THE SPACE OF INTEGRABLE FUNCTIONS WITH RESPECT TO A VECTOR MEASURE

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This memoir owns its existence to the interaction of three important forces of the Universe: chance, work and support. Chance governs in the deepest the movement of the Universe. To it I ow the discovery of Mathematics, the completion of my studies and my relation with Mathematical Analysis. Work expresses the oposition that life, vegetal and animal, opposes against entropy. To it I ow great moments of pleasure and deep anxieties. Support is the concrete expresion of solidarity as an indidual and social power. I have received it from my family, my friends and in my Departament.

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To Lourdes

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Introduction.

At the beginning of this century the works of Lebesgue and other mathematicians created a modern and complete theory of integration which allowed to integrate in a fully satisfactory way a broad class of real functions with respect to a positive measure. Among the several directions of developement of this theory it was the work of Bochner in 1933 who created a Lebesgue type theory in order to integrate vector valued functions with respect to a positive measure.

The study of the formally symmetric situation, namely the integration of scalar funtions with respect to a vector mesaure, had to wait until 1955 when Bartle, Dunford and Schwartz published their paper *"Weak compactness and vector measures"*. In it, they study weakly compact operators with values in a Banach space X and defined on the space of continuous functions over a compact topological space K

$$T: C(K) \longrightarrow X .$$

The technique that they use for this study is to represent the action of the operator as integration with respect to a measure, associated to the operator, with values in the Banach space X. In this way they create a theory for integrating scalar functions with respect to a measure defined on a σ–algebra and with values in a Banach space.

In the early seventies Lewis publishes his papers *"Integration with respect to vector measures"* and *"On integrability and summability in vector spaces"*, where he developes a theory for integrating scalar functions with respect to measures with values in Hausdorff locally convex topological vector spaces. When we restrict to Banach space valued measures this theory is equivalent to the theory of Bartle, Dunford and Schwatrz.

In 1975 Kluvánek and Knowles publish their book "Vector measures and control systems". In it they study the space of real functions which are integrable with respect to a vector measure with values in a Hausdorff locally convex topological vector space.

Our study will be done for measures ν defined over a σ–algebra and with values in a Banach space X. We will consider real functions, which will allow us to endow the space $L^1(\nu)$, of functions which are integrable with respect to ν, with a lattice structure and so, use the tools of the theory of Banach lattices.

Our work started trying to answer an informal question of Prof. J. Diestel: if for the space $L^1(\nu)$ holds an analogue of the theorem of Talagrand on weak sequential completeness of the space $L^1(\mu, X)$, of X valued functions integrable in the sense of Bochner with respect to a positive measure μ, when the Banach space X is weakly sequentially complete. The answer is afirmative (Corollary 2.3).

Next we studied the general problem of relating on the one hand, the properties of the vector masure ν and the Banach space X, and, on the other hand, the properties of the space $L^1(\nu)$ of functions which are integrable with respect to ν. In this setting certain natural questions arise: Does the Banach space X determine the properties of $L^1(\nu)$? If this is true: In what extend does it ocurr? Can $L^1(\nu)$ be reflexive? When is $L^1(\nu)$ an AL–space? The answer, total or partial depending on the case, to these and other questions is the content of the memoir.

In the Preliminaries we establish the notation used through the memoir and we recall the main concepts and results of the theory of vector measures and Banach lattices that we will use. We emphasize the concept of L–weakly compact set in a Banach lattice, due to its importance in our study.

The first chapter is divided into two sections. In the first section we recall the main known results on the theory of integration of real functions with respect to a vector measure ν defined over a σ–algebra and with values in a Banach space X and on the space of those functions $L^1(\nu)$. This is an order continuous Banach lattice with a weak unit.

Our own study of the space $L^1(\nu)$ starts in the second section. The first question is the following: What spaces arise as L^1 of a vector measure? We identify this class precisely: they are the order continuous Banach lattices with weak unit (Theorem 1.5).

We study the existence of a Banach space "universal" in the sense that every space L^1 of a vector measure can be obtained from a vector measure with values in this "universal" space. We prove that this is possible for $L^1(\nu)$ spaces which are separable and have no atoms, with measures taking values in the space c_0 (Theorem 1.20).

There is no usefull identification of the dual space of $L^1(\nu)$. Hence we study the possibility of characterizing weak convergence in $L^1(\nu)$ via weak convergence of integrals over arbitrary sets (Theorem 1.23).

In the second chapter we study several properties of the space $L^1(\nu)$, focusing on the fact that the properties of the Banach space in which the measure takes its values determine, in some extend, the properties of the space $L^1(\nu)$. We obtain, among others, the following results: if the Banach space X does not contain a subspace isomorphic to c_0 neither does $L^1(\nu)$; if X has cotype $q \geq 2$ the $L^1(\nu)$ also has cotype q; is X has the Schur property the $L^1(\nu)$ has the positive Schur property.

We prove a lattice version of the theorem of Dunford–Pettis on weakly compact operators on $L^1[0,1]$, which allows us to prove that is X has the Schur property and the measure ν has σ–finite variation then $L^1(\nu)$ has the Dunford–Pettis property (Theorem 2.10). We also deduce that if the measure ν has σ–finite variation and has no atoms then $L^1(\nu)$ is not reflexive (Theorem 2.11).

As we can obtain any order continuous Banach lattice with weak unit as L^1 of a certain vector measure, it follows, in particular, that we can obtain a Hilbert space. Theorem 2.13 gives a suficient condition in order that a measure taking values in a space of cotype 2 generates a space $L^1(\nu)$ order isomorphic to a Hilbert space.

The last part of the chapter is devoted to the study of a specific lattice property: the subsequence splitting property, whose origins are in the thecniques used by Kadec and Pelczynski in their study of subspaces of the spaces of $L^p[0,1]$. We give suficient conditions so that the space $L^1(\nu)$ has this property (Theorem 2.14).

The third chapter studies the answer to the question: When is $L^1(\nu)$ an AL–space? We prove that this ocurrs precisely when $L^1(\nu)$ is isomorphic to the space $L^1(|\nu|)$ where $|\nu|$ is the variation of the measure ν, which has to be bounded. In the quest for suficient conditions for $L^1(\nu)$ being an AL–space we prove that the domination of the variation by the semivariation is not a suficient condition (Example 3.3) and that those conditions can not be imposed on the Banach space X (Example 3.5).

We study measures with values in spaces with particular properties obtaining suficient conditions if the values are taken in AL–space and a characterization for measures with values in a $C(K)$ space.

The study of the general case is done by identifying the dual of $L^1(\nu)$

with an ideal, in the lattice and algebraic sense, in $L_\infty(|\nu|)$ and by using the Gelfand transform and the characters over $L_\infty(|\nu|)$. We find conditions in terms of the sets of zeros of $L^1(\nu)^*$ and of an ideal in $L^1(\nu)^*$ associated to the set of measures $\{x^*\nu : x^* \in X^*\}$. Theorem 3.16 gives a necessary and suficient condition, in terms of the measure, in order that the space $L^1(\nu)$ be given as a finite number of spaces of the form $L^1(|x^*\nu|)$.

The fourth chapter studies operators with values in $L^1(\nu)$. For this we use the thecnique of associating to each operator a measure, with values in a space of operators, and studying the properties of the operator via the properties of the measure. This measure is bounded, finitely additive and, in general, countably additive in the strong operator topology.

The study is centered in finding the properties of the operators which correspond to "better" properties of the associated measure. We characterize the operators whose associated measure is countably additive in the uniform topology: they are the L–weakly compact operators, that map norm bounded sets into L–weakly compact sets in $L^1(\nu)$ (Theorem 4.5). For measures ν with σ–finite variation the operators whose associated measure has bounded variation are those that factorize with an integral operator through the space $L^1(|\nu|)$ (Theorem 4.6). In the same conditions, if we require the associated measure to have a Bochner integrable density with respect to the variation of the measure ν, the operator must factorize with a nuclear operator (Theorem 4.8). The fact that the previous condition gives a characterization is closely related to the fact that the measure ν has a strongly measurable and Pettis integrable density with respect to its variation (Theorem 4.9).

The last part of the chapter is devoted to applying the previous results to the problem of relating the existence of a subspace isomorphic to ℓ_∞ in the space $\mathcal{L}(Y,X)$, of continuous linear operators between Y and X, with the coincidence of this space with some ideal of operators. In our case the space Y is an arbitrary Banach space. When X is an order continuos Banach lattice with weak unit we prove that if $\mathcal{L}(Y, X)$ does not contain ℓ_∞, then every operator is L–weakly compact (Theorem 4.11). When X is an atomic order continuos Banach lattice we prove that it is equivalent that every operator from Y to X is compact and that $\mathcal{L}(Y, X)$ does not contain ℓ_∞ (Theorem 4.12).

Preliminaries and notation.

In this chapter we establish the notation that will be used through the memoir and collect the main results on vector measures and Banach lattice that we will use. The ones corresponding to the theory of vector measures are taken from chapter I of the book by Diestel and Uhl [DU]. For the theory of Banach lattices we have followed chapters 1.a and 1.b of volume II of the book by Lindenstrauss and Tzafriri [LT].

A *measurable space* is a pair (Ω, Σ) where Ω is an abstract set and Σ is a σ*–algebra* of subsets of Ω. The elements of Σ are known as *measurable sets*. A *partition* of a measurable set A is a finite family $(A_i)_1^n$ of disjoint measurable sets whose union is A.

A *finitely additive measure* is a function ν defined over a σ–algebra and with values in a Banach space X, that satisfies $\nu(\emptyset) = 0$ and if A and B are disjoint measurable sets $\nu(A \cup B) = \nu(A) + \nu(B)$.

A measure is *bounded* if the set of its values is bounded; is *strongly additive* if for every family (A_n) of disjoint measurable sets the series $\sum \nu(A_n)$ is convergent in X; and is *countably additive* is the above series converges to $\nu(\bigcup_n A_n)$. Every countably additive measure is strongly additive and these are bounded. If not especified, we will always understand measure as countably additive measure.

A measure is a *scalar measure* if it takes its values in the field of scalars. For us, all along the memoir, this will be the field of real numbers $\mathbb{R}$. A measure is *positive* if it is scalar and its values are non negative. In all other cases we will have a *vector measure*. We will consider, as usual, measures with values in $[0, +\infty]$, that we will also call positive.

A *measure space* is a triplet $(\Omega, \Sigma, \lambda)$ where (Ω, Σ) is a measurable space and λ is a positive countably additive measure defined over Σ. The measure space is finite if $\lambda(\Omega) < +\infty$ and σ–finite if $\Omega = \cup_n A_n$ where $\lambda(A_n) < +\infty$. A property holds *almost everywhere with respect to* λ if it holds for every point of Ω but for the points of a measurable set Z with $\lambda(Z) = 0$.

The *variation* of a measure ν is the smallest positive measure that dominates the measure ν. It is denoted by $|\nu|$. It is given by

$$|\nu|(A) = \sup \left\{ \sum_1^n \|\nu(A_i)\| : (A_i)_1^n \text{ is a partition of } A \right\}.$$

It can take infinite values.

Let X be a Banach space. We will denote by B_X the unit ball of X, that is $B_X = \{x \in X : \|x\| \leq 1\}$. X^* is the topological dual of X and B_{X^*} its unit ball. In X we will consider the norm topology and the *weak* topology, denoted by $\sigma(X, X^*)$, which is the coarsest topology with respect to which the elements of the dual space are continuous. In X^* we will consider the norm topology, the weak topology and the *weak–$*$* topology, denoted by $\sigma(X^*, X)$, which is the coarsest topology with respect to which the elements of X are continuous.

Let $\nu\colon \Sigma \longrightarrow X$ be a vector measure with values in a Banach space. The *semivariation* of ν is the set function defined on Σ by

$$\|\nu\|(A) = \sup \left\{ |x^*\nu|(A) : x^* \in B_{X^*} \right\},$$

where $|x^*\nu|$ is the variation of the scalar measure

$$A \in \Sigma \longmapsto x^*\nu(A) \in \mathbb{R}\,.$$

The semivariation is not in general a measure. It is monotone, that is, if $B \subset A$ then $\|\nu\|(B) \leq \|\nu\|(A)$. We will consider the following set function defined on Σ

$$|||\nu|||(A) = \sup \{ \|\nu(B)\| : B \subset A, B \in \Sigma \}.$$

Then for every measurable set A we have

$$|||\nu|||(A) \leq \|\nu\|(A) \leq 2 \cdot |||\nu|||(A).$$

Let $\nu\colon \Sigma \longrightarrow X$ be a vector measure. We will say that a positive measure λ is a *control measure* for ν if

$$\lim_{\lambda(A)\to 0} \|\nu\|(A) = 0 \quad \text{and} \quad \lim_{\|\nu\|(A)\to 0} \lambda(A) = 0.$$

Countably additive measures have control measures. An important result of Rybakov (see [DU, IX.2]) states that if ν is a countably additive vector measure, then there exists x_0^* in the unit ball of X^* such that the measure $|x_0^*\nu|$ is a control measure for ν, so it satisfies

$$|x_0^*\nu| \leq \|\nu\| \quad \text{and} \quad \lim_{|x_0^*\nu|(A)\to 0} \|\nu\|(A) = 0.$$

We will say that $|x_0^*\nu|$ is a *Rybakov control measure* for ν. The semivariation of ν and the control measures for ν have the same null sets. A property is said to hold *almost everywhere with respect to* ν if it holds for every point of Ω, but for a set of null semivariation. Thus it is equivalent to saying that it holds almost everywhere with respect to a control measure for ν.

A *Banach lattice* is a Banach space E endowed with an order relation $\leq$ that satisfies

1) if $x, y, z \in E$ and $x \leq y$ then $x + z \leq y + z$,

2) if $x, y \in E$ and $a \in \mathbb{R}$ with $a \geq 0$ then $ax \leq ay$,

3) for every $x, y \in E$ there exists the supremum and the infimum, with respect to the order, of x and y,

4) if $|x| \leq |y|$ then $\|x\| \leq \|y\|$, where $|x| = \sup\{x, -x\}$, is the *módulus* of x.

The dual of a Banach lattice is a Banach lattice for the natural order: $x^* \leq y^*$ if and only if $x^*x \leq y^*x$ for every $x \in E$ with $x \geq 0$.

A set A in E is bounded with respect to the order (*order bounded*) if there exists $x \geq 0$ such that $|z| \leq x$ for every $z \in A$. It is denoted by $[x, y]$ the set of all z such that $x \leq z \leq y$. A Banach lattice is complete with respect to the order (*order complete*) if every order bounded set has supremum. Dual Banach lattice are always order complete.

An *ideal* in a Banach lattice E is a linear subspace F with the property that $y \in F$ whenever $|y| \leq |x|$ for some $x \in F$. A *band* is an ideal F satisfying that if $A \subset F$ and $\sup A$ exists in E, then $\sup A \in F$.

An element $x > 0$ in a Banach lattice is an *atom* if $0 \leq z \leq x$ implies $z = ax$ where a is a real number between 0 and 1. A Banach lattice is said to be atomic if there exists a family $\{x_\alpha\}$ of atoms that is complete, in the sense that if $\inf\{x, x_\alpha\} = 0$ for every α, then $x = 0$.

A linear operator T between two Banach lattices is an *order ismorphism* if it is biyective and preserves the order structure

$$T(\sup\{x, y\}) = \sup\{Tx, Ty\} \quad \text{and} \quad T(\inf\{x, y\}) = \inf\{Tx, Ty\}.$$

An order ismorphism is always a topological isomorphism. Two Banach lattices are said to be *order isometric* if there exist between them a sujective linear isometry that preserves the order.

A crucial property is order continuity. A set A is downward directed if for every $x, y \in A$ there exist $z \in A$ such that $z \leq x$ and $z \leq y$. A Banach lattice has norm continuous with respect to the order (*order continuous*) if every downward directed set with zero as infimum satisfies $\inf\big\{\|x\| : x \in A\big\} = 0$. An important characterization of this property is [LT vol. II, Proposition 1.a.8]:

A Banach lattice is order continuous if and only if every order bounded increasing sequence is convergent.

An element e is a *weak unit* of E if $\inf\{x, e\} = 0$ implies $x = 0$. It is a *unit* or strong unit if $\|x\| \le 1$ if and only if $|x| \le e$.

A *Banach function space* or Köthe function space with respect to a σ–finite measure space $(\Omega, \Sigma, \lambda)$ is [LT vol. II, Definition 1.b.17]: a Banach space E of classes of equal almost everywhere with respect to λ real functions, which are locally integrable and satisfy

1) If $|f(\omega)| \le |g(\omega)|$, f is measurable and $g \in E$, then $f \in E$ and $\|f\| \le \|g\|$.
2) For every $A \in \Sigma$ of finite measure, the characteristic function χ_A is in E.

The *Köthe dual* of a Banach function space E is

$$\big\{\, g\colon S \longrightarrow \mathbb{R} : g \;\text{ is measurable and }\; fg \in L^1(\lambda) \;\text{ for every }\; f \in E \big\}\,.$$

When E is order continuous, E^* coincides with the Köthe dual [LT vol. II, p. 29]. It can also be considered the Köthe bidual of E.

Order continuous Banach lattices with weak unit can be represented as Banach function spaces [LT vol. II, Theorem 1.b.14].

Theorem. *Let E be an order continuous Banach lattice with weak unit. Then there exists a probability space $(\Omega, \Sigma, \lambda)$, an ideal $\tilde{E}$ in $L^1(\Omega, \Sigma, \lambda)$ and a lattice norm $\|\cdot\|_{\tilde{E}}$ in $\tilde{E}$ such that*

1) E is order isometric to $(\tilde{E}, \|\cdot\|_{\tilde{E}})$.
2) $\tilde{E}$ is dense in $L^1(\Omega, \Sigma, \lambda)$ and $L_\infty(\Omega, \Sigma, \lambda)$ is dense in $\tilde{E}$.

3) $\|f\|_1 \le \|f\|_{\tilde{E}} \le 2\|f\|_\infty$ *whenever* $f \in L_\infty(\Omega, \Sigma, \lambda)$.

4) The traspose of the isometry given in 1) maps E^* *onto the Banach lattice* $\tilde{E}^*$ *of all measurable functions* g *such that*

$$\|g\|_{\tilde{E}^*} = \sup\left\{ \int_\Omega fg \, d\lambda : \|f\|_{\tilde{E}} \le 1 \right\} < \infty.$$

The value of the functional given by g *at* $f \in \tilde{E}$ *is* $\int_\Omega fg \, d\lambda$.

An important concept in several parts of this memoir is that of *L–weakly compact set* due to Meyer–Nieberg [M 1, Definition II.1]. A bounded set K is L–weakly compact if for every disjoint sequence (y_n) satisfying $|y_n| \le |x_n|$ with $x_n \in K$, we have that (y_n) tends to zero in norm. In order continuous Banach lattices this concept is equivalent to *almost order boundedness*, that is, for every $\varepsilon > 0$ there exists $x \ge 0$ such that $K \subset [-x, x] + \varepsilon B$ where B is the unit ball [M 1, Satz II.2]. This is the name given by Zaanen [Z, p. 501]. In order continuous Banach function spaces this concept coincides with the concept of bounded *equi–integrabble set*: a bounded set that satisfies

$$\lim_{\mu(A) \to 0} \sup\big\{ \|f \cdot \chi_A\| : f \in K \big\} = 0.$$

Every L–weakly compact set is relatively weakly compact [M 1, Satz II.6]. The Banach lattices in which L–weakly compact sets coincide with relatively weakly compact sets are those in which every infinitedimensional closed sublattice contains a sublattice isomorphic to ℓ^1 [M 2, Satz 14]. Sánchez Henríquez in his Doctoral Thesis proves that this last property is equivalent to the *positive Schur property*: every weakly null positive sequence is convergent [Sa, Teorema 1.16].

In order continuous Banach lattices every relatively compact set is L–weakly compact [M 1, Korollar II.4]. The Banach lattices in which L–weakly compact sets coincide with relatively compact sets are the order continuous and atomic Banach lattices [M 1, Beispiele II.7] and [AS, Satz 1.1].

An operator defined on a Banach space taking values in a Banach lattice is *L–weakly compact* if the image of the unit ball is an L–weakly compact set [M 2, Definition 1 iii)]. Zaanen refers to them as semi–compact operators [Z, p. 529].

Let X_i be Banach spaces $1 \leq i \leq n$. For $1 \leq p < +\infty$ we denote by $\left(\bigoplus_1^n X_i\right)_p$ the space of n–tuples $(x_1, \ldots, x_n)$ with $x_i \in X_i$ endowed with the norm $\|(x_1, \ldots, x_n)\| = \left(\sum_1^n \|x_i\|^p\right)^{1/p}$.

The theory of vector measures and its applications can be seen in [DU]. For Banach lattices see [LT vol. II] and the books by Aliprantis and Burkinshaw [AB], Meyer–Nieberg [M 3] and Schaefer [S].

CHAPTER 1: The space $L^1(\nu)$.

SECTION 1. In this section we present the known results on the theory of integration of real functions with respect to a vector measure and on the space of real functions which are integrable with respect to a vector measure, that are relevant in our memoir.

Let (Ω, Σ) be a measurable space and let X be a Banach space. Consider a countably additive vector measure

$$\nu \colon \Sigma \longrightarrow X\,.$$

The following definition is due to Bartle, Dunford and Schwartz [BDS, Definition 2.5]. They consider functions with real or complex values. We will restrict to real values, which will allow us to use the order estructure of the field of real numbers.

Let f be a simple function with real values. There exists $a_i \in \mathbb{R}$ and measurable sets A_i, $1 \le i \le n$, such that

$$f = \sum_1^n a_i \chi_{A_i}.$$

The *integral of f with respect to ν over a measurable set A* is defined in the following way

$$\int_A f \, d\nu = \sum_1^n a_i \nu(A \cap A_i).$$

It is an element of the Banach space X, which is independent of the representation given to f as a linear combination of characteristic functions.

This definition allows to define the concept of integrability of a measurable function.

Definition 1.1. *Let $f\colon \Omega \longrightarrow \mathbb{R}$ be a measurable function. The function f is integrable with respect to ν if there exists a sequence (φ_n) of simple functions such that*

a) (φ_n) converges to f almost everywhere with respect to ν.

b) The sequence $\left(\int_A \varphi_n \, d\nu\right)$ converges in norm in X, for every $A \in \Sigma$.

