that $\varphi \in H_{1-1}(G)$ satisfies that there is $\xi \in \partial_{\infty}G$ such that $\varphi^n \to \xi$ as $n \to \infty$ uniformly on compact in G. Then $FHC(C_{\varphi})$ is maximal dense-lineable in H(G).
- (b) If φ is a non-elliptic automorphism of \mathbb{D} , then $FHC(C_{\varphi})$ is maximal dense-lineable in $H(\mathbb{D})$.

Proof. (a) As a particular instance of a result by Bès [15] mentioned in Section 1, the set $FHC(C_{\varphi})$ is spaceable if it is assumed that $\varphi \in H_{1-1}(G)$ and there is $\xi \in \partial_{\infty}G$ such that $\varphi^n \to \xi$ as $n \to \infty$ uniformly on compacta in G: indeed, such a φ cannot have fixed points in G. According to Osgood–Carathéodory's theorem (see [25]), there is an isomorphism $f : G \to \mathbb{D}$ that is extendable to a homeomorphism $\overline{G}^{\infty} \to \overline{\mathbb{D}}$, where \overline{G}^{∞} is the closure in \mathbb{C}_{∞} of G. Since the polynomials form a dense set in $H(\mathbb{D})$, the set $\mathcal{D} := \{P \circ f : P \text{ polynomial}\}$ is a dense vector subspace of H(G). Now, the sequence $((C_{\varphi})^n Q)_n$ converges in H(G)for every $Q \in \mathcal{D}$, because if $Q = P \circ f$ then

$$(C_{\varphi})^{n}Q = C_{\varphi^{n}}Q = P \circ f \circ \varphi^{n} \longrightarrow P(f(\xi)) \quad (n \to \infty)$$

uniformly on compact in G. Since $FHC(C_{\varphi}) = FHC((C_{\varphi^n})_n)$ is spaceable and H(G) is a separable F-space, from [12, Theorem 4.10(a)] we can conclude that $FHC(C_{\varphi})$ is maximal dense-lineable in H(G).

(b) This is a special instance of (a) because for every non-elliptic $\varphi \in \operatorname{Aut}(\mathbb{D})$ there is a point $\xi \in \mathbb{T}$ (the unique fixed point if φ is parabolic, and the attractive fixed

boundary point if φ is hyperbolic) such that $\varphi^n \to \xi$ as $n \to \infty$ uniformly on compact in \mathbb{D} (see [32]). \Box

6. In the proof of Theorems 2.6-3.2-3.3, we have not used any property of the lower density except that if A is a subset of N with $\underline{dens}(A) > 0$, then A can be divided into an infinity of disjoint sets A_n with $\underline{dens}(A_n) > 0$. Therefore, our results in Theorems 2.6-3.2-3.3 will also be verified for the upper-frequent hypercyclicity –introduced by Shkarin in [34]– by replacing in the definition of "strongly frequently runaway" (Definition 2.5) the condition (P1) by $\overline{\text{dens}}(A(\nu)) >$ 0 for all $\nu \in \mathbb{N}$. More generally, the notion of \mathcal{A} -hypercyclicity has been recently introduced in a paper of Bès-Menet-Peris-Puig [16], where an operator $T: X \to X$ is said to be \mathcal{A} -hypercyclic if there exists $x \in X$ such that, for every nonempty open set $U \subset X$, we have $N(x, U) \in \mathcal{A}$. Here \mathcal{A} is a nonempty collection of nonempty subsets of \mathbb{N} satisfying appropriate axioms and $N(x, U) := \{n \in \mathbb{N} : T^n x \in U\}.$ The extension of these notions to a sequence $T_n: X \to X \ (n \ge 1)$ is immediate. Therefore, if for every $A \in \mathcal{A}$ there is an infinity of mutually disjoint sets $A_n \subset A$ such that $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then our Theorems 2.6-3.2-3.3 imply corresponding results for \mathcal{A} -hypercyclicity by replacing (P1) by $A(\nu) \in \mathcal{A}$ for all $\nu \in \mathbb{N}$. In particular, mere hypercyclicity is equivalent to \mathcal{A} -hypercyclicity for $\mathcal{A} = \{$ infinite subsets of \mathbb{N} and, in this case, the adapted (P1), (P2) and (P3) are equivalent to the "classical" runaway condition introduced in [10].

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