In this case the integral of f with respect to ν over A is the element of X given by

$$\int_A f \, d\nu = \lim_n \int_A \varphi_n \, d\nu.$$

This definition does not depend on the sequence (φ_n). Lewis in [L 1] studied the integration of complex functions with respect to a measure with values in a Hausdorff locally convex topological vector space. He gives the following definition [L 1, Definition 2.1]. As we have already pointed out we will restrict our study to real valued functions.

Definition 1.2. *Let $f\colon \Omega \longrightarrow \mathbb{R}$ be a measurable function. The function f is integrable with respect to ν if*

a) f is integrable with respect to the measure $|x^*\nu|$, for every $x^* \in X^*$.

b) For every $A \in \Sigma$ there exists an element of X, denoted by $\int_A f \, d\nu$, such that

$$\left\langle x^*, \int_A f \, d\nu \right\rangle = \int_A f \, d x^*\nu, \quad \text{for every } x^* \in X^*.$$

For Banach space valued measures Lewis proves, using Vitali's convergence Theorem, that this definition is equivalent to the one given by Bartle, Dunford and Schwartz, [L 1, Theorem 2.4]. We will use both along the memoir.

Next we recall some of the basic properties of this integration theory, proven by Bartle, Dunford and Schwartz [BDS, Theorem 2.6] and by Lewis [L 1, Theorem 2.2].

Properties 1.3. *The following properties hold:*

1) *Let f be a measurable function essentially bounded with respect to ν, then f is integrable with respect to ν and*

$$\left\| \int_A f \, d\nu \right\| \leq \|f\|_\infty \cdot \|\nu\|(A).$$

2) *If f is integrable with respect to ν the set function*

$$A \in \Sigma \longmapsto \Phi(A) = \int_A f \, d\nu \in X$$

is a countably additive measure, that is absolutely continuous with respect to ν, that is

$$\lim_{\|\nu\|(A) \to 0} \left\| \int_A f \, d\nu \right\| = 0.$$

The semivaration of Φ is given by

$$\|\Phi\|(A) = \sup \left\{ \int_A |f| \, d\,|x^*\nu| : x^* \in B_{X^*} \right\}.$$

It should be remarked that the measure Φ in the previous proposition is absolutely continuous with respect to any control measure for ν. The integration of essentially bounded functions with respect to ν, that appears in 1.3.1), is known as the *Bartle integral* [DU, II.4].

Kluvánek and Knowles in their book [KK] consider the space of real functions which are integrable with respect to a vector measure. Their study is done for measures with values in Hausdorff locally convex topological vector spaces, but the best properties of the space are obtained for Banach space valued measures. We will just show the results obtained for this case.

Bartle, Dunford and Schwartz observed that the set of integrable functions with respect to a vector measure is a vector space. Kluvánek and Knowles prove that

$$\|f\|_\nu = \sup\left\{ \int |f| \, d\,|x^*\nu| : x^* \in B_{X^*} \right\}$$

is a seminorm in this space. Identifying two functions f and g when the set on which they differ has null semivariation

$$f \sim g \iff \|\nu\|\left(\{\omega : f(\omega) \neq g(\omega)\}\right) = 0$$

we obtain a normed space of classes of integrable functions with respect to ν that it is denoted by $L^1(\nu)$. This space is Banach space for the previous norm $\|\cdot\|_\nu$ [KK, II.2, Theorem IV.4.1 and Theorem IV.7.1].

The space $L^1(\nu)$ is a Banach lattice when endowed with the order given by

$$f \leq g \iff f(\omega) \leq g(\omega) \;\; \omega \notin Z \text{ for } Z \in \Sigma, \;\; \|\nu\|(Z) = 0.$$

Moreover, it is an *ideal of measurable functions*, that is, if g is in $L^1(\nu)$ and f is a measurable function such that $|f| \leq |g|$ almost everywhere with respect to ν, then f is in $L^1(\nu)$ and it follows that $\|f\|_\nu \leq \|g\|_\nu$.

Of great importance in our study is the following result [KK, Corollary II.4.2]:

In $L^1(\nu)$ *every order bounded increasing sequence is norm convergent.*

It follows that $L^1(\nu)$ is an order continuous Banach lattice, see Preliminaries. Kluvánek and Knowles did not consider this important consequence. It will turn out to be crucial in our study of the space $L^1(\nu)$. Next we include a proof of this result.

Let λ be a Rybakov control measure for ν. Let (f_n) be an increasing sequence in $L^1(\nu)$ order bounded by $g \in L^1(\nu)$. We can assume that (f_n) is non negative. Let us define the function $f(\omega) = \sup\{f_n(\omega) : n \in \mathbb{N}\}$. It is measurable and bounded by $g \in L^1(\nu)$. As $L^1(\nu)$ is an ideal of measurable functions, it follows that $f \in L^1(\nu)$. Let us see that (f_n) converges to f in $L^1(\nu)$. The sequence (f_n) is increasing and bounded by f in $L^1(\lambda)$, thus it is convergent in $L^1(\lambda)$, thanks to the Monotone Convergence Theorem. As it is increasing it follows that it converges almost everywhere with respect to λ, so, by Egoroff's Theorem, the convergence is almost uniform. Let $\varepsilon > 0$, there exists a measurable set A with $\lambda(A) < \varepsilon$ and there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we have

$$\begin{aligned}\|f - f_n\|_\nu &= \sup\left\{\int_\Omega |f - f_n| \, d|x^*\nu| : x^* \in B_{X^*}\right\} \\ &\leq \varepsilon \cdot \|\nu\|(\Omega \setminus A) + 2 \cdot \sup\left\{\int_A |f| \, d|x^*\nu| : x^* \in B_{X^*}\right\}.\end{aligned}$$

The claim follows as the measure with density f with respect to ν is absolutely continuous with respect to λ. Q.E.D.

It is easy to see that $L^1(\nu)$ as a Banach lattice has a weak unit: consider the function χ_Ω, it is in $L^1(\nu)$. Let $f \in L^1(\nu)$ such that $\inf\{|f|, \chi_\Omega\} = 0$, it follows that $f \equiv 0$. So χ_Ω is a weak unit of $L^1(\nu)$.

Let $x^* \in X^*$. From Definition 1.2 it follows that $L^1(\nu)$ is a linear subspace of $L^1(|x^*\nu|)$, and from the norm in $L^1(\nu)$ it follows that the injection

$$L^1(\nu) \longrightarrow L^1(|x^*\nu|)$$

is continuous and $\|f\|_{L^1(|x^*\nu|)} \leq \|x^*\| \cdot \|f\|_\nu$.

Let λ be a Rybakov control measure for ν. It is of the form $|x_0^*\nu|$ for $x_0^* \in B_{X^*}$. Then we have the following natural injections

$$L_\infty(\lambda) \longrightarrow L^1(\nu) \longrightarrow L^1(\lambda)$$

which are continuous. From Theorem 1.6 it follows that they both have dense range. As ν and λ have the same null sets and as $L^1(\nu)$ is an ideal, it follows that $L^1(\nu)$ is a Banach function space with respect to the measure space $(\Omega, \Sigma, \lambda)$.

The results of Kluvánek and Knowles for countably additive measures with values in a Banach space and the previous considerations are summarized in the next theorem.

Theorem 1.4. *Let $\nu\colon \Sigma \longrightarrow X$ be a vector measure with values in a Banach space. The space $L^1(\nu)$ is an order continuous Banach lattice with weak unit when endowed with the norm*

$$\|f\|_\nu = \sup\left\{ \int |f| \, d\,|x^*\nu| : x^* \in B_{X^*} \right\}$$

and the order

$$f \leq g \iff f(\omega) \leq g(\omega) \;\; \omega \notin Z \text{ for } Z \in \Sigma, \;\; \|\nu\|(Z) = 0.$$

$L^1(\nu)$ is a Banach function space with respect to the measure space $(\Omega, \Sigma, \lambda)$ where λ is a Rybakov control measure for ν.

Equivalent norm 1.5. The norm of a function $f \in L^1(\nu)$ coincides with the semivariation of the measure with density f with respect to ν (Proposition 1.3.2). It follows from the equivalent expression for the semivariation, see Preliminaries, that

$$|||f|||_\nu = \sup\left\{\left\|\int_A f \, d\nu\right\| : A \in \Sigma\right\}$$

is a norm equivalent to $\|\cdot\|_\nu$ in $L^1(\nu)$. In general it is not a lattice norm (condition 4 in the definition of Banach lattice). We have that

$$|||f|||_\nu \leq \|f\|_\nu \leq 2 \cdot |||f|||_\nu .$$

Lewis proves the following result [L 2, Theorem 3.5].

Theorem 1.6. *The simple functions are a dense set in* $L^1(\nu)$.

It is important that the theory has a "good" theorem of dominated convergence [BDS, Theorem 2.8] and [L 1, Theorem 2.2].

Theorem 1.7. *Let* (f_n) *be a sequence in* $L^1(\nu)$ *that converges almost everywhere with respect to* ν *to a function* f *and let* $g \in L^1(\nu)$ *such that* $|f_n| \leq g$ *for every* n. *Then* $f \in L^1(\nu)$ *and* (f_n) *converges to* f *in* $L^1(\nu)$.

Scalar integrability 1.8. Lewis in [L 1, Definition 2.5] defines the functions with generalized integral as those that satisfy condition a) in Definition 1.2. We will refer to them as *functions which are scalarly integrable with respect to* ν. It is easy to see that for these functions it holds that, for every $A \in \Sigma$

$$\int_A f \, d\nu \in X^{**},$$

as it is a pointwise limit of integrals of simple functions. Moreover, for these functions it follows from the Uniform Boundedness Principle that

$$\sup\left\{\int |f| \, d|x^*\nu| : x^* \in B_{X^*}\right\} < +\infty.$$

Lewis proves the following characterization of scalarly integrable functions which are integrable.

Theorem 1.9. *Let f be a function which is scalarly integrable with respect to the measure $\nu\colon \Sigma \longrightarrow X$. Consider the set function*

$$A \in \Sigma \longmapsto \Phi(A) = \int_A f \, d\nu \in X^{**} .$$

Then the following conditions are equivalent

a) *The function f is integrable with respect to ν.*

b) *Φ is a countably additive measure.*

c) *Φ is absolutely continuous with respect to ν, that is,*

$$\lim_{\|\nu\|(A)\to 0} \left\| \int_A f \, d\nu \right\| = 0.$$

Lewis characterizes the Banach spaces X with the property that for $X-$valued measures integrability and scalar integrability are equivalent as those spaces in which every weakly unconditionally Cauchy series is unconditionally convergent [L 1, Theorem 5.1]. Taking in account the results of Bessaga and Pelczynski [BP], the following characterization holds.

Theorem 1.10. *Let X be a Banach space. The following conditions are equivalent:*

a) *For every measure ν with values in X, if f is a function which is scalarly integrable with respect to ν, then f is integrable with respect to ν.*

b) *X does not contain a subspace isomorphic to c_0.*

Integration operator 1.11. A basic tool in the study of the space $L^1(\nu)$ is the *integration operator* defined by

$$f \in L^1(\nu) \longmapsto \nu(f) = \int f \, d\nu \in X \,,$$

which is linear and continuous, satisfying $\|\nu(f)\| \leq \|f\|_\nu$.

Let us consider the measure $|\nu|$ variation of the measure ν. Lewis studies the relation between the spaces $L^1(|\nu|)$ and $L^1(\nu)$ [L 2, Theorem 4.1, Theorem 4.2 and Corollary 4.3].

Theorem 1.12. *Let X be a Banach space, ν a measure with values in X and $|\nu|$ its variation. Then we have*

1) *If f is in $L^1(|\nu|)$ then f is in $L^1(\nu)$ and we have $\|f\|_\nu \leq \|f\|_{L^1(|\nu|)}$.*
2) *Let $f \in L^1(\nu)$ and let Φ be the measure with density f with respect to ν, then $f \in L^1(|\nu|)$ if and only if the measure Φ has bounded variation, and in this case*
$$|\Phi|(A) = \int_A |f| \, d\,|\nu| \qquad \text{for every } \; A \in \Sigma.$$

3) *X is finitedimensional if and only if for every X–valued measure ν we have that every function in $L^1(\nu)$ is in $L^1(|\nu|)$.*

It follows that when X is finitedimensional the spaces $L^1(\nu)$ and $L^1(|\nu|)$ coincide and their norms are equivalent.

Observations 1.13. **1.** A measure space $(\Omega, \Sigma, \lambda)$ is said to be *localizable* if for every measurable set A having non null measure there exists a measurable set $B \subseteq A$ such that $0 < \lambda(B) < +\infty$. Let $\nu\colon \Sigma \longrightarrow X$ be a countably additive measure. From the existence of a control measure for ν it follows, by applying an Exhaustion Lemma [DU, Lemma III.2.4], that the variation of ν is localizable iff it is σ–finite. The importance of this fact in our study relays on the fact that where the variation is not localizable the space $L^1(|\nu|)$ is just the zero vector. That is, if $|\nu|$ is not σ–finite there exist a measurable set A with $|\nu|(A) > 0$, such that for every measurable set $B \subset A$ we have either $|\nu|(B) = 0$ or $|\nu|(B) = +\infty$. Thus if ν_A is the restiction of ν to A, then we have $L^1(|\nu_A|) = \emptyset$.

2. Consider a measure $\nu\colon \Sigma \longrightarrow X$ where X is a closed subspace of a Banach space Y. The semivariation of ν does not change by considering that the measure takes its values in Y. It follows from Definition 1.1 that the concept of integrability also does not change and so neither does change the space $L^1(\nu)$. We conclude that, in order to study the space $L^1(\nu)$, we can consider that the measure ν takes its values in the space $[\nu(\Sigma)]$, the closure of the linear span of the range of ν.

3. Kluvánek and Knowles [KK, II.7] consider the direct sum of an arbitrary family of measures. We will use the direct sum of two measures $(\Omega_i, \Sigma_i, \nu_i)$ with values in Banach spaces X_i, $i = 1, 2$, defined by

$$A \in \Sigma \longmapsto \nu_1 \oplus \nu_2(A) = \nu_1(A \cap \Omega_1) \oplus \nu_1(A \cap \Omega_2) \in X_1 \oplus X_2\,,$$

where Ω is the disjoint union of the spaces Ω_1 and Ω_2 and Σ is the σ–algebra of sets A of Ω such that $A \cap \Omega_i \in \Sigma_i$, $i = 1, 2$. In this situation we have $L^1(\nu_1 \oplus \nu_2) = L^1(\nu_1) \oplus L^1(\nu_2)$, [KK, Theorem II.7.2].

4. For finite σ–algebras the resulting space $L^1(\nu)$ has finite dimension, equal to the cardinality of the σ–algebra. Thus we will not consider this case in our study.

Chapter IV.10 of the book by Dunford and Schwartz [DS] is devoted to this integration theory. Besides the already mentioned authors, Debiéve [D] and Thomas [Th] have also worked in this integration theory. Egghe [E] and Okada [O] have studied several aspects of the space $L^1(\nu)$. Other authors have considered in their works the space $L^1(\nu)$: Drewnowski [Dr]; Ghoussoub and Saab [GS]; Kalton, Turret and Uhl [KTU]. This integration theory is a particular case of the bilinear integrals studied by Bartle [B], Brooks and Dinculeanu [BD] and Dobrakov [Do 1 and 2].

SECTION 2. In this second section we start with our own study of the space $L^1(\nu)$.

Dual space 1.14. The order continuity of the space $L^1(\nu)$, Theorem 1.4, allows to represent the dual space of $L^1(\nu)$ through the theory of Banach lattices.

$L^1(\nu)$ is a Banach function space with respect to the measure space $(\Omega, \Sigma, \lambda)$ where λ is a Rybakov control measure for ν. As $L^1(\nu)$ is order continuous it follows that the space $L^1(\nu)^*$ can be represented as the Köthe dual of $L^1(\nu)$, see Preliminaries:

$$L^1(\nu)^* = \{ g\colon \Omega \longrightarrow \mathbb{R} : g \text{ is measurable, and } gf \in L^1(\lambda) \text{ for every } f \in L^1(\nu) \},$$

where the action of such functions over $L^1(\nu)$ is given by

$$f \in L^1(\nu) \longmapsto \int gf \, d\lambda \in \mathbb{R}.$$

As the space $L^1(\nu)$ contains the characteristic functions of measurable sets, it follows that the functions of $L^1(\nu)^*$ are in the space $L^1(\lambda)$. Moreover, it follows from the previous representation, that $L^1(\nu)^*$ is an *ideal of measurable functions* in $L^1(\lambda)$, that is, if $g \in L^1(\nu)^*$ and $h \in L^1(\lambda)$ such that $|h| \leq |g|$, then $h \in L^1(\nu)^*$.

The following natural inclusions hold

$$L_\infty(\lambda) \longrightarrow L^1(\nu)^* \longrightarrow L^1(\lambda),$$

where the operators are continuous injections, that satisfy: if $g \in L_\infty(\lambda)$ then $\|g\|_\infty \le \|g\|_{L^1(\nu)^*}$, and if $g \in L^1(\nu)^*$ then $\|g\|_{L^1(\nu)^*} \le \|\nu\|(\Omega) \cdot \|g\|_{L^1(\lambda)}$.

Especially relevant in the space $L^1(\nu)^*$ are the functionals given by

$$f \in L^1(\nu) \longmapsto \varphi_{x^*}(f) = \int f \, d\, x^*\nu \in \mathbb{R},$$

where $x^* \in X^*$. We have $\|\varphi_{x^*}\| \le \|x^*\|$. Consider the scalar measure $x^*\nu$. It is absolutely continuous with respect to λ. Thus there exists, by the Radon–Nikodym Theorem, a function h_{x^*} in $L^1(\lambda)$ such that

$$x^*\nu(A) = \int_A h_{x^*} \, d\,\lambda \quad \text{for every } \ A \in \Sigma.$$

It follows that for every $f \in L^1(\nu)$

$$\varphi_{x^*}(f) = \int f h_{x^*} \, d\,\lambda,$$

so the functional φ_{x^*} can be identified with the function h_{x^*} in $L^1(\lambda)$.

Later on we will consider in $L^1(\nu)^*$ the *lattice ideal* generated by the functions $\{h_{x^*} : x^* \in X^*\}$, that we will denote by $\mathcal{I}$. That is

$$\mathcal{I} = \left\{ h \in L^1(\nu)^* : \text{ there exists } x_1^*, \dots, x_n^* \in X^* \text{ with } |h| \le \sum_1^n |h_{x_i^*}| \right\}.$$

Consider the Köthe bidual of $L^1(\nu)$:

$$\left\{ g\colon \Omega \longrightarrow \mathbb{R} : g \text{ is measurable, and } gh \in L^1(\lambda) \quad \text{for every } \ h \in L^1(\nu)^* \right\}.$$

As the functions h_{x^*} are in $L^1(\nu)^*$ it follows that the Köthe bidual of $L^1(\nu)$ is included in the space of scalarly integrable functions with respect to ν. On the other hand, if f is an scalarly integrable function with respect to ν, there exists a sequence of simple functions that converge almost everywhere to f; as it also holds that $\|f\|_\nu < +\infty$, see 1.8, it follows from [LT vol. II, p. 30] that f is in the Köthe bidual of $L^1(\nu)$. Hence both spaces coincide.

The first problem that arises in the study of the space $L^1(\nu)$ is to determine what spaces are obtained as L^1 of a vector measure. It also arises in a natural way the question of whether the space $L^1(\nu)$ can be reflexive or a Hilbert space. The following theorem gives a complete answer to these problems, showing that the class of spaces obtained as L^1 of a vector measure coincides with the class of order continuous Banach lattices with weak unit.

Theorem 1.15. *Let E be an order continuous Banach lattice with weak unit. There exists a vector measure ν, with values in E, such that the space $L^1(\nu)$ is order isomorphic and isometric to E.*

PROOF. In these conditions, E is order isomorphic and isometric to a Banach function space with respect to a probability space $(\Omega, \Sigma, \lambda)$, see Preliminaries.

Consider the measure

$$A \in \Sigma \longmapsto \nu(A) = \chi_A \in E\,.$$

It is well defined as E is a Banach function space with respect to $(\Omega, \Sigma, \lambda)$. It is finitely additive. Let (A_n) be a sequence of disjoint measurable sets. Denote $B_n = \cup_1^n A_i$ for every n, and $B = \cup_1^\infty A_i$, they are measurable sets. The sequence of sets (B_n) is increasing, so the sequence $\nu(B_n) = \chi_{B_n}$ is increasing in E. As $B_n \subset B$ for every n, it follows that $\nu(B)$ is a bound in E for $\nu(B_n)$. From the order continuity of E we deduce that the sequence $\big(\nu(B_n)\big)$ is convergent in E, to its supremum, that is $\nu(B)$. Thus, the measure ν is countably additive.

As E is order continuous the dual space coincides with the Köthe dual (see Preliminaries):

$$E^* = \big\{\, g\colon S \longrightarrow \mathbb{R} \,:\, g \text{ is measurable and } gf \in L^1(\lambda) \text{ for every } f \in E \big\}\,,$$

where the action of these elements is given by integration with respect to λ.

Let $g \in E^*$. The measure $g\nu$ is

$$A \in \Sigma \longmapsto g\nu(A) = \int_A g \, d\lambda \in \mathbb{R}\,.$$

That is, is the measure with density g with respect to λ. A function $f\colon \Omega \longrightarrow \mathbb{R}$ is scalarly integrable with respect to ν if it is integrable with respect to all measures $g\,d\lambda$ for every $g \in E^*$.

Let $f \in E$, it follows that f is scalarly integrable with respect to ν. Let $A \in \Sigma$. The functional

$$g \in E^* \longmapsto \int_A gf \, d\lambda = \langle g, f \cdot \chi_A \rangle \in \mathbb{R}\,,$$

defines an element of E, since $f \cdot \chi_A$ belongs to E, for every $A \in \Sigma$. It follows that $f \in L^1(\nu)$ and

$$\int_A f \, d\nu = f \cdot \chi_A \quad \text{for every } A \in \Sigma.$$

On the other hand, if $f \in L^1(\nu)$ the previous funcional defines an element of E for every $A \in \Sigma$, so $f \in E$.

In fact we have an isometry between E and $L^1(\nu)$:

$$\begin{aligned} \|f\|_\nu &= \sup\left\{ \int |f| \, d\,|g\nu| : g \in B_{E^*} \right\} \\ &= \sup\left\{ \int |f||g| \, d\lambda : g \in B_{E^*} \right\} \\ &= \sup\left\{ |\langle f, g \rangle| : g \in B_{E^*} \right\} \\ &= \|f\|_E. \end{aligned}$$

We deduce from the proof that the integration operator is the identity operator from E to E. Q.E.D.

It follows from the previous theorem that the spaces $L^p[0,1]$ for $1 \leq p < +\infty$, Orlicz spaces satisfying the Δ_2 condition and Banach space with an unconditional basis, are among the spaces of the form $L^1(\nu)$.

In order to represent the order continuous Banach lattices without a weak unit we can use the theory of integration with respect to vector measures defined on δ–rings, first studied by Lewis [L 2] and later developed by Masani and Niemi [MN, 1 y 2]. Let ν be a countably additive vector measure defined over a δ–ring. The space $L^1(\nu)$ of real functions which are integrable with respect to ν in the sense of Lewis, Definition 1.2, is a Banach space, that Masani and Niemi denote by $P_{1,\nu}$ [MN 2, Theorem 4.7.c)]. It is a Banach lattice when endowed with the "almost everywhere with respect to ν" order. By using the existence of a control measure for ν [Br, Theorem 1] it is easy to prove that $L^1(\nu)$ is an order continuous Banach lattice, in a similar way as done in the case of measures defined over σ–algebras, see the discussion previous to Theorem 1.4. In fact we have the following extension of Theorem 1.15.

Let E be an order continuous Banach lattice. There exists a countably additive vector measure ν defined over a δ–ring and with values in E, such that the space $L^1(\nu)$ is order isomorphic and isometric to E.

The proof is based on the fact that as E is order continuous it can expressed as an unconditional sum of Banach lattices E_α, which are order continuous and have a weak unit [LT vol. II, Proposition 1.a.9]. For every E_α, in virtue of Theorem 1.15, there exists a space Ω_α, a σ–algebra Σ_α and a countably additive measure $\nu_\alpha\colon \Sigma_\alpha \longrightarrow E_\alpha$ such that $L^1(\nu_\alpha)$ is order isomorphic and isometric to

E_α. Let Ω be the disjoint union of the spaces Ω_α and consider the δ–ring

$$D = \left\{ A = \bigcup_{i \in I} A_i : I \subset \mathcal{A} \text{ is finite}, \; A_i \in \Sigma_i \right\}.$$

The measure

$$A = \bigcup_{i \in I} A_i \in D \longmapsto \nu(A) = \sum_{i \in I} \nu_i(A_i) \in E \,,$$

is well defined, is countably additive on D, thanks to the order continuity of E, and it can be verified that the identity map is a biyection that preserves order and norm between the spaces $L^1(\nu)$ y E.

Let us see an example of the previous representation. If Γ is an uncountable set, the space $\ell^1(\Gamma)$ is an order continuous Banach lattice without a weak unit. It is obtanied as $L^1(\nu)$ for the following measure ν, defined over the δ–ring D of finite subsets of Γ,

$$A \in D \longmapsto \nu(A) = \sum_{\gamma \in A} e_\gamma \in \ell^1(\Gamma) \,,$$

where e_γ is the characteristic function of the point γ.

We now study the posibility of obtaining spaces $L^1(\nu)$ from measures taking values in a certain fixed Banach space. We will see in Theorem 1.20 that if the measure ν is separable and has no atoms then the space $L^1(\nu)$ can be obtanied from a measure with values in the Banach space c_0. First we will study atoms and separability in $L^1(\nu)$.

Proposition 1.16. *Let $f \in L^1(\nu)$. Then f is an atom in $L^1(\nu)$ if and only if f is a multiple of χ_A, where $A \in \Sigma$ is an atom of ν.*

PROOF. Let f be an atom in $L^1(\nu)$. Suppose f is not constant where it is non null. Then there exists $a \in \mathbb{R}$, $a > 0$, such that the sets $A = \{\omega : f(\omega) \geq a\}$ and

$B = \{\omega : 0 < f(\omega) < a\}$ have both non null semivariation. Then $0 < f \cdot \chi_A < f$, but $f \cdot \chi_A$ is not a multiple of f, contradicting the fact that f is an atom in $L^1(\nu)$. So f is a multiple of the characteristic function of the set $A = \{\omega : f(\omega) > 0\}$. If A were not an atom of ν, there would exist $B \subset A$ such that both B and $A \setminus B$ have non null semivariation. The function $f \cdot \chi_B$ gives then a contradiction with f being an atom in $L^1(\nu)$. A similar argument proves that if A is an atom of ν, then χ_A is an atom in $L^1(\nu)$. Q.E.D.

Corollary 1.17. *$L^1(\nu)$ is atomic if and only if ν is purely atomic.*

Given a measurable space (Ω, Σ) and a measure $\nu : \Sigma \longrightarrow X$ we have the following associated pseudometric space (Σ, d_ν) where for $A, B \in \Sigma$ we define $d_\nu(A, B) = \|\nu\|(A \triangle B)$. Let λ be a control measure for ν. The space (Σ, d_ν) is homeomorphic to the pseudometric space (Σ, d_λ), associated to the measure space $(\Omega, \Sigma, \lambda)$. A vector measure ν defined over Σ is said to be *separable* if so is the space (Σ, d_ν).

Proposition 1.18. *$L^1(\nu)$ is separable if and only if the pseudometric space (Σ, d_ν) is separable.*

PROOF. Let λ be a Rybakov control measure for ν. The separability of the space (Σ, d_ν) is equivalent to the separability of the space (Σ, d_λ). It is well known that the separability of (Σ, d_λ) is equivalent to the separability of the space $L^1(\lambda)$. Suppose that $L^1(\nu)$ is separable. As the injection $L^1(\nu) \longrightarrow L^1(\lambda)$ is continuous and has dense image, it follows that $L^1(\lambda)$ is separable and so (Σ, d_ν) is separable. In order to prove the converse let (A_n) be a sequence of measurable sets that is dense in (Σ, d_ν). It is easy to verify that the simple functions with rational coeficients and supported over the sets (A_n) are dense in $L^1(\nu)$. Q.E.D.

The integration operator $\nu : L^1(\nu) \longrightarrow X$ is continuous. If we denote by

$[\nu(\Sigma)]$ the clousure of the linear span of range of ν then we have that $\nu\colon L^1(\nu) \longrightarrow [\nu(\Sigma)]$ is continuous and has dense image, as it contains the integrals of the simple functions. On the other hand we have seen that it is just the space $[\nu(\Sigma)]$ which determines the space $L^1(\nu)$, see 1.13. Thus we have the following proposition.

Proposition 1.19. *Let $\nu\colon\Sigma \longrightarrow X$ be a vector measure such that $L^1(\nu)$ is separable. Then there exists a linear subspace $Y \subset X$, separable, with $\nu(\Sigma) \subset Y$ such that $\nu\colon\Sigma \longrightarrow Y$ generates the same space $L^1(\nu)$.*

Theorem 1.20. *Let $\nu\colon\Sigma \longrightarrow X$ be a separable vector measure without atoms. Then there exists a measure $\mu\colon\Sigma \longrightarrow c_0$ such that the space $L^1(\mu)$ is order isomorphic and isometric to $L^1(\nu)$.*

PROOF. We have seen that if the measure ν is separable so is $L^1(\nu)$; as the simple functions are dense in $L^1(\nu)$ there exists a sequence (f_n) of simple functions which is dense in $L^1(\nu)$. Let λ be a Rybakov control measure for ν. Recall that $L^1(\nu)^*$ can be identified with a lattice ideal in $L^1(\lambda)$.

The proof will be perfomed in four steps.

FIRST STEP. There exists a sequence (h_n) of simple functions which is dense in the unit ball of $L^1(\nu)^*$ when we consider the weak–$*$ topology.

Let g be an element of the unit ball of $L^1(\nu)^*$. There exists a sequence (φ_n) of simple functions that converges pointwise to g satisfying $|\varphi_n| \le |g|$. Then we have that for every $f \in L^1(\nu)$

$$\langle \varphi_n, f\rangle = \int \varphi_n f\, d\lambda \quad \text{converges to} \quad \int g f\, d\lambda = \langle g, f\rangle.$$

Thus (φ_n) converges to g in the weak–$*$ topology of $L^1(\nu)^*$. As $|\varphi_n| \le |g|$, it follows that φ_n is in $L^1(\nu)^*$ and $\|\varphi_n\| \le \|g\| \le 1$. So the simpe functions with norm less or equal than one form a set which is weak–$*$ dense in unit ball of $L^1(\nu)^*$.

As the space $L^1(\nu)$ is separable, the unit ball of $L^1(\nu)^*$ is metrizable for the weak–$*$ topology. It follows that the unit ball of $L^1(\nu)^*$ is separable in the weak–$*$ topology. Thus there exists s sequence of simple functions which is weak–$*$ dense.

SECOND STEP. There exists an increasing sequence (Σ_n) of finite sub–σ–algebras of Σ and there exists a sequence (g_n) of simple functions satisfying

a) for every n, the functions f_n and g_n are measurable with respect to Σ_n,

b) for every n, $|g_n| = |h_n|$,

c) (g_n, Σ_n) is a martingale difference sequence, that is, for every n the conditional expectation of g_n with respect to Σ_{n-1} is null.

Set $g_1 = h_1$. Suppose already defined the functions $g_1, \dots, g_{n-1}$ and the σ–algebras $\Sigma_1, \dots, \Sigma_{n-2}$. Set

$$\Sigma_{n-1} = \sigma(f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1}),$$

the smallest σ–algebra with respect to which are measurable all the functions $f_1, \dots, f_{n-1}$ and $g_1, \dots, g_{n-1}$. As these are simpe functions, Σ_{n-1} is finite and so it is generated by a finite number of atoms.

Let A be a constancy set of the simple function h_n. Let B be atom of Σ_{n-1}. In order to define g_n it suffices to do so over each non empty set $A \cap B$. Suppose that $A \cap B$ is non empty. As the measure ν has no atoms, neither does λ. Thus, there exists two measurable disjoint sets C_1 and C_2 whose union is $A \cap B$ and such that $\lambda(C_1) = \lambda(C_2)$.

Define g_n in $A \cap B$ in the following way:

$$g_n(\omega) = \begin{cases} h_n(\omega) & \text{if } \omega \in C_1, \\ -h_n(\omega) & \text{if } \omega \in C_2. \end{cases}$$

The function g_n is simple and satisfies b). To verify that c) is satisfied it suffices to prove that if A is a set of constancy of h_n and B is an atom of Σ_{n-1}, then the integral of g_n over $A \cap B$ is null.

Let $\omega_0 \in A \cap B$:

$$\begin{aligned} \int_{A\cap B} g_n \, d\lambda &= \int_{C_1} g_n \, d\lambda + \int_{C_2} g_n \, d\lambda \\ &= \int_{C_1} h_n \, d\lambda - \int_{C_2} h_n \, d\lambda \\ &= h_n(\omega_0) \cdot \Big(\lambda(C_1) - \lambda(C_2) \Big) \\ &= 0 \end{aligned}$$

THIRD STEP. The following set function is a countably additive measure:

$$A \in \Sigma \longmapsto \mu(A) = \left(\int_A g_n \, d\lambda \right)_0^\infty \in c_0 \,,$$

where $g_0 \equiv 1$. We have that if f is in $L^1(\nu)$ then f is in $L^1(\mu)$.

Let f be in $L^1(\nu)$. Given $\varepsilon > 0$ by the density of the sequence (f_n) there exists $n_0 \in \mathbb{N}$ such that $\|f - f_{n_0}\|_\nu < \varepsilon$. Then

$$\begin{aligned} |\int g_n f \, d\lambda| &\le |\int g_n (f - f_{n_0}) \, d\lambda| + |\int g_n f_{n_0} \, d\lambda| \\ &= |\langle g_n, f - f_{n_0} \rangle| + |\int g_n f_{n_0} \, d\lambda| \\ &\le \varepsilon + |\int g_n f_{n_0} \, d\lambda|. \end{aligned}$$

For $n > n_0$ the function f_{n_0} is measurable with respect to Σ_{n-1}, thus, as (g_n, Σ_n) is a martingale difference sequence, the second integral is null. This proves that for every function f in $L^1(\nu)$

$$\left(\int f g_n \, d\lambda\right)_0^\infty \in c_0. \tag{1}$$

Considering $f = \chi_A$ for every measurable set A it follows that the measure μ is well defined.

Let $a^* = (a_n)$ be in $\ell^1 = c_0^*$. As the functions (g_n) are in the unit ball of $L^1(\nu)^*$ and the sequence (a_n) is summable, it follows that the series $\sum a_n g_n$ is absolutely summable in $L^1(\nu)^*$, thus absolutely summable in $L^1(\lambda)$. It follows that

$$a^*\mu(A) = \sum_0^\infty a_n \int_A g_n \, d\lambda = \int_A \left(\sum_0^\infty a_n g_n\right) d\lambda. \tag{2}$$

Thus the measure $a^*\mu$ is countably additive. So the measure μ is weakly countably additive, hence by the Orlicz–Pettis theorem it is countably additive.

Let f be in $L^1(\nu)$. From (2) it follows that f is integrable with respect to the measure $a^*\mu$. Thus f is scalarly integrable with respect to μ. From (1) it follows that $\int_A f \, d\mu$ is in c_0, for every $A \in \Sigma$. Hence f is in $L^1(\mu)$.

FOURTH STEP. The inclusion of $L^1(\nu)$ in $L^1(\mu)$ is surjective and norm preserving.

Let f be in $L^1(\mu)$. Let $x^* \in X^*$ be fixed. For every $n \in \mathbb{N}$ set $f_n = f \cdot \chi_{A_n}$, where $A_n = \{\omega : |f(\omega)| \le n\}$. As f_n is bounded, it is in $L^1(\nu)$. Let h_{x^*} be the Radon–Nikodym derivative of the measure $x^*\nu$ with respect to λ, it is in the unit ball of $L^1(\nu)^*$. Let (h_{n_i}) be a subsequence of (h_n) that converges in the weak–$*$ topology to $|h_{x^*}|$.

Then

$$\begin{aligned}\int |f_n|\, d\,|x^*\nu| &= \int |f_n||h_{x^*}|\, d\lambda \\ &= \lim_i \int |f_n| h_{n_i}\, d\lambda \\ &\leq \lim_i \int |f_n||h_{n_i}|\, d\lambda \\ &= \lim_i \int |f_n||g_{n_i}|\, d\lambda.\end{aligned} \tag{3}$$

Let e_n be the n–th vector of the canonical basis of ℓ^1. The measure $|e_1\mu|$ is the measure with density g_0 with respect to λ. It follows that λ is a Rybakov control measure for μ.

The operator

$$f \in L^1(\mu) \longmapsto \int f g_n\, d\lambda \in \mathbb{R}$$

is the composition of the integration operator with respect to μ with e_n, so it is a continuous linear functional with norm less or equal than one. It follows that

$$\int |f_n||g_{n_i}|\, d\lambda = \big\langle |f_n|, |g_{n_i}| \big\rangle \leq \|f_n\|_\mu \leq \|f\|_\mu. \tag{4}$$

From inequalities (3) and (4) we have

$$\int |f_n|\, d|x^*\nu| \leq \|f\|_\mu \text{ for every } n \in \mathbb{N}.$$

It follows that

$$\int |f|\, d|x^*\nu| \leq \|f\|_\mu.$$

So f is integrable with respect to the measure $|x^*\nu|$. Thus f is scalarly integrable with respect to ν and

$$\sup\left\{\int |f|\, d|x^*\nu| : x^* \in B_{X^*}\right\} \leq \|f\|_\mu. \tag{5}$$

In order to prove that f is in $L^1(\nu)$ we just have to apply the previous procedure to the functions $f_n - f_m$, which are in $L^1(\nu)$, obtaining from (5)

$$\|f_n - f_m\|_\nu \leq \|f_n - f_m\|_\mu.$$

From the order continuity of $L^1(\mu)$ if follows that (f_n) is a convergent sequence in $L^1(\mu)$ and so (f_n) is a Cauchy sequence in $L^1(\nu)$, which converges almost everywhere with respect to λ to f. Thus $f \in L^1(\nu)$ and from (5) we have $\|f\|_\nu \leq \|f\|_\mu$.

On the other hand, for $a^* = (a_n)$ in the unit ball of ℓ^1 we have seen that the measure $a^*\mu$ is the measure with density $\sum a_n g_n$ with respect to λ. We also know that the function $\sum a_n g_n$ is in $L^1(\nu)^*$ and has norm less or equal than one. It follows then that

$$\begin{aligned}\int |f|\, d|a^*\mu| &= \int |f| \cdot |\sum a_n g_n|\, d\lambda \\ &= \left\langle |f|, |\sum a_n g_n| \right\rangle \\ &\leq \|f\|_\nu\,.\end{aligned}$$

Taking supremum over the unit ball of ℓ^1 we obtain $\|f\|_\mu \leq \|f\|_\nu$. Q.E.D.

In order to study the previous theorem in the case of purely atomic measures we first study the space $L^1(\nu)$ obtained for these measures.

Purely atomic countably additive measures can be considered to be defined over the σ–algebra $\mathcal{P}(\mathbb{N})$ of subsets of the natural numbers. This is due to the fact that there can only be at most a countable number of atoms of non null measure. This follows from the fact that for every n there can only be a finite number of atoms, A_k, such that $\|\nu(A_k)\| \geq 1/n$, if this would not be the case the series $\sum \nu(A_k)$ would not be convergent, contradicting that $\nu(\cup A_k) = \sum \nu(A_k)$. Thus these measures are of the form

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum_{n \in A} x_n,$$

where $\sum x_n$ is a unconditionally convergent series.

Proposition 1.21. *Let $\nu\colon \mathcal{P}(\mathbb{N}) \longrightarrow X$ be a countably additive measure. Let $\nu(n) = x_n \in X$. Then $L^1(\nu)$ is the following sequence space*

$$\Big\{(a_n): \text{ the series } \sum a_n x_n \text{ converges unconditionally in } X\Big\},$$

the norm being

$$\|(a_n)\|_\nu = \sup\Big\{\sum |a_n x^* x_n| : x^* \in B_{X^*}\Big\}.$$

The space of scalarly integrable functions with respect to ν is

$$\Big\{(a_n): \text{ the series } \sum a_n x_n \text{ is weakly unconditionally Cauchy in } X\Big\}.$$

PROOF. Let $x^* \in X^*$. Consider the measure $x^*\nu$

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto x^*\nu(A) = \sum_{n\in A} x^* x_n \in \mathbb{R}.$$

Its variation is the measure

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto |x^*\nu|(A) = \sum_{n\in A} |x^* x_n| \in \mathbb{R}.$$

Let (a_n) be a real sequence. It is integrable with respect to the measure $|x^*\nu|$ if and only if

$$\sum_1^\infty |a_n||x^* x_n| < +\infty.$$

It follows that (a_n) is scalarly integrable with respect to ν if and only if the series $\sum a_n x_n$ is weakly unconditionally Cauchy in X.

In order to (a_n) be integrable with respect to ν it must be that for every $A \subset \mathbb{N}$ the funtional

$$x^* \in X^* \longmapsto \sum_{n \in A} a_n x^* x_n \in \mathbb{R}$$

defines an element of the space X, that is, by Orlicz–Pettis theorem, the series $\sum_A a_n x_n$ must converge unconditinally in X. The norm in $L^1(\nu)$ follows directly from the expression of the measure $|x^*\nu|$. Q.E.D.

From the previous proposition it follows that for measures defined over $\mathcal{P}(\mathbb{N})$ the sequence $\left(\chi_{\{n\}}\right)$ is an unconditional basis in $L^1(\nu)$. The order in $L^1(\nu)$ is the coordinatewise order. We also have $\|\chi_{\{n\}}\|_\nu = \|\nu(n)\|$.

The next proposition collects several results previously seen.

Proposition 1.22. *Let E be an order continuous Banach lattice with weak unit. The following properties are equivalent:*

a) E is purely atomic.

b) E is order isomorphic to $L^1(\nu)$ where ν is purely atomic.

c) In E relatively compact sets and L–weakly compact sets coincide.

d) E is a Banach space with an unconditional basis.

PROOF. The equivalence between a) and b) follows from Theorem 1.15 and Corollary 1.17. For the equivalence between a) and c) see the Preliminaries.

Let (y_n) be an unconditional basis for E. E is a Banach lattice for the coordinatewise order

$$\sum a_n y_n \leq \sum b_n y_n \iff a_n \leq b_n \text{ for every } n \in \mathbb{N},$$

and the equivalent norm

$$\left|\left|\left|\sum a_n y_n\right|\right|\right|_E = \sup\left\{\left\|\sum_{n\in A} a_n y_n\right\| : A \subset \mathbb{N}\right\}.$$

Let (a_n) be such that $a_n > 0$ for every n and such that $\sum a_n y_n$ is in E. The measure

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum_{n\in A} a_n y_n \in E$$

is countably additive. It follows from Proposition 1.21 that $L^1(\nu)$ is the following sequence space

$$\left\{(b_n): \quad \text{the series} \quad \sum b_n a_n y_n \quad \text{converges unconditionally in} \quad E\right\}.$$

The operator

$$(b_n) \in L^1(\nu) \longmapsto \sum b_n a_n y_n \in E$$

is an order preserving bijection. It is an isometry when we consider in $L^1(\nu)$ the equivalent norm $|||\cdot|||_\nu$. Q.E.D.

From Proposition 1.22 it follows that the space c_0 can be obtained as $L^1(\nu)$ from a measure $\nu\colon \mathcal{P}(\mathbb{N}) \longrightarrow c_0$.

To obtain the space ℓ^1 we just have to consider the measure $\nu\colon \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{R} \subset c_0$ definided by $\nu(n) = a_n$ where $a_n > 0$ for every n and $(a_n) \in \ell^1$ is given. Then

$$L^1(\nu) = \left\{(b_n) : \sum |a_n b_n| < +\infty\right\} \quad \text{and} \quad \|(b_n)\|_\nu = \sum |a_n b_n|.$$

Thus, the operator

$$(x_n) \in \ell^1 \longmapsto (x_n/a_n) \in L^1(\nu)$$

is an order and norm preserving bijection.

Consider a Banach space E with a normalized unconditional basis (y_n), not containing subspaces isomorphic to c_0 or ℓ^1, that is, E is reflexive. In E we consider the order given by the coordinates with respect to the basis (y_n). Then E is not isomorphic to $L^1(\nu)$ for a measure $\nu\colon \mathcal{P}(\mathbb{N}) \longrightarrow c_0$.

Suppose the contrary. Let $T\colon E \longrightarrow L^1(\nu)$ be an order preserving isomorphism. Consider the associated integration operator $\nu\colon L^1(\nu) \longrightarrow c_0$. For every n let $Ty_n = f_n \in L^1(\nu)$ and $\nu(f_n) = z_n \in c_0$. Then we have that the series $\sum a_n y_n$ converges unconditionally in E if and only if the series $\sum a_n z_n$ converges unconditionally in c_0. The sufficiency follows from the continuity of the operator $\nu \circ T$. As y_n is an atom in E, f_n is an atom in $L^1(\nu)$, thus it is a multiple of $\chi_{\{n\}}$ and so z_n is a multiple of $\nu(n)$. The necessity follows then from the identification of the space $L^1(\nu)$ given in Proposition 1.21.

As f_n is an atom in $L^1(\nu)$, it follows that $\|z_n\| = \|f_n\|_\nu$, thus there exists a constant $C > 0$ such that $\|z_n\| \geq C$ for every n. As the space E has no subspace isomorphic to ℓ^1, the basis (y_n) is shrinking, thus it is weakly null. From continuity it follows that the sequence (z_n) is weakly null in c_0. The Bessaga and Pelczynski Selection Principle [LT vol. I, Proposition 1.a.12] guarantees the existence of a subsequence (z_{n_k}) which is a basic sequence equivalent to a block basis in c_0, thus it is equivalent to the canonical basis of c_0 [LT vol. I, Proposition 2.a.1]. It follows that (z_{n_k}) is an unconditional basic sequence. Consider the restriction of the operator $\nu \circ T$ to the clousure in E of the linear span of the sequence (y_{n_k}). It is clear from what we have seen, that it is an isomorphism onto the clousure in c_0 of the linear span of (z_{n_k}). This last space is isomorphic to c_0. Thus E has a subspace isomorphic to c_0, which contradicts our assumptions.

Let us consider again the dual space of $L^1(\nu)$. In 1.14 we have seen a representation of the space $L^1(\nu)^*$. It is not too usefull for our study, in that it does not depend explicitly on the Banach space X in which the measure takes its values. Egghe in [E, Theorem 2.5] gives a representation of $L^1(\nu)^*$ for the case of measures with bounded variation as a quotient of the space $L_\infty(|\nu|, X^*)$; unfortunately neither the proof nor the result are correct [O, Example 2].

With the idea of finding tools for studying the space $L^1(\nu)^*$ and the weak topology in $L^1(\nu)$, let us consider the following condition. Let (f_α) be a net in $L^1(\nu)$ and let $f \in L^1(\nu)$. Consider the condition

$$\int_A f_\alpha \, d\nu \ \text{ converges weakly to } \ \int_A f \, d\nu \ \text{ in } X \text{ for every } A \in \Sigma. \qquad (*)$$

Weak convergence in $L^1(\nu)$ always implies condition $(*)$ due to the continuity of the integration operator and of the restriction operator $f \in L^1(\nu) \longmapsto f \cdot \chi_A \in L^1(\nu)$, for every $A \in \Sigma$. But the condition $(*)$ is not equivalent to the weak convergence of the net. This is easily seen by considering the space $L^1[0,1]$ obtained from the Lebesgue measure on [0,1].

We will consider condition $(*)$ for bounded nets. We have the following result, where $\mathcal{I}$ is the ideal generated in $L^1(\nu)^*$ by the functions $\{h_{x^*} : x^* \in X^*\}$, see 1.14.

Theorem 1.23. *Consider the following conditions:*

a) *$L^1(\nu)$ has no complemented subspace isomorphic to the space ℓ^1.*

b) *The ideal $\mathcal{I}$ is dense in $L^1(\nu)^*$.*

c) *Weak convergence in $L^1(\nu)$ of bounded nets is characterized by the weak convergence in X of the integrals over arbitrary sets, that is, if (f_α) is a bounded net in $L^1(\nu)$ then*

$$f_\alpha \xrightarrow{w} f \text{ in } L^1(\nu) \iff \int_A f_\alpha \, d\nu \xrightarrow{w} \int_A f \, d\nu \ \text{ in } X, \text{ for every } A \in \Sigma.$$

Then a) implies b) and b) implies c).

PROOF. a) $\Rightarrow$ b) Suppose $L^1(\nu)$ has no subspace isomorphic to ℓ^1. It follows from results of Bessaga and Pelczynski [BP, Theorem 4] that $L^1(\nu)^*$ has no subspace isomorphic to ℓ_∞. As $L^1(\nu)^*$ is an order complete Banach lattice, as it is a dual Banach lattice, it follows that it is order continuous [AB, Theorem 14.9]. In order continuous Banach lattices closed ideals are bands [M 3, Corollary 2.4.4]. As $L^1(\nu)$ is order continuous it follows that the bands in $L^1(\nu)^*$ are closed for the weak–$*$ topology [M 3, Corollary 2.4.7]. Thus the clousure of the ideal $\mathcal{I}$, which is an ideal itself, is closed in the weak–$*$ toplogy of $L^1(\nu)^*$.

On the other hand, the ideal $\mathcal{I}$ is total for the weak–$*$ topology. To see this let $f \in L^1(\nu)$ be such that $\langle h, f\rangle = 0$ for every $h \in \mathcal{I}$. It follows that for every $x^* \in X^*$ and for every $A \in \Sigma$

$$\int_A f \, d\, x^*\nu = \int_A f h_{x^*} \, d\,\lambda = \big\langle h_{x^*}\chi_A, f\big\rangle = 0.$$

From where we deduce that $f \equiv 0$, what proves our claim. From being total it follows, as it is a linear subspace, that it is dense in $L^1(\nu)$ for the weak–$*$ topology.

Thus the clousure of the ideal $\mathcal{I}$ is weak–$*$ closed and weak–$*$ dense, thus it coincides with $L^1(\nu)^*$.

b) $\Rightarrow$ c) We have to prove that condition $(*)$ implies weak convergence. It suffices to prove it for $f = 0$. Let (f_α) be a bounded net in $L^1(\nu)$ such that

$$\int_A f_\alpha \, d\, x^*\nu \longrightarrow 0 \quad \text{for every } x^* \in X^* \text{ and for every } A \in \Sigma.$$

Fix $x^* \in X^*$. By considering the Hahn decomposition of the measure $x^*\nu$, we deduce that the net (f_α) is weakly null in $L^1(|x^*\nu|)$.

Let λ be a Rybakov control measure for ν. Let h be an element of the ideal

$\mathcal{I}$. Then there exists $x_1^*, \ldots, x_n^*$ in X^* such that

$$|h| \leq \sum_1^n |h_{x_i^*}|.$$

Let μ be the measure with density $|h|$ with respect to λ. From the previous inequality we have the following continuous injection

$$\left[\bigoplus_1^n L^1(|x_i^*\nu|)\right]_1 \longrightarrow L^1(\mu)\,.$$

Thus the net (f_α) is weakly null in $L^1(\mu)$. It follows that for every $A \in \Sigma$

$$\int_A f_\alpha |h| \; d\,\lambda \longrightarrow 0.$$

By decomposing h in its positive and negative parts we deduce that

$$\langle h, f_\alpha \rangle = \int_A f_\alpha h \; d\,\lambda \longrightarrow 0. \tag{6}$$

From the density of the ideal $\mathcal{I}$ in $L^1(\nu)^*$ and from the boundedness of the net (f_α), it follows that (6) is satisfied for every $h \in L^1(\nu)^*$, thus (f_α) tends weakly to zero in $L^1(\nu)$. Q.E.D.

It should be observed in relation with condition a) in the previous theorem that, as $L^1(\nu)$ is an order continuous Banach lattice, it follows from a result of Tzafriri [T, Theorem 16] that whenever it contains a subspace isomorphic to ℓ^1 in fact it contains a complemented subspace isomorphic to ℓ^1. The implication a) $\Rightarrow$ c) has been proved by Okada for sequences of functions [O, Corollary 16]. For sequences of functions it follows from the Nikodym boundedness Theorem [DU, Theorem I.3.1] that condition $(*)$ implies norm boundedness of the sequence.

Condition c) does not imply in general condition a). Consider the space ℓ^1 obtained from the measure with values in ℓ^1 given in the proof of Theorem 1.15, it satisfies c) as the integration operator is the identity operator.

If the measure ν has relatively compact range, for every $f \in L^1(\nu)$ the measure with density f with respect to ν has also relatively compact range. It follows from the results of Lewis [L 3, Corollary 3.3] that, for measures with relatively compact range, condition $(*)$ characterizes weak convergence of sequences in $L^1(\nu)$. If in addition the space X^* has the Radon–Nikodym property, then, from the work of Graves and Ruess on convergence of measures, it follows that the previous result extends to bounded nets [GR, Corollary 7.3].

The next example exhibits a measure ν for which condition $(*)$ does not characterize weak convergence of sequences in $L^1(\nu)$.

Example 1.24. Let $\mathcal{M}$ be the σ–algebra of Lebesgue measurable sets of the interval $[0,+\infty)$ and let m be the Lebesgue measure on the interval. Let r_n be the Rademacher functions, defined in $[0,+\infty)$ by $r_n(t) = \operatorname{sign}(\sin(2^n \pi t))$. Consider the measure

$$A \in \mathcal{M} \longmapsto \nu(A) = \sum_1^\infty \frac{1}{2^k} \nu_k(A) \in \ell^2 ,$$

where the measures ν_k are defined in the following way

$$\nu_k(A) = \left(\overbrace{0, \dots, 0}^{k-1}, \int_{A \cap [k-1,k]} r_k \, dm, \int_{A \cap [k-1,k]} r_{k+1} \, dm, \dots \right) .$$

Each measure ν_k is well defined, countably additive and satisfies

$$\|\nu_k(A)\|_2 \le \|\chi_{A \cap [k-1,k]}\|_{L^2([k-1,k])} = m(A \cap [k-1,k])^{1/2}.$$

Thus the measure ν is well defined and countably additive. Consider in $L^1(\nu)$ the sequence (f_n) where $f_n = 2^n \cdot \chi_{[n-1,n]}$. As the function f_n is supported on the interval $[n-1,n]$, we have:

$$\|f_n\|_\nu = \|\nu_n\|([n-1,n]) \le 1.$$

For every $A \in \mathcal{M}$ we have:

$$\int_A f_n \, d\nu = \nu_n(A \cap [n-1, n]).$$

This vector, with norm less or equal than one, is in the subspace generated by the vectors $e_n, e_{n+1}, \ldots$ of the canonical basis of ℓ^2, Thus the sequence $(\int_A f_n \, d\nu)$ tends weakly to zero in ℓ^2.

Let $a_1, \ldots, a_N$ be scalars. For every n, $1 \leq n \leq N$, consider the set

$$A_n = \{t \in [n-1, n] : r_N(t) = \operatorname{sign}(a_n)\}.$$

Then we have that

$$\int_{A_n} r_k \, dm = \begin{cases} 0, & \text{si } k \neq N; \\ (1/2) \cdot \operatorname{sign}(a_n), & \text{si } k = N. \end{cases}$$

Thus we have that $\nu_n(A_n) = (1/2) \cdot \operatorname{sign}(a_n) \cdot e_N$. Let $A = \cup_1^N A_n$. Then

$$\begin{aligned} \left\| \sum_1^N a_n f_n \right\|_\nu &\geq \left\| \sum_1^N a_n \int_A f_n \, d\nu \right\|_2 \\ &= \left\| \sum_1^N a_n \nu_n(A_n) \right\|_2 \\ &= \left\| \sum_1^N a_n (1/2) \cdot \operatorname{sign}(a_n) \cdot e_N \right\|_2 \\ &= (1/2) \sum_1^N |a_n|. \end{aligned}$$

As the sequence (f_n) is bounded, it follows from what we have previously seen that (f_n) is a sequence equivalent in $L^1(\nu)$ to the canonical basis of ℓ^1. Thus (f_n) does not tend weakly to zero in $L^1(\nu)$.

The measure ν has unbounded variation (to see this just take into account that ν is absolutely continuous with respect to the Lebesgue measure and that the space ℓ^2 has the Radon–Nikodym property). This is not relevant, as the same construction can be done with values in c_0 and the resulting measure has bounded variation.

CHAPTER 2: Properties of the space $L^1(\nu)$.

In the previous chapter have seen that every space $L^1(\nu)$ that is separable and has no atoms can be obtained from a measure with values in c_0. It follows that the properties of $L^1(\nu)$ do not determine, in general, the properties of the Banach space X. On the contrary, the properties of ν and X do determine the properties of the space $L^1(\nu)$: in what extend this occurs is the object of study of this chapter.

The space $L^1(\nu)$ is weakly compactly generated, that is, there exists a relatively weakly compact set whose linear span is dense in $L^1(\nu)$. This result can be deduced from the general theory of Banach lattices, as $L^1(\nu)$ is order continuous [BVL]. We include a direct proof of this result that is based on the properties of the range of a vector measure and as such emphasizes the role of the theory of vector measures in the study of the space $L^1(\nu)$.

Theorem 2.1. $L^1(\nu)$ *is weakly compactly generated.*

PROOF. Consider the following set function:

$$A \in \Sigma \longmapsto \Phi(A) = \chi_A \in L^1(\nu) \ .$$

It is clearly a finitely additive vector measure. As $\|\Phi(A)\| = \|\chi_A\|_\nu = \|\nu\|(A)$ for every $A \in \Sigma$, and the semivariation of ν is absolutely continuous with respect to a control measure, it follows that Φ is countably additive. Thus its range is a relatively weakly compact set in $L^1(\nu)$ [DU, Corollary I.2.7]. The linear span of the range of Φ is the set of simple functions, which is dense in $L^1(\nu)$, Theorem 1.6, so the theorem is proved. Q.E.D.

Let us consider the sequential completeness of the space $L^1(\nu)$ for the weak topology. For Banach lattices the property of being weakly sequentially complete is equivalent to not containing a subspace isomorphic to c_0 [LT vol. II, Theorem 1.c.4]. The following theorem shows that the property of not containing subspaces isomorphic to c_0 is transmited from the Banach space X to the space $L^1(\nu)$. We include three proofs of this result, each one of them using different techniques.

Theorem 2.2. *Let X be a Banach space that has no subspace isomorphic to c_0. Then $L^1(\nu)$ has no subspace isomorphic to c_0.*

FIRST PROOF. As $L^1(\nu)$ is a Banach lattice, it does not contain a subspace isomorphic to c_0 if and only if every norm bounded increasing sequence is norm convergent [LT vol. II, Theorem 1.c.4]. Let (f_n) be a norm bounded increasing sequence in $L^1(\nu)$. Let $x^* \in X^*$, the sequence (f_n) is norm bounded and increasing in the space $L^1(|x^*\nu|)$. As $|x^*\nu|$ is a positive measure, the Theorem of Beppo–Levi guarantees that (f_n) is convergent in $L^1(|x^*\nu|)$ to a function $f_{x^*} \in L^1(|x^*\nu|)$. Let λ be a Rybakov control measure for ν. The above argument shows that (f_n) converges in $L^1(\lambda)$ to a function $f \in L^1(\lambda)$. As the sequence is increasing, there exists a measurable set Z with $\lambda(Z) = 0$ such that $(f_n(\omega))$ converges to $f(\omega)$ for $\omega \notin Z$. As the measure ν is absolutely continuous with respect to λ, for every $x^* \in X^*$ we have $|x^*\nu|(Z) = 0$. It follows that $f = f_{x^*}$ in $L^1(|x^*\nu|)$. So $f \in L^1(|x^*\nu|)$ for every $x^* \in X^*$. Thus the function f is scalarly integrable.

The Banach space X has no subspace isomorphic to c_0. It follows from the characterization of Lewis, Theorem 1.10, that the functions which are scalarly integrable with respect to ν, are in fact integrable. Hence $f \in L^1(\nu)$.

The sequence (f_n) is increasing and is order bounded in $L^1(\nu)$ by the function f. From the order continuity of $L^1(\nu)$ (Theorem 1.4) it follows that (f_n) is norm convergent, in $L^1(\nu)$. Q.E.D.

SECOND PROOF. We will proof the counterreciprocal result. Assume that $L^1(\nu)$ contains a subspace isomorphic to c_0. Let (f_n) be a sequence in $L^1(\nu)$ that is a basic sequence equivalent in $L^1(\nu)$ to the canonical basis of c_0. It satisfies that

$$\text{there exists } M > 0 \text{ such that } \|f_n\|_\nu \geq M \text{ for every } n \in \mathbb{N}, \tag{1}$$

and that $\sum f_n$ is a weakly unconditionally Cauchy series in $L^1(\nu)$.

Let λ be a Rybakov control measure for ν. The injection

$$L^1(\nu) \longrightarrow L^1(\lambda)$$

is continuous, thus the series $\sum f_n$ is weakly unconditionally Cauchy in $L^1(\lambda)$. As the space $L^1(\lambda)$ has no subspace isomorphic to c_0, due to a result of Bessaga y Pelczynski [BP, Theorem 5] we deduce that the series $\sum f_n$ is unconditionally convergent in $L^1(\lambda)$. Hence the sequence (f_n) tends to zero in norm in $L^1(\lambda)$. It follows that there exists a subsequence (f_{n_k}) that tends to zero almost everywhere with respect to λ.

Suppose that for every measurable set A we have

$$\int_A f_{n_k}\, d\nu \longrightarrow 0.$$

Then the sequence (f_{n_k}) tends to zero almost everywhere with respect to λ and satisfies that $(\int_A f_{n_k}\, d\nu)$ tends to zero in X for every measurable set A. It follows, from the Theorem of Egoroff and the Theorem of Vitali–Hahn–Saks [DU, Corollary I.4.10], that (f_{n_k}) tends to zero in norm in $L^1(\nu)$, which contradicts (1).

Thus, there exist a measurable set A such that the sequence $(\int_A f_{n_k}\, d\nu)$ does not tend to zero in X. We can assume, by passing to a subsequence if necessary, that there exists a constant $C > 0$ such that

$$\left\| \int_A f_{n_k}\, d\nu \right\| \geq C \quad \text{for every } k \in \mathbb{N}. \tag{2}$$

As the series $\sum f_n$ is weakly unconditionally Cauchy in $L^1(\nu)$ and the map

$$f \in L^1(\nu) \longmapsto \int_A f\, d\nu \in X$$

is continuous, it follows that the series $\sum \int_A f_{n_k}\, d\nu$ is weakly unconditionally Cauchy in X and that the sequence $(\int_A f_{n_k}\, d\nu)$ tends weakly to zero in X. This conditions, added to (2), guarantee, by the Selection Principle of Bessaga and Pelczynski [BP, Theorem 5], that there exists a subsequence of $(\int_A f_{n_k}\, d\nu)$, in X, that is a basic sequence equivalent to the canonical basis of c_0. Q.E.D.

The previous proof shows that if $L^1(\nu)$ has a subspace isomorphic to c_0, there exists a subspace of $L^1(\nu)$ isomorphic to c_0 on which the integration operator is an isomorphism. If (g_n) is the subsequence of (f_n) such that the sequence $(\int_A g_n\, d\nu)$ is equivalent to the canonical basis of c_0 in X, the mentioned subspace is the clousure in $L^1(\nu)$ of the linear span of the functions $\{g_n \cdot \chi_A\}$.

THIRD PROOF. Suppose that X has no subspace isomorphic to c_0. It follows from the characterization of Lewis, Theorem 1.10, that the space of integrable functions with respect to ν coincides with the space of scalarly integrable functions. We know, 1.14, that this last space is the Köthe bidual of $L^1(\nu)$. Thus $L^1(\nu)$ coincides with its Köthe bidual. This is equivalent to $L^1(\nu)$ satisfying the *Fatou property* (see [LT vol. II, p. 30]), which is the following

> Let (f_n) be an increasing and norm bounded sequence in $L^1(\nu)$. Then if $f = \sup f_n$ we have that f is in $L^1(\nu)$ and $(\|f_n\|_\nu)$ converges to $\|f\|_\nu$.

Thus $L^1(\nu)$ is an order continuous Banach lattice with the Fatou property. Let (f_n) be an increasing and norm bounded sequence in $L^1(\nu)$. From Fatou's property it follows that $f = \sup f_n$ is in $L^1(\nu)$. Hence (f_n) is an increasimg sequence which is order bounded. As $L^1(\nu)$ is order continuous it follows that (f_n) is convergent. Thus, $L^1(\nu)$ has no subspace isomorphic to c_0. Q.E.D.

Prof. I. Dobrakov has pointed out to the author that the following Corollary to Theorem 2.2 can be deduced from results of J. K. Brooks and N. Dinculeanu on bilinear integration [BD, Corollary 8.10 and Theorem 9.1].

Corollary 2.3. *Let X be a weakly sequentially complete Banach space. Then $L^1(\nu)$ is weakly sequentially complete.*

The reverse implication to Theorem 2.2 is not true as the following example shows. From Theorem 1.20 it follows that there exists a measure ν with values in c_0 for which $L^1(\nu) = L^1[0,1]$, space that does not contain c_0. In the next example we explicitly build a measure satisfying the above conditions and for which the linear span of the range is dense in c_0.

Example 2.4. Let $\mathcal{M}$ be the σ–algebra of Lebesgue measurable sets of the interval [0,1] and let m be the Lebesgue measure on the interval. Let us consider the measure

$$A \in \mathcal{M} \longmapsto \nu(A) = \left(\int_A r_n \, dm \right) \in c_0 \, ,$$

where r_n are the Rademacher functions defined by $r_n(t) = \operatorname{sign}(\operatorname{sen}(2^n \pi t))$ for $n = 0, 1, \ldots$. The Riemann–Lebesgue Lemma shows that ν is well defined. From the bound

$$\left\| \left(\int_A r_n \, dm \right) \right\|_\infty \leq m(A) \quad \text{for every} \quad A \in \mathcal{M}$$

it follows that ν is countably additive, has bounded variation and $|\nu| \leq m$. Considering $x^* = e_k$ the k–th vector of the canonical basis of $c_0^* = \ell^1$, we have that $x^*\nu(A) = \int_A r_k \, dm$, so $|x^*\nu| \equiv m$. Hence we deduce that the variation of the measure ν is the Lebesgue measure m.

Let $x^* = (a_n) \in \ell^1$. As the sequence (a_n) is summable and the functions r_n bounded in $L^1[0,1]$ we have

$$\langle x^*, \nu(A) \rangle = \sum a_n \int_A r_n \, dm = \int_A \left(\sum a_n r_n \right) \, dm. \tag{3}$$

Thus the measure $x^*\nu$ is the measure with density $\sum a_n r_n$ with respect to m. Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a measurable function. Suppose that it is scalarly integrable with respect to ν. For $x^* = e_k$ we have $|x^*\nu| = m$, so $f \in L^1[0,1]$. On the other hand from (3) we deduce the bound $|x^*\nu|(A) \leq \|x^*\| \cdot m(A)$. Thus, if $f \in L^1[0,1]$ it follows that $f \in L^1(|x^*\nu|)$ for every $x^* \in X^*$, so f is scalarly integrable. That is, the space of scalarly integrable functions with respect to ν is $L^1[0,1]$.

Let $f \in L^1(\nu)$. Consider the integration operator

$$f \in L^1(\nu) \longmapsto \nu(f) = \int f \, d\nu \in c_0 \, .$$

Let $e_k \in \ell^1$. Then

$$\left\langle e_k, \int f \, d\nu \right\rangle = \int f \, d\, e_k \nu = \int f \, r_k \, dm.$$

It follows that $\nu(f) = \int f \, d\nu = \left(\int f \, r_n \, dm\right)_0^\infty$. Given f scalarly integrable, in order to have integrability we need for every measurable set A to have $\int_A f \, d\nu \in c_0$, which is true, by the Riemann–Lebesgue Lemma, for every $f \in L^1[0,1]$. Hence $L^1(\nu)$ and $L^1[0,1]$ coincide as sets.

The norm in $L^1(\nu)$ is given by the following expression

$$\|f\|_\nu = \sup\left\{ \int_0^1 |f| \left| \sum a_n r_n \right| \, dm : \|(a_n)\|_1 \leq 1 \right\}.$$

Considering $(a_n) = e_k$ we deduce that $\|f\|_{L^1[0,1]} \leq \|f\|_\nu$. The reverse inequality follows from $|\sum a_n r_n| \leq 1$ for $\|(a_n)\|_1 \leq 1$. Thus the space $L^1(\nu)$ is order isometric to $L^1[0,1]$.

The linear span of the range of the measure ν is dense in c_0. To see this consider the sets $A_n = \{t \in [0,1] : r_n(t) = 1\}$. We have $\nu(A_n) = 1/2 \cdot e_n$ where e_n is the n–th vector of the canonical basis of c_0.

Next we study how cotype translates from the Banach space X to $L^1(\nu)$. A Banach space has *cotype* q, for $2 \le q < +\infty$, if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for any elements $x_1, \dots, x_n$ in X we have

$$\left(\sum_1^n \|x_i\|^q \right)^{1/q} \le C \cdot \int_0^1 \left\| \sum_1^n r_i(t) \cdot x_i \right\| \, dt,$$

where (r_n) is the sequence of Rademacher functions on the interval [0,1]. For Banach lattices the property of having cotype $q > 2$ is equivalent to satisfying a *lower q–estimate* [LT vol. II, p. 88]: there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and any pairwise disjoint elements $x_1, \dots, x_n$ in X we have

$$\left(\sum_1^n \|x_i\|^q \right)^{1/q} \le C \cdot \left\| \sum_1^n x_i \right\|.$$

For $q = 2$ the situation is different. In this case satisfying a lower 2–estimate does not imply having cotype 2 [LT vol. II, Example 1.f.19]. Having cotype 2 is equivalent to being 2–concave [LT vol. II, Theorem 1.f.16]. A Banach lattice is *q–concave* if there exists a constant $C >$ such that for every $n \in \mathbb{N}$ and for any elements $x_1, \dots, x_n$ in X we have

$$\left(\sum_1^n \|x_i\|^q \right)^{1/q} \le C \cdot \left\| \left(\sum_1^n |x_i|^q \right)^{1/q} \right\|.$$

Theorem 2.5. *Let X be a Banach space with cotype q for $q \ge 2$ and let ν be a measure with values in X. Then $L^1(\nu)$ has cotype q.*

PROOF. Suppose that $q > 2$. We will see that $L^1(\nu)$ satisfies a lower q–estimate. Let $f_1, \dots, f_n$ be disjoint functions in $L^1(\nu)$. Let $(A_i)_1^n$ be disjoint measurable sets such that each A_i is contained in the support of f_i. Then

$$\int_{\cup A_i} (\sum_1^n f_i) \, d\nu = \sum_1^n \int_{A_i} f_i \, d\nu.$$

Denote $x_i = \int_{A_i} f_i \, d\nu \in X$. Let $(\theta_i)_1^n$ be an arbitrary choice of signs $\theta_i = \pm 1$. As the functions f_i are disjoint it follows that

$$\left\| \sum_1^n \theta_i \cdot x_i \right\| \leq \left\| \sum_1^n \theta_i \cdot f_i \right\|_\nu = \left\| \sum_1^n f_i \right\|_\nu .$$

Averaging over all posible choices of signs we get

$$\frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\| \sum_1^n \theta_i \cdot x_i \right\| \leq \left\| \sum_1^n f_i \right\|_\nu . \tag{4}$$

Let us consider the Rademacher functions $r_1, \ldots, r_n$. The average that appears in the first member of (4) can be written as

$$\frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\| \sum_1^n \theta_i \cdot x_i \right\| = \int_0^1 \left\| \sum_1^n r_i(t) \cdot x_i \right\| dt. \tag{5}$$

As the Banach space X has cotype q there exists a positive constant C such that

$$\left(\sum_1^n \|x_i\|^q \right)^{1/q} \leq C \cdot \int_0^1 \left\| \sum_1^n r_i(t) \cdot x_i \right\| dt. \tag{6}$$

From (4), (5) and (6) it follows that

$$\left(\sum_1^n \left\| \int_{A_i} f_i \, d\nu \right\|^q \right)^{1/q} \leq C \cdot \left\| \sum_1^n f_i \right\|_\nu .$$

Consider the previous inequality, taking supremum over all posible choices of sets $(A_i)_1^n$, and considering the equivalent norm $||| \cdot |||_\nu$ in $L^1(\nu)$, we deduce that

$$\left(\sum_1^n \|f_i\|_\nu^q \right)^{1/q} \leq 2\,C \cdot \left\| \sum_1^n f_i \right\|_\nu .$$

Hence $L^1(\nu)$ satisfies a lower q–estimate and thus it has cotype q.

Let $q = 2$. We will prove that $L^1(\nu)$ has cotype 2 by showing that it is 2–concave. Let $f_1, \ldots, f_n$ be in $L^1(\nu)$. Set $f = \left(\sum_1^n |f_i|^2\right)^{1/2}$, it is in $L^1(\nu)$. Consider the ideal generated by f in $L^1(\nu)$

$$I(f) = \left\{ g \in L^1(\nu) \, : \exists \lambda > 0, 0 \leq |g| \leq \lambda f \right\}$$

with the norm

$$\|g\|_\infty = \inf\Big\{\lambda \ge 0 : |g| \le \lambda \cdot f/\|f\|\Big\}$$

The completion of $(I(f), \|\cdot\|_\infty)$ is an AM–space with unit, so in virtue of a result of Kakutani [LT vol. II, Theorem 1.b.6] is order isometric to a space $C(K)$. As for every function g in $I(f)$ we have $\|g\|_\infty \le \|g\|_\nu$, the inyection

$$j : C(K) \longrightarrow L^1(\nu)$$

has norm one and $\|f\|_\infty = \|f\|_\nu$. Consider the composition of this injection with the integration operator $\nu : L^1(\nu) \longrightarrow X$, that is

$$\nu \circ j : C(K) \longrightarrow X \,.$$

As X has cotype 2, by Grothendieck's Theorem [P, Theorem 5.14], the operator $\nu \circ j$ is 2–summing. Thus there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for any functions $g_1, \dots, g_n$ in $C(K)$ we have

$$\left(\sum_1^n \|\nu \circ j(g_i)\|^2\right)^{1/2} \le C \cdot \sup\left\{\left(\sum_1^n |\langle \mu, g_i\rangle|^2\right)^{1/2} : \mu \in C(K)^*, \|\mu\| \le 1\right\}.$$

The supremum in the previous inequality equals $\|(\sum_1^n |g_i|^2)^{1/2}\|_\infty$.

Consider measurable sets $(A_i)_1^n$ and set $g_i = f_i \cdot \chi_{A_i}$. From the previous expression we have

$$\begin{aligned}\left(\sum_1^n \left\|\int_{A_i} f_i \, d\nu\right\|^2\right)^{1/2} &\le C \cdot \left\|\left(\sum_1^n |f_i \cdot \chi_{A_i}|^2\right)^{1/2}\right\|_\infty \\ &\le C \cdot \left\|\left(\sum_1^n |f_i|^2\right)^{1/2}\right\|_\infty \\ &= C \cdot \left\|\left(\sum_1^n |f_i|^2\right)^{1/2}\right\|_\nu .\end{aligned}$$

Taking supremum over all posible choices of sets $(A_i)_1^n$ and considering the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$ it follows that

$$\left(\sum_1^n \|f_i\|_\nu^2\right)^{1/2} \leq 2\,C\cdot \left\| \left(\sum_1^n |f_i|^2\right)^{1/2} \right\|_\nu .$$

Thus $L^1(\nu)$ is 2–concave and so it has cotype 2. Q.E.D.

A Banach space X is said to have *type* p, for $1 \leq p \leq 2$, if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and any elements $x_1, \ldots, x_n$ in X we have

$$\int_0^1 \left\| \sum_1^n r_i(t)\cdot x_i \right\| \, dt \leq C\cdot \left(\sum_1^n \|x_i\|^p\right)^{1/p} .$$

The case $p = 1$ corresponds to the triangle inequality for the norm. In this case X is said to have trivial type or to have no type. Hilbert spaces have type 2.

Type is not inherited by the space $L^1(\nu)$ from the Banach space X, this is easily seen by considering the space $L^1[0,1]$, which has no type, obtained from the Lebesgue measure (with values in $\mathbb{R}$).

For especial Banach spaces we can obtain results on type for $L^1(\nu)$. We first prove an auxiliary result, which has independent interest. The next proposition shows how the properties of the integration operator are reflected in the space $L^1(\nu)$.

Proposición 2.6. *Consider the integration operator* $\nu\colon L^1(\nu) \longrightarrow X$. *If* ν *is compact then the space* $L^1(\nu)$ *contains a complemented subspace isomorphic to* ℓ^1.

PROOF. Let λ be a Rybakov control measure for ν. Consider the trapose of the integration operator $\nu^*\colon X^* \longrightarrow L^1(\nu)^*$, it is compact. Let f be in $L^1(\nu)$.

Then

$$\langle \nu^*(x^*), f\rangle = \langle x^*, \nu(f)\rangle = \int f\, dx^*\nu = \int f h_{x^*}\, d\lambda\,,$$

where h_{x^*} is the Radon–Nikodym derivative of the measure $x^*\nu$ with respect to λ, which is an element of $L^1(\nu)^*$. Thus the norm in $L^1(\nu)$ can be written in the following way

$$\|f\|_\nu = \sup\left\{\int |f||h|\, d\lambda : h \in \nu^*(B_{X^*})\right\}.$$

Let f be in $L^1(\nu)$ and let A be a measurable set. We have

$$\begin{aligned} \|f\cdot\chi_A\|_\nu &= \sup\left\{\int_A |f||h|\, d\lambda : h \in \nu^*(B_{X^*})\right\} \\ &= \sup\Big\{\langle |f|, |h\cdot\chi_A|\rangle : h \in \nu^*(B_{X^*})\Big\} \\ &\le \|f\|_\nu \cdot \sup\Big\{\|h\cdot\chi_A\|_{L^1(\nu)^*} : h \in \nu^*(B_{X^*})\Big\}. \end{aligned} \tag{7}$$

Suppose $L^1(\nu)$ has no complemented subspace ismorphic to ℓ^1. Then, by a result of Bessaga and Pelczynski [BP, Theorem 4] it follows that $L^1(\nu)^*$ has no subspace isomorphic to ℓ_∞. As $L^1(\nu)^*$ is a dual Banach lattice it is order complete, it follows that it is order continuous [AB, Theorem 14.9]. In order continuous Banach lattices relatively compact sets are L–weakly compact and in Banach function spaces these coincide with equi–integrable sets, see Preliminaries. As $\nu^*(B_{X^*})$ is compact in $L^1(\nu)^*$, we have

$$\lim_{\lambda(A)\to 0} \sup\Big\{\|h\cdot\chi_A\|_{L^1(\nu)^*} : h \in \nu^*(B_{X^*})\Big\} = 0. \tag{8}$$

It follows from (7) and (8) that in $L^1(\nu)$ norm bounded sets are equi–integrable, thus L–weakly compact. The infinite dimensional Banach lattices in which relatively weakly compact sets are L–weakly compact are characterized by Meyer–Nieberg, see Preliminaries, for satisfying that every infinite dimensional sublattice contains a subspace isomorphic to ℓ^1. On the other hand, the unit ball of $L^1(\nu)$ being bounded is L–weakly compact and so relatively weakly compact. Thus $L^1(\nu)$ is reflexive, which contradicts the fact of containing subspaces isomorphic to ℓ^1. Q.E.D.

Theorem 2.7. *Let $\nu\colon\Sigma \longrightarrow \ell^p$ for $1 \leq p < 2$. Then the space $L^1(\nu)$ does not have type 2.*

PROOF. Suppose $L^1(\nu)$ has type 2. As ℓ^p has cotype 2, for $1 \leq p < 2$, it follows from Theorem 2.5 that $L^1(\nu)$ has cotype 2. From a theorem of Kwapién [P, Theorem 3.3] it follows that $L^1(\nu)$ is a Hilbert space. Consider the integration operator $\nu\colon L^1(\nu) \longrightarrow \ell^p$. The operator ν is compact as it is so for its restiction to any separable subspace, thanks to Pitt–Rosenthal's theorem, which asserts that every operator from ℓ^q into ℓ^p for $q > p$ is compact. From the previous proposition it follows that $L^1(\nu)$ contains a complemented subspace isomorphic to ℓ^1 which is a contradiction with being a Hilbert space. Q.E.D.

Dunford and Pettis proved that weakly compact operators defined over $L^1[0,1]$ map relatively weakly compact sets into relatively compact sets. This property was later isolated and named as the *Dunford–Pettis property* for Banach spaces by Grothendieck. Among the spaces that satisfy it are the AL–spaces and the AM–spaces. We study suficient conditions on the measure ν and the Banach space X in order to obtain a space $L^1(\nu)$ with the Dunford–Pettis property. Taking into account that in $L^1[0,1]$ relatively weakly compact sets and L–weakly compact sets coincide, the next theorem can be considered, in a certain sense, as an extension of the theorem of Dunford and Petis.

Proposition 2.8. *Let $\nu\colon\Sigma \longrightarrow X$ be a vector measure with σ–finite variation, Y a Banach space and $T\colon L^1(\nu) \longrightarrow Y$ a weakly compact operator. Then T maps L–weakly compact sets into relatively compact sets.*

PROOF. Let λ be a Rybakov control measure for ν. Consider the measure

$$A \in \Sigma \longmapsto F(A) = \chi_A \in L^1(\nu) \ .$$

It is countably additive and absolutely continuous with respect to λ. We will see that the average range of F is locally bounded, that is, for every measurable set A with $\lambda(A) > 0$ there exists a measurable set $B \subset A$ with $\lambda(B) > 0$ such that the set of averages

$$\left\{ \frac{F(D)}{\lambda(D)} : D \in \Sigma, D \subset B \right\}$$

is bounded in $L^1(\nu)$.

The measure $|\nu|$ is σ–finite and absolutely continuous with respect to λ, thus there exists a function $h \geq 0$ locally integrable with respect to λ such that $|\nu|(A) = \int_A h\, d\lambda$ for every measurable set A (both sides of the equality infinite if one of them is). Also, as $|\nu|$ is σ–finite, there exists a partition (A_n) of Ω such that $|\nu|(A_n) < +\infty$ for every n.

Let A be a measurable set with $\lambda(A) > 0$. There exists $n \in \mathbb{N}$ such that $\lambda(A \cap A_n) > 0$. Also there exists $k \in \mathbb{N}$ such that the set $B = \{\omega \in A \cap A_n : h(\omega) \leq k\}$ satisfies $|\nu|(B) > 0$. Then we have

$$|\nu|(D) = \int_D h\, d\lambda \leq k \cdot \lambda(D) \qquad \text{for every measurable set} \quad D \subset B.$$

It follows that

$$\left\| \frac{\chi_D}{\lambda(D)} \right\|_\nu = \frac{\|\nu\|(D)}{\lambda(D)} \leq \frac{|\nu|(D)}{\lambda(D)} \leq k.$$

Consider the measure defined by

$$A \in \Sigma \longmapsto G(A) = T(\chi_A) \in Y\,.$$

It is countably additive and absolutely continuous with respect to λ. With the measure G we can represent the operator T: if $\varphi = \sum_1^n a_i \chi_{A_i}$ is a simple function, we have

$$T(\varphi) = \sum_1^n a_i T(\chi_{A_i}) = \sum_1^n a_i G(A_i) = \int \varphi\, dG.$$

From the continuity of the operator T and the fact of being G absolutely continuous with respect to λ it follows, in view of Definition 1.1, that

$$\text{if } f \in L^1(\nu) \text{ then } f \in L^1(G) \text{ and } T(f) = \int f\, dG. \tag{9}$$

As $G = T \circ F$ and the operator T is weakly compact, it follows that the average range of G is locally relatively weakly compact. In these conditions by the vector Radon–Nikodym Theorem (see [DU, Theorem III.2.18]) there exists a function $g\colon \Omega \longrightarrow Y$ λ–measurable and Pettis integrable with respect to λ, such that

$$G(A) = \text{ Pettis–}\int_A g\, d\lambda.$$

From Definition 1.2 it follows that if f is in $L^1(G)$, then the function fg is Pettis integrable with respect to λ and

$$\int f\, dG = \text{ Pettis–}\int fg\, d\lambda. \tag{10}$$

From (9) and (10) we deduce that the operator T can be represented as

$$T(f) = \int fg\, d\lambda.$$

Let K be an L–weakly compact set in $L^1(\nu)$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \lambda(A) < \delta \text{ then } \|f \cdot \chi_A\|_\nu < \varepsilon \text{ for every } f \in K. \tag{11}$$

The density g is λ–measurable, thus there exists a sequence of simple functions g_n that conveges to g almost everywhere with respect to λ. By Egoroff's Theorem the convergence is almost uniform, thus for $\delta > 0$ there exists a measurable set A with $\lambda(A) < \delta$ such that on $\Omega \setminus A$ the convergence is uniform. Let $n \in \mathbb{N}$ be such that $\|g(\omega) - g_n(\omega)\| < \varepsilon$ for every $\omega \in \Omega \setminus A$.

Then for f in K we have

$$\begin{aligned} Tf =& T(f\cdot\chi_A) + T(f\cdot\chi_{\Omega\setminus A}) \\ =& T(f\cdot\chi_A) + \int_{\Omega\setminus A} f g_n\,d\lambda + \int_{\Omega\setminus A} f(g-g_n)\,d\lambda \end{aligned}$$

From (11) it follows that $\|T(f\cdot\chi_A)\| < \varepsilon\|T\|$. On the other hand

$$\|\int_{\Omega\setminus A} f(g-g_n)\,d\lambda\| \le \int_{\Omega\setminus A} |f|\|g-g_n\|\,d\lambda \le \varepsilon\int |f|\,d\lambda \le \varepsilon\cdot\|f\|_\nu\,.$$

K is bounded as it is L–weakly compact, thus we have seen that for every $\varepsilon > 0$ there exists a simple function $\varphi\colon\Omega\longrightarrow Y$ such that the distance between the sets $T(K)$ and $\{\int f\varphi\,d\lambda : f\in K\}$ is less than ε. This last set is relatively compact as, if $\varphi = \sum_1^n y_i\chi_{A_i}$ with $y_i\in Y$, then

$$\int f\varphi\,d\lambda = \sum_1^n y_i\int_{A_i} f\,d\lambda$$

and the coefficients $\int_{A_i} f\,d\lambda$ are bounded by $\sup\{\|f\|_\nu : f\in K\}$. Hence $T(K)$ is relatively compact in Y. Q.E.D.

A Banach space is said to have the *Schur property* if weak convergence of a sequence implies its norm convergence. For Banach lattices we also consider the positive Schur property, defined in the Preliminaries.

Proposition 2.9. *Let X be a Banach space with the Schur property. Then $L^1(\nu)$ has the positive Schur property.*

PROOF. We have to prove that in $L^1(\nu)$ L–weakly compact sets and relatively weakly compact sets coincide. Suppose that there exists a set $M\subset L^1(\nu)$ that is relatively weakly compact but it is not L–weakly compact. Thus there exists functions f_n in M and disjoint measurable sets A_n such that for a certain $\varepsilon > 0$

$$\|f_n \cdot \chi_{A_n}\|_\nu \geq \varepsilon \quad \text{for every } \ n \in \mathbb{N}. \tag{12}$$

As $\{f_n : n \in \mathbb{N}\}$ is a relatively weakly compact set, by the Theorem of Eberlein–Smulian, there exists a subsequence (f_{n_k}) which converges weakly in $L^1(\nu)$ to a function $f \in L^1(\nu)$. It follows that for every $A \in \Sigma$

$$\int_A f_{n_k}\, d\nu \ \text{ converges weakly in } X \text{ to } \ \int_A f\, d\nu.$$

As X is a Schur space, the previous convergence is in norm. Let μ and μ_k be the measures with densities f and f_{n_k} with respect to ν, respectively. They are countably additive and satisfy that $\mu_k(A)$ converge to $\mu(A)$ in norm, for every $A \in \Sigma$. By the Vitali–Hahn–Saks Theorem (see [DU, Corollary I.5.6]) it follows that $\{\mu_k\}$ is a uniformily countably additive family. This implies that

$$\lim_n \sup_k \|\mu_k\|(A_n) = 0,$$

which contradicts (12). Q.E.D.

Theorem 2.10. *Let X a Banach space with the Schur property and let ν be a measure with σ–finite variation. Then $L^1(\nu)$ has the Dunford–Pettis property.*

PROOF. From the previous proposition, in $L^1(\nu)$ L–weakly compact sets and relatively weakly compact set coincide. By Proposition 2.8 every weakly compact operator defined over $L^1(\nu)$ maps L–weakly compact sets into relatively compact sets. Q.E.D.

From Proposition 2.8 it follows a result that emphasizes, once more, the role that the measure plays in determining the space $L^1(\nu)$.

Theorem 2.11. *Let ν be a vector measure with no atoms and σ–finite variation. Then $L^1(\nu)$ is not reflexive.*

PROOF. Suppose by way of contradiction that $L^1(\nu)$ is reflexive. Consider the identity operator $I\colon L^1(\nu) \longrightarrow L^1(\nu)$, it is weakly compact. Let $K = \{f \in L^1(\nu) \,:\, |f| \leq 1\}$. It is an L–weakly compact set as it is order bounded by the function χ_Ω. From Proposition 2.8 it follows that K is relatively compact in $L^1(\nu)$. Let λ be a Rybakov control measure for ν. The injection of $L^1(\nu)$ into $L^1(\lambda)$ is continuous, so K is relatively compact in $L^1(\lambda)$.

As ν has no atoms, neither does λ. Thus in $L^1(\lambda)$ we can construct a Rademacher type sequence (r_n), satisfying, in particular, $|r_n| = 1$ and $\|r_n - r_m\|_{L^1(\lambda)} = \lambda(\Omega)$. But $r_n \in K$ and this contradicts K being compact in $L^1(\lambda)$.
Q.E.D.

We have seen that the measure ν has no atoms if and only if the space $L^1(\nu)$ has no atoms, Proposition 1.16. Thus it follows from the previous Theorem that the spaces $L^p[0,1]$ for $1 < p < +\infty$ can only be obtained, order isomorphically, from vector measures whose variation on every measurable set is either null or infinite.

The condition of non existence of atoms is necessary: it is enough to consider the measure $A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum a_n e_n \in \ell^2$ where (a_n) is a positive sequence in ℓ^2. We have $L^1(\nu) = \ell^2$, Proposition 1.22. The condition of σ–finiteness of the variation is also necessary: it is enough to consider the measure $A \in \mathcal{M}[0,1] \longmapsto \nu(A) = \chi_A \in L^2[0,1]$. We have $L^1(\nu) = L^2[0,1]$ and $|\nu|(A)$ is either 0 or $+\infty$.

$\mathbf{L^p}$–spaces 2.12. Among the most important Banach lattices are the abstract L^p–spaces, $1 \leq p < +\infty$. A Banach lattice is said to be an *abstract L^p–space* if the p–th power of the norm is additive over disjoint elements, that is, for every pair of disjoint elements x and y we have

$$\|x+y\|^p = \|x\|^p + \|y\|^p.$$

These spaces are order continuous Banach lattices. In virtue of a result of Kakutani and Bohnenblust [LT vol. II, Theorem 1.b.2] these spaces are order isomorphic and isometric to a "concrete" L^p–space, that is: if E is an abstract L^p–space there exists a measure space (S, σ, μ) such that E is order isomorphic and isometric to the space $L^p(S, \sigma, \mu)$. When E has a weak unit, then the measure μ can be taken to be finite.

We will study this situation for the space $L^1(\nu)$. We will see the relation of the measure space (S, σ, μ) given by the theorem of Kakutani and Bohnenblust, with the vector measure ν.

Suppose $L^1(\nu)$ is order isomorphic to an abstract L^p–space E. That is, there exists an isomorphism $T\colon L^1(\nu) \longrightarrow E$ which preserves order. Let $f_1, \ldots, f_n$ be disjoint functions in $L^1(\nu)$. Then, considering that the functions Tf_i are disjoint in E, which is an abstract L^p–space, it follows that

$$\begin{aligned} \left\| \sum_1^n f_i \right\|_\nu^p &\leq \|T^{-1}\|^p \cdot \left\| \sum_1^n Tf_i \right\|^p \\ &= \|T^{-1}\|^p \cdot \left(\sum_1^n \|Tf_i\|^p \right) \\ &\leq \|T\|^p \cdot \|T^{-1}\|^p \cdot \left(\sum_1^n \|f_i\|_\nu^p \right). \end{aligned}$$

Analogously we obtain a similar lower bound. Thus there exist two positive constants C_1 and C_2 such that for any pairwise disjoint functions $f_1, \ldots, f_n$ in $L^1(\nu)$ we have

$$C_1 \cdot \sum_1^n \|f_i\|_\nu^p \leq \left\| \sum_1^n f_i \right\|_\nu^p \leq C_2 \cdot \sum_1^n \|f_i\|_\nu^p. \tag{13}$$

Let $(A_i)_1^n$ be a partition of Ω. Apply (13) to the functions $f_i = \chi_{A_i}$. We have

$$C_1 \cdot \sum_1^n \|\nu\|(A_i)^p \leq \|\nu\| \left(\bigcup_1^n A_i \right)^p \leq C_2 \cdot \sum_1^n \|\nu\|(A_i)^p. \tag{14}$$

Consider the following set function

$$A \in \Sigma \longmapsto \tau(A) = \sup \left\{ \sum_1^n \|\nu\|(A_i)^p : (A_i)_1^n \text{ is a partition of } A \right\} \in \mathbb{R}\,.$$

It is well defined, by (14), that is, it is finite. It is superadditive and satisfies

$$C_1 \cdot \tau(A) \leq \|\nu\|(A)^p \leq \tau(A).$$

Consider now the set function

$$A \in \Sigma \longmapsto \mu(A) = \inf \left\{ \sum_1^n \tau(A_i) : (A_i)_1^n \text{ is a partition of } A \right\} \in \mathbb{R}\,.$$

It is well defined and it is finite by (14) and the relation between $\|\nu\|^p$ and τ. It is additive and satisfies

$$C_1 \cdot \mu(A) \leq \|\nu\|(A)^p \leq C_2 \cdot \mu(A) \quad \text{for every} \quad A \in \Sigma. \tag{15}$$

The semivariation is absolutely continuous with respect to a control measure, it follows that so is the measure μ. Hence μ is countably additive.

Consider the space $L^p(\Omega, \Sigma, \mu)$. Let $f = \sum_1^n a_i \chi_{A_i}$ be a simple function with $(A_i)_1^n$ disjoint. Then from (13) we have

$$C_1 \cdot \sum_1^n |a_i|^p \|\nu\|(A_i)^p \leq \|f\|_\nu^p \leq C_2 \cdot \sum_1^n |a_i|^p \|\nu\|(A_i)^p.$$

from (15) we have

$$(C_1)^2 \cdot \sum_1^n |a_i|^p \mu(A_i) \leq \|f\|_\nu^p \leq (C_2)^2 \cdot \sum_1^n |a_i|^p \mu(A_i).$$

Thus, over the simple functions the norms of $L^1(\nu)$ and of $L^p(\Omega, \Sigma, \mu)$ are equivalent. As the simple functions are dense in both spaces, we deduce that they are order isomorphic.

Consider now the following set function

$$A \in \Sigma \longmapsto \|\nu\|(A)^p \in \mathbb{R}\,.$$

It is easy to verify that the measure μ previously builted is, in the space of real countably additive measures defined over Σ, on the one hand a lower bound of the set of measures that mayorize $\|\nu\|(\cdot)^p$, and on the other hand, an upper bound of the set of measures that minorize $\|\nu\|(\cdot)^p$. When $L^1(\nu)$ is an abstract L^p–space it follows from the previous study that $L^1(\nu) = L^p(\Omega, \Sigma, \|\nu\|^p)$, where in this case $\|\nu\|^p$ is a countably additive measure. In the case $p = 1$ the semivariation turns out to be additive and so it coincides with the variation of the measure. We will study this case, that of AL–spaces, in detail in the next chapter.

The case $p = +\infty$ corresponds to the AM–spaces, which are those Banach lattices in which for every pair of disjoint elements x and y we have

$$\|x + y\| = \sup\{\|x\|, \|y\|\}.$$

An AM–space which is order continuous is order isomorphic and isometric to $c_0(\Gamma)$ for a certain set of indexes Γ [LT vol. II, Lemma 1.b.10]. The existence of a weak unit implies that Γ is countable. It follows that if $L^1(\nu)$ is an AM–space the only posibility is c_0.

It is of interest to find conditions in order to have the space $L^1(\nu)$ order ismorphic to a Hilbert space. The example of the Lebesgue measure on [0,1],with values in $\mathbb{R}$, which gives the space $L^1[0,1]$, shows that these conditions have to be more restrictive than being the Banach space in which the measure takes its values a Hilbert space.

A sequence (x_n) in a Banach space is said to be 2–*lacunary* if there exists a constant $C > 0$ such that for every sequence (α_n) in ℓ^2 we have

$$\left\| \sum_1^\infty \alpha_n x_n \right\| \leq C \cdot \left(\sum_1^\infty \alpha_n^2 \right)^{1/2} .$$

Theorem 2.13. *Let X be a Banach space with cotype 2. Let $\nu\colon \Sigma \longrightarrow X$ be a measure satisfying that for every partition $(A_n)_1^\infty$, the sequence*

$$\left(\frac{\nu(A_n)}{\|\nu(A_n)\|} \right)$$

is 2–lacunary. Then $L^1(\nu)$ is order isomorphic to a Hilbert space.

PROOF. From our hypothesis given any partition $(A_n)_1^\infty$ there exists a constant $K = K(A_n)$, depending on the partition, such that for every sequence (α_n) in ℓ^2 we have

$$\left\| \sum_1^\infty \alpha_n \frac{\nu(A_n)}{\|\nu(A_n)\|} \right\| \leq K \cdot \left(\sum_1^\infty \alpha_n^2 \right)^{1/2} . \tag{16}$$

Let λ be a control measure for ν. We will see that for every $A \in \Sigma$ with $\lambda(A) > 0$ there exists a measurable set $B \subset A$ with $\lambda(B) > 0$ such that

$$\mathcal{K}(B) = \sup \{ K(B_n) : (B_n) \text{ is a partition of } B \} < +\infty.$$

Assume by way of contradiction that this is not the case. Then there exists a measurable set A with $\lambda(A) > 0$ such that for every $B \subset A$ with $\lambda(B) > 0$ we have $\mathcal{K}(B) = +\infty$. Let (A_n) be a partition of A such that $\lambda(A_n) > 0$. For every $n \in \mathbb{N}$, as $\mathcal{K}(A_n) = +\infty$, there exist a partition (A_i^n) of A_n such that $K(A_i^n) > n$. That is, there exists $(\alpha_i^n) \in \ell^2$ such that

$$\left\| \sum_{i=1}^\infty \alpha_i^n \frac{\nu(A_i^n)}{\|\nu(A_i^n)\|} \right\| > n \cdot \left(\sum_{i=1}^\infty |\alpha_i^n|^2 \right)^{1/2} .$$

Thus there exists an index $i(n)$ such that

$$\left\| \sum_{i=1}^{i(n)} \alpha_i^n \frac{\nu(A_i^n)}{\|\nu(A_i^n)\|} \right\| > n \cdot \left(\sum_{i=1}^{i(n)} |\alpha_i^n|^2 \right)^{1/2} .$$

Let us consider the partition

$$A_1^1,\, A_2^1,\, \dots,\, A_{i(1)}^1,\, \bigcup_{i(1)}^{\infty} A_i^1,\, A_1^2,\, A_2^2,\, \dots,\, A_{i(2)}^2,\, \bigcup_{i(2)}^{\infty} A_i^2,\, \dots .$$

It is partition of A and it is easy to see that, by its contruction, the associted sequence is not 2–lacunary.

We are now in the conditions for applying an Exhaustion Lemma [DU, Lemma III.2.4] to the following property (P) "ν has (P) on A if $\mathcal{K}(A) < +\infty$". It follows that there exists a partition (B_n) of Ω such that $\mathcal{K}(B_n) < +\infty$.

An argument similar to the previous one shows that in fact we have $K = \sup_n \mathcal{K}(B_n) < +\infty$.

From the previous discussion and from (16) it follows that for every partition (A_n) and for every sequence $(a_n) \in \ell^2$, by considering $\alpha_n = a_n \|\nu(A_n)\|$, we have

$$\begin{aligned} \left\| \sum_1^{\infty} a_n \nu(A_n) \right\| &= \left\| \sum_1^{\infty} \alpha_n \frac{\nu(A_n)}{\|\nu(A_n)\|} \right\| \\ &\leq K \cdot \left(\sum_1^{\infty} a_n^2 \|\nu(A_n)\|^2 \right)^{1/2} \\ &\leq K \cdot \left(\sum_1^{\infty} a_n^2 \|\nu\|(A_n)^2 \right)^{1/2} . \end{aligned} \tag{17}$$

Let g be a simple function. We can write it as $g = \sum_1^n a_i \chi_{A_i}$ where the sets

A_i are disjoint. Let $B \in \Sigma$, then by (17) we have

$$\begin{aligned}\| \int_B g \, d\nu \| &= \left\| \sum_1^n a_i \nu(A_i \cap B) \right\| \\ &\le K \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i \cap B)^2 \right)^{1/2} \\ &\le K \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} .\end{aligned}$$

Considering the equivalent norm $|||\cdot|||$ in $L^1(\nu)$ we deduce that

$$\left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu \le 2K \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} . \tag{18}$$

From having X cotype 2 we deduce, by Theorem 2.5, that $L^1(\nu)$ has cotype 2, so it is 2–concave. Thus there exists a constant $C > 0$ such that for any scalars $a_1, \dots, a_n$ and any measurable sets $B_1, \dots, B_n$ we have

$$\left(\sum_1^n a_i^2 \|\nu\|(B_i)^2 \right)^{1/2} \le C \cdot \left\| \left(\sum_1^n |a_i \chi_{B_i}|^2 \right)^{1/2} \right\|_\nu .$$

Set $B_i = A_i$, as the sets A_i are disjoint, we have

$$\left\| \left(\sum_1^n |a_i \chi_{A_i}|^2 \right)^{1/2} \right\|_\nu = \left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu .$$

Hence

$$\left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} \le C \cdot \left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu . \tag{19}$$

It follows that for a simple function $g = \sum_1^n a_i \chi_{A_i}$ where the sets A_i are disjoint, we have

$$1/C \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} \le \left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu \le 2K \cdot \left(\sum_1^n a_n^2 \|\nu\|(A_i)^2 \right)^{1/2} . \tag{20}$$

From the previous inequality evaluated at $a_1 = \ldots = a_n = 1$ we obtain, for disjoint measurable sets $(A_i)_1^n$,

$$(1/C)^2 \cdot \sum_1^n \|\nu\|(A_i)^2 \leq \|\nu\|(\bigcup_1^n A_i)^2 \leq (4K)^2 \cdot \sum_1^n \|\nu\|(A_i)^2.$$

This inequality, similar to (14) for $p = 2$, allows to apply the same argument as in 2.12, to obtain the measure μ associated to the square of the semivariation. From (20) it follows that $L^1(\nu)$ is order ismorphic to the space $L^2(\Omega, \Sigma, \mu)$, which is a Hilbert space. Q.E.D.

The last part of the chapter is devoted to study a more especific Banach lattice property which goes back to the study by Kadec y Pelczynski in [KP] of the subspaces of $L^p[0,1]$. The following definition is due to Weis [W].

Definition. [W, Definition 2.1] *Let X be an order continuous Banach lattice with weak unit. X has the subsequence splitting property if given a bounded sequence (f_n) in X there exists a subsequence (f_{n_k}) and there exists sequences (g_k) and (h_k) such that*

a) $f_{n_k} = g_k + h_k$ for every k.

b) g_k and h_k are disjoint, for every k.

c) The sequence (g_k) is equi–integrable.

d) (h_k) are pairwise disjoint.

The spaces $L^p[0,1]$ satisfy this property. The space c_0 is an example of a Banach lattice not satisfying the property. Figiel, Ghoussoub and Johnson have constructed a p–convex and reflexive Banach lattice not satisfying the property, see [W]. Weis in [W] does the construction that follows in order to characterize the Banach lattices satisfying the subsequence splitting property.

Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$. Consider the ultraproduct of X by $\mathcal{U}$:

$$X_{\mathcal{U}} = \ell_\infty(X)/M \quad \text{where} \quad M = \Big\{(f_n) \in \ell_\infty(X) : \lim_{\mathcal{U}} \|f_n\| = 0\Big\}.$$

We will denote by $[f_n]$ the equivalence class of the sequence (f_n). The space $X_{\mathcal{U}}$ is a Banach lattice for the following norm and order

$$\|[f_n]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f_n\|,$$
$$\inf\{[f_n],[g_n]\} = [\inf\{f_n, g_n\}].$$

For details on ultraproducts see Heinrich [H].

We will denote by $\mathbf{1}$ the weak unit of X. Let $[\mathbf{1}]$ be the equivalence class of constant sequence $\mathbf{1}$. Let $\tilde{X}$ be the band generated by $[\mathbf{1}]$ in $X_{\mathcal{U}}$, that is $\tilde{X} = [\mathbf{1}]^{\perp\perp}$. Weis gives the following characterization.

Theorem. [W, Theorem 2.5] *Let X be an order continuous Banach lattice with weak unit. The following conditions are equivalent:*

1) X has the subsequence splitting property.

2) $\tilde{X}$ has order continuous norm.

3) $\tilde{X}$ has no subspace isomorphic to c_0.

In this context we have following result.

Theorem 2.14. *Let X be an order continuous Banach lattice with weak unit such that X and X^* have the subsequence splitting property. Let $\nu\colon\Sigma \longrightarrow X$ be a vector measure whose range is L–weakly compact in X. Then $L^1(\nu)$ has the subsequence splitting property.*

PROOF. In order to prove the result we will construct a measure $\tilde{\nu}$ such that the space $\tilde{L}^1(\nu)$ is contained order isomorphically in the space $L^1(\tilde{\nu})$. This last space is order continuous, Theorem 1.14. It follows that $\tilde{L}^1(\nu)$ is order continuous and from the charaterization of Weis the theorem will be proved.

Let us construct the measure $\tilde{\nu}$. Let λ be a Rybakov control measure for ν. For this measure we have

$$L_\infty(\Omega, \Sigma, \lambda) \longrightarrow L^1(\nu) \longrightarrow L^1(\Omega, \Sigma, \lambda),$$

where all injections are continuos.

Dacunha-Castelle and Krivine in [DK 1] and [DK 2] prove that the ultraproduct of the space $L^1(\Omega, \Sigma, \lambda)$ can be identified in the following way

$$L^1(\Omega, \Sigma, \lambda)_{\mathcal{U}} = L^1(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda}) \oplus \Delta'$$

where $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda})$ is a measure space and the elements of Δ' are disjoint from $[\chi_\Omega]$. Thus it follows that

$$\tilde{L}^1(\Omega, \Sigma, \lambda) = L^1(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda}).$$

The same procedure can be done with $L_\infty(\Omega, \Sigma, \lambda)$. This allows to identify $\tilde{L}^1(\nu)$ as a function space by using the ultraproduct of the injections of the previous diagram

$$L_\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda}) \longrightarrow \tilde{L}^1(\nu) \longrightarrow L^1(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda}),$$

being the injections continuous. See Weis [W].

The σ–algebra $\tilde{\Sigma}$ is isomorphic to the Boolean ring $\{[\chi_{A_n}] : A_n \in \Sigma\}$ in the space $L^1(\Omega, \Sigma, \lambda)_{\mathcal{U}}$. Thus every measurable set $\tilde{A} \in \tilde{\Sigma}$ can be identified with a sequence (A_n) for $A_n \in \Sigma$, identifing two sequences (A_n) and (B_n) if $\lim_{\mathcal{U}} \lambda(A_n \triangle B_n) = 0$.

The measure $\tilde{\lambda}$ is defined as follows:

$$\tilde{A} = (A_n) \in \tilde{\Sigma} \longmapsto \tilde{\lambda}(\tilde{A}) = \lim_{\mathcal{U}} \lambda(A_n) \in \mathbb{R}\,.$$

A function $\tilde{f}$ in $L^1(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\lambda})$ is an element $[f_n]$ in $L^1(\Omega, \Sigma, \lambda)_{\mathcal{U}}$, being the integration with respect to $\tilde{\lambda}$ defined as follows, for $\tilde{A} = (A_n) \in \tilde{\Sigma}$

$$\int_{\tilde{A}} \tilde{f}\, d\tilde{\lambda} = \lim_{\mathcal{U}} \int_{A_n} f_n\, d\lambda.$$

For more details see Heinrich [H].

We define the measure $\tilde{\nu}$ as follows

$$\tilde{A} = (A_n) \in \tilde{\Sigma} \longmapsto \tilde{\nu}(\tilde{A}) = [\nu(A_n)] \in X_{\mathcal{U}}\,.$$

As the measure ν is bounded, $\tilde{\nu}$ is well defined. It is finitely additive. Let $\varepsilon > 0$. As ν is absolutely continuous with respect to λ, there exists $\delta > 0$ such that if $\lambda(A) < \delta$ then $\|\nu\|(A) < \varepsilon$. Let $\tilde{A} = (A_n) \in \tilde{\Sigma}$ such that $\tilde{\lambda}(\tilde{A}) < \delta$, this is $\lim_{\mathcal{U}} \lambda(A_n) < \delta$. Then there exists $V \in \mathcal{U}$ such that for every $n \in V$ we have $\lambda(A_n) < \delta$. Thus for every $n \in V$ we have $\|\nu(A_n)\| \leq \|\nu\|(A_n) < \varepsilon$. Thus $\|\tilde{\nu}(\tilde{A})\|_{\mathcal{U}} < \varepsilon$. Hence $\tilde{\nu}$ is absolutely continuous with respect to $\tilde{\lambda}$ and we deduce that $\tilde{\nu}$ is countably additive.

From our hypothesis the range of the measure ν is L–weakly compact in X. Using the following result of Weis we deduce that the measure $\tilde{\nu}$ takes its values in $\tilde{X}$.

Proposition. [W, Proposition 1.5] *Let (f_n) be equi–integrable in X. Then $[f_n]$ is an order continuous element in $\tilde{X}$.*

Let us see that $\tilde{L}^1(\nu)$ is in $L^1(\tilde{\nu})$. It would be enough to prove that the elements of $\tilde{L}^1(\nu)$ are scalarly integrable with respect to $\tilde{\nu}$, as, from our hypothesis X has the subsequence property and so from the theorem of Weis it follows that $\tilde{X}$ has no subspace isomorphic to c_0. The characterization of Lewis, Theorem 1.10, guarantees then that integrability with respect to $\tilde{\nu}$ is equivalent to scalar integrability.

Weis proves that if X and X^* satisfy the subsequence splitting property then $\tilde{X}^* = \widetilde{X^*}$ and the norms of both spaces coincide [W, Corollary 2.7]. Hence the elements of $\tilde{X}^*$ can be expressed as $\tilde{x}^* = [x_n^*]$ for $x_n^* \in X^*$ and (x_n^*) a bounded sequence.

The measure $\tilde{x}^*\tilde{\nu}$ is absolutely continuous with respect to $\tilde{\lambda}$ as it is so for $\tilde{\nu}$. Thus it has a Radon–Nikodym derivative with respect to $\tilde{\lambda}$, $h_{\tilde{x}^*}$ in $L^1(\tilde{\lambda})$. For $\tilde{A} = (A_n) \in \tilde{\Sigma}$, we have

$$\begin{aligned}\langle \tilde{x}^*, \tilde{\nu}(\tilde{A})\rangle &= \langle [x_n^*], [\nu(A_n)]\rangle \\ &= \lim_{\mathcal{U}} \langle x_n^*, \nu(A_n)\rangle \\ &= \lim_{\mathcal{U}} \int_{A_n} dx_n^*\nu \\ &= \lim_{\mathcal{U}} \int_{A_n} h_{x_n^*}\, d\lambda \\ &= \int_{\tilde{A}} \tilde{h}\, d\tilde{\lambda},\end{aligned}$$

where $\tilde{h} = [h_{x_n^*}]$, being $h_{x_n^*}$ Radon–Nikodym derivative of the measure $x_n^*\nu$ with respect to λ.

Let $\tilde{F} = [f_n] \in \tilde{L}^1(\nu)$. Its norm in this space is $\|\tilde{F}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f_n\|_\nu$. Then we have

$$\begin{aligned}\int |\tilde{F}|\, d|\tilde{x}^*\tilde{\nu}| &= \int |\tilde{F}||\tilde{h}|\, d\tilde{\lambda} \\ &= \lim_{\mathcal{U}} \int |f_n||h_{x_n^*}|\, d\lambda \\ &\leq \lim_{\mathcal{U}} \|f_n\|_\nu \cdot \|x_n^*\| \\ &= \|\tilde{F}\|_{\mathcal{U}} \cdot \|\tilde{x}^*\|_{\mathcal{U}}\end{aligned}$$

Hence $\tilde{F}$ is integrable with respect to $\tilde{x}^*\tilde{\nu}$. We deduce that $\tilde{F}$ is integrable with respect to $\tilde{\nu}$ and

$$\|\tilde{F}\|_{L^1(\tilde{\nu})} \leq \|\tilde{F}\|_{\mathcal{U}}.$$

Given $\varepsilon > 0$, using the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$, we can find for every n a measurable set A_n such that

$$\|\int_{A_n} f_n \, d\nu\| \geq 1/2 \cdot (1-\varepsilon)\|f_n\|_\nu \,.$$

Let $\tilde{A} = (A_n)$ in $\tilde{\Sigma}$. Then

$$\begin{aligned}
\|\tilde{F}\|_{L^1(\tilde{\nu})} &\geq \|\int_{\tilde{A}} \tilde{F} d\tilde{\nu}\| \\
&= \lim_{\mathcal{U}} \|\int_{A_n} f_n d\nu\| \\
&\geq 1/2 \cdot (1-\varepsilon) \lim_{\mathcal{U}} \|f_n\|_\nu \\
&= 1/2 \cdot (1-\varepsilon)\|\tilde{F}\|_{\mathcal{U}}
\end{aligned}$$

Thus both norms are equivalent and hence $L^1(\nu)$ is order isomorphic to a subspace of $L^1(\tilde{\nu})$. This completes the proof. Q.E.D.

CHAPTER 3: When is $L^1(\nu)$ an AL–space ?

Theorem 1.15 shows that every order continuous Banach lattice with weak unit is obtained as L^1 of a certain vector measure. Thus among the function spaces that appear as L^1 of a vector measure there are reflexive, even Hilbert spaces. These spaces are, in their properties, very different from the classical spaces $L^1(S,\sigma,\mu)$ where μ is a positive finite measure. Thus it arises a natural question: Under what conditions on the vector measure or on the Banach space in which the measure takes its values, the space $L^1(\nu)$ is of the form $L^1(S,\sigma,\mu)$? And in this case: Which is the relation between the measure space (S,σ,μ) and the vector measure ν?

There is more flexibility if we require $L^1(\nu)$ to be *order isomorphic* to a space of the form $L^1(S,\sigma,\mu)$. Moreover we can limit the requirements to the existence of a positive constant C such that for every pair of functions f and g in $L^1(\nu)$ with disjoint support we have:

$$C \cdot (\|f\|_\nu + \|g\|_\nu) \leq \|f+g\|_\nu \leq \|f\|_\nu + \|g\|_\nu ,$$

as in this case $L^1(\nu)$ would be order isomorphic to an AL–space, that is a Banach lattice in which the norm is additive for disjoint elements. Due to a theorem of Kakutani [LT vol. II, Theorem 1.b.2] every AL–space is order isomorphic to a space $L^1(S,\sigma,\mu)$ for a certain measure space (S,σ,μ), where μ is a positive measure that is finite if the space has a weak unit.

Thus the question can be restated in the following way:

When can $L^1(\nu)$ be equivalently renormed so that with the new norm and the same order is a Banach lattice in which the norm is additive for disjoint functions?

Lewis proved that the formal identity is a continuous injection of the space $L^1(|\nu|)$ into $L^1(\nu)$ and that the elements f of $L^1(|\nu|)$ are characterized in $L^1(\nu)$ for having bounded variation the measure with density f with respect to ν, Theorem 1.12. The space $L^1(|\nu|)$ has in this problem we are studying a more important role than at first could be expected, as the following proposition shows. The definition of $\mathcal{L}_1$–spaces is due to Lindenstrauss and Pelczynski [LP, Definition 3.1].

Proposition 3.1. *The following conditions are equivalent:*

a) *$L^1(\nu)$ is an $\mathcal{L}_1$–space.*

b) *$L^1(\nu)$ is isomorphic to an AL–space.*

c) *$L^1(\nu)$ is order isomorphic to an AL–space.*

d) *In $L^1(\nu)$ every positive summable sequence is absolutely summable.*

e) *The integration operator maps positive summable sequences into absolutely summable sequences.*

f) *The transposed of the integration operator maps norm bounded sets into order bounded sets.*

g) *The natural injection of $L^1(|\nu|)$ into $L^1(\nu)$ is an onto (order) isomorphism.*

In this conditions the measure ν has bounded variation.

PROOF. As $L^1(\nu)$ is an order continuous Banach lattice, the equivalence of a), b) and c) follows from the results of Abramovich and Wojtaszczyk on uniqueness of order [AW, p. 781]. The equivalence of c) and d) is a classical result of the theory of Banach lattices [S, Theorem IV.2.7].

The operators that satisfy condition e) are called *cone absolutely summing*, see [S IV.3]. We will see that conditions d) and e) are equivalent. Condition e) follows from d) due to the continuity of the integration operator. Let (f_n) be a positive summable sequence in $L^1(\nu)$. By using the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$ we find, for every n, a measurable set A_n such that

$$\|f_n\|_\nu \le 4\cdot \|\int_{A_n} f_n \, d\nu\| = 4\cdot \|\nu(f_n\chi_{A_n})\|.$$

The sequence $(f_n\chi_{A_n})$ is positive and summable. From condition e) is follows that the sequence $(\nu(f_n\chi_{A_n})$ is absolutely summable. From the previous inequality it follows that (f_n) is absolutely summable in $L^1(\nu)$, so d) is satisfied.

To see that condition f) follows from c) consider that $L^1(\nu)^*$ is an AM–space with unit and so its unit ball is order bounded. Condition g) implies trivially a). So we just have to prove that f) implies g).

Let λ be a Rybakov control measure for ν. Consider $L^1(\nu)^*$ in $L^1(\lambda)$. The norm in $L^1(\nu)$ can be written as:

$$\|f\|_\nu = \sup\left\{\int |f||h_{x^*}|\, d\lambda : x^* \in X^*\right\},$$

where h_{x^*} is the Radon–Nikodym derivative of the scalar measure $x^*\nu$ with respect to λ. Let $\nu^*\colon X^* \longrightarrow L^1(\nu)^*$ be the transposed of the integration operator. In terms of ν^* we have

$$\int f\, dx^*\nu = \langle x^*, \nu(f)\rangle = \langle \nu^*(x^*), f\rangle.$$

So $\nu^*(x^*) = h_{x^*}$ for every $x^* \in X^*$. Consider the set $\nu^*(B_{X^*})$. By f) is order bounded in $L^1(\nu)^*$. As $L^1(\nu)^*$ is a dual Banach lattice, it is order complete, so there exists h in $L^1(\nu)^*$, the supremum of $\nu^*(B_{X^*})$. Thus we have:

$$\|f\|_\nu \le \int |f|h\, d\lambda = \|f\|_{L^1(\mu)} \tag{1}$$

where μ is the measure with density h with respect to λ. Also we have

$$\|f\|_{L^1(\mu)} = \int |f| h \, d\lambda \le \langle |f|, h\rangle \le \|f\|_\nu \cdot \|h\|_{L^1(\nu)^*}. \tag{2}$$

We deduce that $L^1(\nu)$ is (order) isomorphic to $L^1(\mu)$.

Let A be a measurable set, consider the function χ_A. From inequality (1) we have:

$$\|\nu(A)\| \le \|\nu\|(A) = \|\chi_A\|_\nu \le \mu(A).$$

This inequality implies that μ is a positive measure that dominates the vector measure ν, thus it also dominates its variation

$$|\nu|(A) \le \mu(A) \text{ for every measurable set } A. \tag{3}$$

Thus ν has bounded variation.

Let g be a simple function. It follows from inequality (2) and from (3) that

$$\|g\|_{L^1(|\nu|)} \le \|g\|_{L^1(\mu)} \le \|g\|_\nu \cdot \|h\|_{L^1(\nu)^*}.$$

As simple functions are dense in $L^1(|\nu|)$, we deduce that the previous inequality holds for every f in $L^1(|\nu|)$, and so the natural injection of $L^1(|\nu|)$ into $L^1(\nu)$ is continuous, closed and has dense range. Thus both spaces are isomorphic. Q.E.D.

Consequences 3.2. From the previous proposition we deduce two important consequences. Firstly we identify the measure space we refered to at the begining of the chapter, it is $(\Omega, \Sigma, |\nu|)$. This allows us to reduce our problem, without loss of generality, to study when $L^1(\nu)$ coincides with $L^1(|\nu|)$.

Secondly we obtain the following necessary condition for $L^1(\nu)$ being an AL–space:

$$\text{there exists } C > 0 \text{ such that } |\nu|(A) \le C \cdot \|\nu\|(A) \text{ for every } A \in \Sigma, \tag{L 1}$$

that is *the domination of the variation by the semivariation*. This follows from the isomorphism between $L^1(|\nu|)$ and $L^1(\nu)$, as for $A \in \Sigma$ we have that $\|\chi_A\|_\nu = \|\nu\|(A)$.

The previous proposition shows that bounded variation is a necessary condition. Hence we will assume through this chapter that the measure ν has bounded variation.

Condition (L 1) is restrictive, as the following example shows.

Example 3.3. *Let X be an infinite dimensional Banach space. There exist a measure ν with values in X and bounded variation such that there exists no constant $C > 0$ satisfying*

$$|\nu|(A) \leq C \cdot \|\nu\|(A) \quad \text{for every} \quad A \in \Sigma.$$

PROOF. By the Theorem of Dvoretsky–Rogers there exists a sequence (x_n) in X such that the series $\sum x_n$ is unconditionally convergent but not absolutely convergent. Suppose that there exists a sequence of scalars (α_n) satisfying the following requirements:

(a) $0 \leq \alpha_n \leq 1$ for every n,

(b) $\sum_1^\infty \alpha_n \|x_n\| < +\infty$,

(c) $\left(\sum_{i \geq n} \alpha_i \|x_i\|\right) \cdot \left(\sup \left\{ \sum_{i \geq n} |x^* x_i| : x^* \in B_{X^*} \right\}\right)^{-1} \to +\infty.$

Consider the measure defined by

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum_{n \in A} \alpha_n x_n \in X\,.$$

As it is defined by an unconditional series, it is well defined and countably additive. Its variation and semivariation are given by the following expressions:

$$|\nu|(A) = \sum_{n\in A} \alpha_n \|x_n\|$$
$$\|\nu\|(A) = \sup\left\{\sum_{n\in A} \alpha_n |x^* x_n| : x^* \in B_{X^*}\right\}.$$

Condition (b) implies that the total variation of the measure ν is finite. From condition (a) follows that the semivariation is bounded by

$$\|\nu\|(A) \leq \sup\left\{\sum_{n\in A} |x^* x_n| : x^* \in B_{X^*}\right\}$$

Let $A_n = \{n, n+1, n+2, \ldots\}$. The it follows that

$$\frac{|\nu|(A_n)}{\|\nu\|(A_n)} = \left(\sum_{i\geq n} \alpha_i \|x_i\|\right) \cdot \left(\sup\left\{\sum_{i\geq n} \alpha_i |x^* x_i| : x^* \in B_{X^*}\right\}\right)^{-1}$$
$$\geq \left(\sum_{i\geq n} \alpha_i \|x_i\|\right) \cdot \left(\sup\left\{\sum_{i\geq n} |x^* x_i| : x^* \in B_{X^*}\right\}\right)^{-1}$$

that tends to infinity with n by condition (c). Hence the result would be proved.

The existence of a sequence (α_n) satisfying the required conditions follows from the next lemma.

Lemma 3.4. *Let (γ_n) be a positive sequence decreasing to zero and (β_n) a positive sequence such that $\sum \beta_n = +\infty$. Then there exists a sequence (a_n) decreasing to zero, such that:*

1) $\beta_n \geq a_n - a_{n+1}$,
2) $\dfrac{a_n}{\gamma_n}$ *tends to* $+\infty$.

PROOF. We will see that we can define, in an inductive process, two increasing sequences of non negative integers (n_k) and (m_k) such that, if we denote $J_k = \{m_k, m_k + 1, \dots, m_{k+1} - 1\}$, we have that $\cup J_k = \mathbb{N}$ and

$$i)\ \sum_{n \in J_k} \beta_n \geq \frac{1}{2^k}$$
$$ii)\ \frac{1}{2^{n_k}} \geq \sqrt{\gamma_n} > \frac{1}{2^{n_{k+1}}} \quad \text{for} \quad n \in J_k.$$

The sequence $(\sqrt{\gamma_n})$ decreases to zero. Dividing by $\sup_n \sqrt{\gamma_n}$ if necessary, we can assume that $\sqrt{\gamma_n} \leq 1$. Set $m_1 = 1$ and $n_1 = 0$. Suppose that n_{k-1} and m_{k-1} are already defined. As the series $\sum \beta_n$ diverges, there exists m'_k the first integer for which

$$\sum_{n=m_{k-1}}^{m'_k - 1} \beta_n \geq \frac{1}{2^{k-1}}. \tag{4}$$

Let n_k be the first integer for which

$$\sqrt{\gamma_n} > \frac{1}{2^{n_k}} \quad \text{for every} \quad m_{k-1} \leq n < m'_k. \tag{5}$$

Let m_k be the biggest integer $m'_k \leq m_k$ for which (5) holds. Inequality (4) still is satisfied with m_k instead of m'_k. We also deduce that

$$\frac{1}{2^{n_k}} \geq \sqrt{\gamma_n} \quad \text{for every} \quad n \geq m_k.$$

We will now define the sequence (a_n) we are looking for. Let $A_k = \sum_{n \in J_k} \beta_n$. We then define:

$$a_1 = 1,$$
$$a_{n+1} = a_n - \frac{\beta_n}{2^k A_k} \quad \text{for} \quad m_k < n < m_{k+1}.$$

The sequence (a_n) is strictly decreasing as β_n and are A_k positive. From condition i) it follows that $A_k 2^k \geq 1$, so

$$a_n - a_{n+1} = \frac{\beta_n}{2^k A_k} \leq \beta_n.$$

We then have 1). Let us see that (a_n) is positive and convergent to zero. On the one hand

$$\begin{aligned}\sum_{n=1}^{\infty}(a_n - a_{n+1}) &= \sum_{k=1}^{\infty}(\sum_{n\in J_k}(a_n - a_{n+1}))\\ &= \sum_{k=1}^{\infty}(\sum_{n\in J_k}\frac{\beta_n}{2^k A_k})\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}\\ &= 1.\end{aligned}$$

On the other hand, as the series is telescopic, we have that

$$\sum_{i=1}^{n}(a_i - a_{i+1}) = a_1 - a_{n+1} = 1 - a_{n+1}.$$

So $a_n \geq 0$ for every n. We will see to finish that condition 2) is satisfied. Given $n \in \mathbb{N}$, let k_0 be such that $n \in J_{k_0}$. Then by ii) we have

$$\begin{aligned}\frac{a_n}{\gamma_n} &= (\sum_{i=n}^{\infty}(a_i - a_{i+1}))\gamma_n^{-1}\\ &\geq (\sum_{k=k_0+1}^{\infty}(\sum_{i\in J_k}\frac{\beta_i}{2^k A_k}))\gamma_n^{-1}\\ &= (\sum_{k=k_0+1}^{\infty}\frac{1}{2^k})\gamma_n^{-1}\\ &= \frac{1}{2^{k_0}\gamma_n}\\ &\geq \frac{1}{2^{n_{k_0}}\gamma_n}\\ &\geq \frac{1}{\sqrt{\gamma_n}},\end{aligned}$$

that tends to infinity as $(\sqrt{\gamma_n})$ decreases to zero. Q.E.D.

Let us apply the previous Lemma to the following sequences $\beta_n = \|x_n\|$ and $\gamma_n = \sup\{\sum_{i\geq n}|x^* x_i| : x^* \in B_{X^*}\}$. Let (a_n) be the sequence given by the Lemma. Define the sequence $\alpha_n = (a_n - a_{n+1})/\beta_n$. Then $0 \le \alpha_n \le 1$ by condition 1). On the other hand

$$\sum_{n=1}^{\infty} \alpha_n \|x_n\| = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 < +\infty$$

so (b) is deduced. Let us see that (c) is satisfied:

$$\begin{aligned}(\sum_{i\ge n} \alpha_i \|x_i\|) \cdot (\sup\{\sum_{i\ge n} |x^* x_i| : x^* \in B_{X^*}\})^{-1} &= (\sum_{i\ge n} \alpha_i \beta_i) \cdot (\gamma_n)^{-1} \\ &= (\sum_{i\ge n} (a_i - a_{i+1})) \cdot (\gamma_n)^{-1} \\ &= a_n \cdot \gamma_n^{-1}\end{aligned}$$

that tends to infinity by 2). Q.E.D.

Conditions in order to have as $L^1(\nu)$ an AL–space can be impossed, at first, on the Banach space X and on the measure ν. The next example shows that it is not sufficient to imposse them exclusively on the Banach space. It also shows that condition (L 1), although necessary, it is not sufficient.

Example 3.5. *Let X be an infinite dimensional Banach space. There exists a measure ν with values in X, satisfying condition (L 1), such that $L^1(\nu)$ is not order isomorphic to an AL–space.*

PROOF. Let (x_n) be a sequence in X given by the Theorem of Dvoretsky–Rogers, that is $\sum x_n$ converges unconditionally but not absolutely. Consider the following measure:

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum_{n\in A} \frac{x_n}{2^n \cdot \|x_n\|} \in X\,.$$

It is well defined and countably additive by the unconditional convergence of the series that defines it. The absolute convergence of the series proves that the measure has bounded variation, as

$$|\nu|(A) = \sum_{n\in A} \frac{1}{2^n}.$$

Thus the total variation is 1. The spaces $L^1(\nu)$ and $L^1(|\nu|)$ are the sequence spaces given by

$$L^1(\nu) = \left\{(b_n) : \sum \frac{b_n x_n}{2^n \cdot \|x_n\|} \text{ converges inconditionally in } X\right\}$$

$$L^1(|\nu|) = \left\{(b_n) : \sum \frac{b_n}{2^n} \text{ is absolutely summable } \right\}.$$

Thus $(2^n \cdot \|x_n\|)$ is a sequence in $L^1(\nu)$ but not in $L^1(|\nu|)$. It follows from Proposition 3.1 that $L^1(\nu)$ is not order isomorphic to an AL–space.

Let us see that condition (L 1) is satisfied. Let A be a subset of $\mathbb{N}$. Let $n_0 = \min\{n : n \in \ A\}$. Then

$$|\nu|(A) = \sum_{n\in A} \frac{1}{2^n} \leq \sum_{n\geq n_0} \frac{1}{2^n} = \frac{2}{2^{n_0}}.$$

On the other hand we have

$$\|\nu\|(A) \geq \sup\{\|\nu(B)\| : B \subset A\} \geq \|\nu(\{n_0\})\| = \frac{1}{2^{n_0}}.$$

That is, for every measurable set A we have

$$|\nu|(A) \leq 2 \cdot \|\nu\|(A). \qquad \text{Q.E.D.}$$

For measures with values in certain spaces we can obtain simple sufficient conditions, even characterizations.

Consider measures with values in AL–spaces. Recall that an operator between Banach lattices is *regular* if it is the difference of two linear, continuous and positive operators. We have the following result.

Proposition 3.6. *Let ν be a measure with values in an AL–space. The following conditions are equivalent:*

a) *$L^1(\nu)$ is order isomorphic to an AL–space.*

b) *The integration operator is regular.*

PROOF. a) $\Rightarrow$ b) It follows from the fact that every operator defined on an AL–space and with values in a Banach lattice which is complemented in its bidual by a positive projection, is regular [S, Theorem IV.1.5].

b) $\Rightarrow$ a) Let $\nu: \Sigma \longrightarrow L$, where L is an AL–space. Suppose that the operator $\nu: L^1(\nu) \longrightarrow L$ is regular. It follows that the transposed $\nu^*: L^* \longrightarrow L^1(\nu)^*$ is also regular. Thus it maps order bounded sets in L^* into order bounded sets in $L^1(\nu)^*$. L^* is an AM–space with unit as it is the dual of an AL–space. Thus its unit ball is order bounded. It follows that ν^* maps norm bounded sets into order bounded sets and so, by Proposition 3.1, we have that $L^1(\nu)$ is an AL–space. Q.E.D.

From the previous Proposition we can deduce the following sufficient condition.

Corollary 3.7. *Let ν be a measure with values in an AL–space. Then $L^1(\nu)$ is an AL–space if ν has a Hahn decomposition, that is, there exists a measurable set A such that $\nu(B) \geq 0$ if $B \subset A$ and $\nu(B) \leq 0$ if $B \subset \Omega \setminus A$.*

PROOF. It suffices to prove the result for positive measures as in the general case the measure ν is the direct sum of the measures ν_1 and ν_2, restriction of ν to A and $\Omega \setminus A$ respectively, and so $L^1(\nu)$ is order ismorphic to the space $L^1(\nu_1) \oplus L^1(\nu_2)$, see 1.13. For positive measures the integration operator is positive and the results follows from the previous proposition. Q.E.D.

A result of Diestel y Faires [DF, Theorem 2.1] states that for measures with values in an AL–space it is equivalent bounded variation and regularity, that is being the difference of two positive measures. This is not the situation of the previous Corollary. The reason is that if the positive measures are not disjointly suported there is no assurance that the integration operator will be regular. As a counterexample just consider the one given by Example 3.5 when the Banach space X is an AL–space.

We consider now measures with values in spaces $C(K)$ of continuous functions over a compact Hausdorff topological space K. This case includes measures with values in AM–spaces as, by a theorem of Kakutani [LT vol. II, Theorem 1.b.6], these spaces are order isometric to a sublattice of a space $C(K)$.

We will consider purely atomic measures. They can be defined over the σ–algebra $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$. We have the following result.

Theorem 3.8. *Let $\nu\colon \mathcal{P}(\mathbb{N}) \longrightarrow C(K)$ where K is a compact Hausdorff topological space. Let $f_n = \nu(\{n\})$ and let $|f_n|$ be the modulus of f_n in $C(K)$. The following conditions are equivalent:*

a) $L^1(\nu)$ is order isomorphic to an AL–space.

b) The measure ν satisfies the condition

$$0 \notin \overline{co}\left\{\frac{|f_n|}{\|f_n\|} : n \in \mathbb{N}\right\}. \tag{L 2}$$

PROOF. Proposition 3.1 shows that a) is equivalent to $L^1(\nu)$ being isomorphic to $L^1(|\nu|)$. The measure $|\nu|$ is the following

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto |\nu|(A) = \sum_{n\in A} \|f_n\| \in \mathbb{R}\,.$$

The natural injection of the space $L^1(|\nu|)$ into $L^1(\nu)$ is always continuous. As ν has bounded variation and simple functions are dense the injection has dense range. Thus it is equivalent to prove that the injection is open, that is there exists a constant $C > 0$ such that for every function F in $L^1(|\nu|)$ we have

$$\|F\|_1 \leq C \cdot \|F\|_\nu .$$

Again by density of simple functions in $L^1(|\nu|)$ it suffices to prove the previous inequality for simpe functions. Moreover it suffices to prove it for simple functions with finite support (in $\mathbb{N}$) and then extend the result by an approximation argument. Lastly observe that it suffices to consider positive functions, as a function and its modulus have the same norm in both spaces.

The previous argument shows that $L^1(\nu)$ is an AL–space if and only if there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ for arbitrary positive scalars $a_1, \ldots, a_N$ we have

$$C \cdot \left\| \sum_1^N a_n \chi_{\{n\}} \right\|_1 \leq \left\| \sum_1^N a_n \chi_{\{n\}} \right\|_\nu .$$

Consider in $L^1(\nu)$ the equivalent norm given in 1.5

$$|||F|||_\nu = \sup \left\{ \| \int_B F \, d\nu \, \| : B \subset \mathbb{N} \right\},$$

we have

$$C \cdot \sum_1^N a_n \|f_n\| \leq \sup \left\{ \left\| \sum_{n \in B} a_n f_n \right\| : B \subset \{1, \ldots, N\} \right\}.$$

Dividing the previous expression by $\sum_1^N a_n \|f_n\|$ we obtain

$$C \leq \sup \left\{ \left\| \sum_{n \in B} \frac{a_n \|f_n\|}{\sum_1^N a_n \|f_n\|} \cdot \frac{f_n}{\|f_n\|} \right\| : B \subset \{1, \ldots, N\} \right\}.$$

Denote $g_n = f_n/\|f_n\|$. Then $L^1(\nu)$ is an AL–space if and only if there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ and every $\alpha_1, \ldots, \alpha_n$ positive with $\sum_1^N \alpha_n = 1$, we have

$$C \leq \sup \left\{ \left\| \sum_{n \in B} \alpha_n g_n \right\| : B \subset \{1, \ldots, N\} \right\}.$$

Let us study the supremum that appears in the previous expression. On the one hand given B in $\{1, \ldots, N\}$

$$\left\| \sum_{n \in B} \alpha_n g_n \right\| \leq \left\| \sum_1^N \alpha_n |g_n| \right\|.$$

So this last expression bounds the mentioned supremum. On the other hand there exists a point $t_0 \in K$ where the function $\sum_1^N \alpha_n |g_n|$ attains its supremum, thus

$$\left\| \sum_1^N \alpha_n |g_n| \right\| = \sum_1^N \alpha_n |g_n(t_0)| = \sum_{B_1} \alpha_n g_n(t_0) + \sum_{B_2} \alpha_n g_n(t_0),$$

let $B_1 = \{n, g_n(t_0) \geq 0, 1 \leq n \leq N\}$ and B_2 the complement of B_1 in $\{1, \ldots, N\}$.

Then

$$\begin{aligned} \left\| \sum_1^N \alpha_n |g_n| \right\| &\leq \left\| \sum_{B_1} \alpha_n g_n \right\| + \left\| \sum_{B_2} \alpha_n g_n \right\| \\ &\leq 2 \cdot \sup \left\{ \left\| \sum_{n \in B} \alpha_n g_n \right\| : B \subset \{1, \ldots, N\} \right\}. \end{aligned}$$

Thus both expressions are equivalent and so $L^1(\nu)$ is an AL–space if and only if there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ and every $\alpha_1, \ldots, \alpha_n$ positive with $\sum_1^N \alpha_n = 1$, we have that

$$C \leq \left\| \sum_1^N \alpha_n \frac{|f_n|}{\|f_n\|} \right\|.$$

This is precisely condition (L 2). Q.E.D.

We now start the study of the general case for a measure with values in a Banach space X. Example 3.5 shows that a priori conditions have to be impossed on the vector measure. We have seen that bounded variation for ν is a necessary condition. This implies that the natural injection

$$L^1(|\nu|) \longrightarrow L^1(\nu)$$

has dense range, as simple functions are dense in $L^1(\nu)$. Thus the transposed is an injective map, so $L^1(\nu)^*$ can be identified with a linear subspace of $L_\infty(|\nu|)$ the dual space of $L^1(|\nu|)$. In fact we have the following proposition.

Proposition 3.9. *Let ν be a vector measure with bounded variation. Then $L^1(\nu)^*$ can be identified with a lattice ideal in $L_\infty(|\nu|)$.*

PROOF. Let λ be a Rybakov control measure for ν. We have seen in 1.14 that $L^1(\nu)^*$ is a lattice ideal in $L^1(\lambda)$. We have to prove that if g is a function in $L_\infty(|\nu|)$ that belongs to $L^1(\nu)^*$ and if h is in $L_\infty(|\nu|)$ such that $|h| \leq |g|$ but for a $|\nu|$–null set, then h is in $L^1(\nu)^*$. As λ and $|\nu|$ have the same null sets, it follows, on the one hand that $|h| \leq |g|$ but for a λ–null set, and on the other hand, in this case of bounded variation, we have the following injections

$$L^1(\nu)^* \longrightarrow L_\infty(|\nu|) \longrightarrow L^1(\lambda)\,.$$

Hence h is in $L^1(\nu)^*$. Q.E.D.

In $L_\infty(|\nu|)$ algebraic and lattice ideals coincide, so from the previous proposition it follows that $L^1(\nu)^*$ is an algebraic ideal in $L_\infty(|\nu|)$.

From Proposition 3.1 we know that $L^1(\nu)$ being an AL–space is equivalent to being sujective the natural injection of $L^1(|\nu|)$ into $L^1(\nu)$, and so it is equivalent to identifying, in the way we have seen, the space $L^1(\nu)^*$ with $L_\infty(|\nu|)$.

This happens precisely when $L^1(\nu)^*$ is closed in $L_\infty(|\nu|)$. From the previous proposition $L^1(\nu)^*$ is a lattice ideal in $L_\infty(|\nu|)$, thus they coincide if and only if there exists an element in $L^1(\nu)^*$ that *dominates a constant function*. This idea is precised in the discussion that follows.

Let μ be a finite measure over (Ω, Σ). $L_\infty(\mu)$ is a Banach algebra. Let Δ be the set of linear and multiplicative functionals (characters) over $L_\infty(\mu)$. Endowed with the induced $\sigma(L_\infty, L^1)$ topology is a topological compact Hausdorff space. Let us consider the Gelfand transform, defined in the following way:

$$f \in L_\infty(\mu) \longmapsto f\,\hat{} \in C(\Delta)$$

defined by

$$s \in \Delta \longmapsto f\,\hat{}(s) = s(f) \in \mathbb{R}\,.$$

The Theorem of Gelfand states that the previous map is an algebraic isomorphism that preserves the order and the norm.

The next Lemma describes the action of a character over a function. Although it is known we include a prove as we have not found any precise reference. Let $\mu^{-1}(0)$ be the family of measurable sets with measure zero. Let $\Sigma/\mu^{-1}(0)$ be the quotient σ–algebra.

Lema 3.10. *Let s be a character over $L_\infty(\Omega, \Sigma, \mu)$, then there exists an ultrafilter $\mathcal{U}$ in $\Sigma/\mu^{-1}(0)$ such that*

$$s(f) = \lim_{A \in \mathcal{U}} \frac{\int_A f \, d\mu}{\mu(A)}$$

for every $f \in L_\infty(\Omega, \Sigma, \mu)$. Conversely, for every ultrafilter in $\Sigma/\mu^{-1}(0)$ the previous expression defines a character over $L_\infty(\Omega, \Sigma, \mu)$.

PROOF. Let $\mathcal{U}$ be an ultrafilter in $\Sigma/\mu^{-1}(0)$. In order to simplify notation we will identify measurable sets with their equivalence class in the quotient σ–algebra. Consider the map s defined in the statement of the Lemma. Let f be in $L_\infty(\Omega,\Sigma,\mu)$. As

$$\left|\int_A f\,d\mu\right| \le \|f\|_\infty \cdot \mu(A),$$

it follows that the net $\left(\mu(A)^{-1}\cdot\int_A f\,d\mu\right)_{A\in\mathcal{U}}$ is bounded and so there exists the limit by the ultrafilter $\mathcal{U}$. So s is well defined. Linearity of integration and limits guarantee the linearity of s. The previous bound shows that s is a bounded functional over $L_\infty(\Omega,\Sigma,\mu)$. Let us see that it is multiplicative. Let $B\in\mathcal{U}$. Then

$$s(\chi_B) = \lim_{A\in\mathcal{U}} \frac{\mu(B\cap A)}{\mu(A)} = \lim_{A\in\mathcal{U},A\subset B} \frac{\mu(B\cap A)}{\mu(A)} = 1$$

Analogously we obtain that if $B\notin\mathcal{U}$ then $s(\chi_B)=0$. Let $f\in L_\infty(\Omega,\Sigma,\mu)$. We have that for $B\in\mathcal{U}$,

$$\begin{aligned}
s(f\cdot\chi_B) &= \lim_{A\in\mathcal{U}} \frac{\int_A f\cdot\chi_B\,d\mu}{\mu(A)}\\
&= \lim_{A\in\mathcal{U},A\subset B} \frac{\int_A f\cdot\chi_B\,d\mu}{\mu(A)}\\
&= \lim_{A\in\mathcal{U},A\subset B} \frac{\int_A f\,d\mu}{\mu(A)}\\
&= \lim_{A\in\mathcal{U}} \frac{\int_A f\,d\mu}{\mu(A)}\\
&= s(f)\cdot 1\\
&= s(f)\cdot s(\chi_B)
\end{aligned}$$

Analogously, for $B\notin\mathcal{U}$ we obtain $s(f\cdot\chi_B)=0=s(f)\cdot s(\chi_B)$. Thanks to the linearity of s we deduce that it is multiplicative over the simple functions. As simple functions are dense, we deduce that s is multiplicative.

Conversely let s be a character. Consider the family $\mathcal{U}=\{A\in\Sigma/\mu^{-1}(0):s(\chi_A)=1\}$. It is well defined as $s(0)=0$. As for every measurable set A we have that $s(\chi_A)=0$ or 1, it follows that $\mathcal{U}$ is an ultrafilter. We have seen that this ultrafilter defines a character, which coincides with s over the characteristic functions and so over the simple functions. Hence it coincides with s. Q.E.D.

Consider now the measure space $(\Omega, \Sigma, |\nu|)$. We define the following sets in the space Δ of characters over $L_\infty(|\nu|)$.

Definition 3.11. *Let H be the set of characters that are null over $L^1(\nu)^*$. That is, the set of zeros of the image in $C(\Delta)$, by the Gelfand transform, of the ideal $L^1(\nu)^*$.*

Given x^* in X^* let g_{x^*} be the Radon–Nikodym derivative, with respect to $|\nu|$, of the scalar measure $x^*\nu$. As we have that $|x^*\nu|(A) \leq \|x^*\| \cdot |\nu|(A)$ for every measurable set A, it follows that g_{x^*} is in $L_\infty(|\nu|)$. On the other hand the map:

$$f \in L^1(\nu) \longmapsto \int f \, dx^*\nu = \int f g_{x^*} d|\nu| \in \mathbb{R}$$

defines a countinuous linear functional. Hence g_{x^*} is in $L^1(\nu)^*$.

Definition 3.12. *Let $\mathcal{I}$ be the lattice ideal generated by the set $\{g_{x^*} : x^* \in X^*\}$ in $L^1(\nu)^*$.*

There exists a biyective map between the ideal $\mathcal{I}$ just defined and the one that appears in Theorem 1.23, as the measures λ and $|\nu|$ have the same null sets.

Definition 3.13. *Let H^* be the set of characters that are null over $\mathcal{I}$. That is, the set of zeros of the image in $C(\Delta)$, by the Gelfand transform, of the ideal $\mathcal{I}$*

The usefulness of the previous definitions in what respects to our problem is shown in the next propositions. Obviously we have $H \subseteq H^*$.

Proposition 3.14. *The following conditions are equivalent:*

a) $L^1(\nu)$ is order isomorphic to an AL–space.

b) The set H is empty.

PROOF. If $L^1(\nu)$ is order isomorphic to an AL–space from Proposition 3.1 it follows that $L^1(\nu)$ is order isomorphic to $L^1(|\nu|)$ and so H is empty. Conversely, if H is empty, it follows that $L^1(\nu)^*$ is dense in $L_\infty(|\nu|)$. As $L^1(\nu)^*$ is an (algebraic) ideal its clousure in $L_\infty(|\nu|)$ is also an ideal. An ideal is a proper ideal if and only if it is so its clousure. Then from being $L^1(\nu)^*$ dense in $L_\infty(|\nu|)$ it follows that $L^1(\nu)^*$ is not a proper ideal, so $L^1(\nu)^*$ coincides with $L_\infty(|\nu|)$. Hence $L^1(\nu)$ is order isomorphic to $L^1(|\nu|)$. Q.E.D.

Proposition 3.15. *The following conditions are equivalent:*

a) *There exists a finite partition $(A_i)_1^n$ and there exists elements $x_1^*, \dots, x_n^*$ in X^* such that the identity map is an (order) isomorphism between $L^1(\nu)$ and the space $L^1(\mu)$ where $\mu = \sum_1^n \mu_i$ and every measure μ_i is the restriction of the measure $|x_i^*\nu|$ to the trace of Σ over the set A_i.*

b) *The set H^* is empty.*

PROOF. a) $\Rightarrow$ b) From Proposition 3.1 it follows that $L^1(|\nu|)$ is order isomorphic to $L^1(\mu)$ via the identity. Thus there exists a constant $C > 0$ such that the following inequality holds

$$|\nu|(A) \leq C \cdot \sum_1^n |x_i^*\nu|(A \cap A_i) \quad \text{for every } A \in \Sigma, \tag{6}$$

by integration with respect to $|\nu|$ we deduce that

$$\chi_\Omega \leq C \cdot \sum_1^n |g_{x_i^*}| \cdot \chi_{A_i} \quad \text{a.e. } |\nu|.$$

So the unit of $L_\infty(|\nu|)$ is in the ideal $\mathcal{I}$ and so H^* is empty.

b) $\Rightarrow$ a) If H^* is empty so is H, so $L^1(\nu)$ is an AL–space and by Proposition 3.1 is isomorphic to $L^1(|\nu|)$. On the other hand the ideal $\mathcal{I}$ is dense in $L_\infty(|\nu|)$ and so, as we have seen in the previous proposition, $\mathcal{I}$ coincides with $L_\infty(|\nu|)$.

That is $L_\infty(|\nu|)$ is the lattice ideal generated by the set $\{g_{x^*} : x^* \in X^*\}$. Thus there exists $x_1^*, \ldots, x_n^*$ in X^* such that

$$\chi_\Omega \leq \sum_1^n |g_{x_i^*}| \ \ |\nu| \text{ a.e.}$$

Let us see that there exists disjoint measurable sets $(A_i)_1^n$ with $|\nu|(\cup A_i) = |\nu|(\Omega)$ and a constant $C > 0$ such that

$$\chi_\Omega \leq C \cdot \sum_1^n |g_{x_i^*}| \cdot \chi_{A_i} \ \ |\nu| \text{ a.e..}$$

Set $B_i = [|g_{x_i^*}| < 1/n]$ for $1 \leq i \leq n$. Then we have that $\cap_i B_i$ has $|\nu|$–null measure. Set $A_1 = B_1$ and $A_{i+1} = B_{i+1} \setminus \cup_1^i B_j$ if $i \geq 1$ and set $C = n$. By integrating with respect to $|\nu|$ in the previous inequality we obtain inequality (6). As we always have the inequality

$$\sum_1^n |x_i^* \nu|(A \cap A_i) \leq \max_i \|x_i^*\| \cdot |\nu|(A) \ \text{ for every } A \in \Sigma,$$

it follows that the identity map is an (order) isomorphism from $L^1(\nu)$ onto $L^1(\mu)$, for the measure μ of b). Q.E.D.

We deduce that in this case the space $L^1(\nu)$ is order isomorphic to the space $\left[\bigoplus_1^n L^1(|x_i^*\nu|, \Sigma_{A_i})\right]_1$.

The previous proposition allows to give a partial answer to the problem we are studying. For this consider the following condition on the vector measure $\nu\colon \Sigma \longrightarrow X$:

$$0 \notin \overline{co}\left\{\frac{\nu(A)}{|\nu|(A)} : |\nu|(A) \neq 0, A \in \Sigma\right\}. \qquad \text{(L 3)}$$

Then we have the following result:

Theorem 3.16. *The following conditions are equivalent:*

a) *$L^1(\nu)$ is order isomorphic via the identity map to $L^1(\Omega, \Sigma, \mu)$ where the measure μ is as in Proposition 3.15.*

b) *There exists a finite partition $(B_j)_1^k$ of the measure space such that the restriction of the measure ν to each set B_j satisfies condition (L 3).*

PROOF. a) $\Rightarrow$ b) By hypothesis, and using the notation of Proposition 3.15, there exists a constant $C > 0$ such that

$$|\nu|(A) \leq C \cdot \sum_1^n |x_i^* \nu|(A \cap A_i) \text{ for every } A \in \Sigma.$$

Let us consider the Hahn decomposition of the measure $x_i^* \nu$ restricted to the set A_i. There exists disjoint measurable sets A_i^1 and A_i^2 whose union is A_i and such that $x_i^* \nu(B) \geq 0$ for every measurable set $B \subset A_i^1$, and $x_i^* \nu(B) \leq 0$ for every measurable set $B \subset A_i^2$. Then we have

$$\left| \frac{x_i^* \nu(A)}{|\nu|(A)} \right| \geq \frac{1}{C} \quad \text{for every measurable set } A \subset A_i^k,$$

for $1 \leq i \leq n$ and $k = 1, 2$. Thus the restriction of the measure ν to each set A_i^k satisfies condition (L 3). As the sets A_i are disjoint so are the sets A_i^k.

b) $\Rightarrow$ a) Let ν_j, $1 \leq j \leq k$, be the restriction of the measure ν to B_j. As the measures ν_j have disjoint supports, we have

$$\|f\|_\nu \leq \sum_1^k \|f\|_{\nu_j} \leq k \cdot \|f\|_\nu \, .$$

So the space $L^1(\nu)$ is order isomorphic to the space $(\bigoplus_1^k L^1(\nu_j))_1$. It suffices to prove the result for each space $L^1(\nu_j)$, so we can assume, without loss of generality, that the measure ν satisfies condition (L 3). From Proposition 3.15

it suffices to prove that H^* is empty. Suppose that is not the case. There would exist a character $s \in \Delta$ such that $s(g_{x^*}) = 0$ for every $x^* \in X^*$. By Lemma 3.10 there exists an ultrafilter $\mathcal{U}$ in $\Sigma/|\nu|^{-1}(0)$ such that, for every $x^* \in X^*$

$$0 = \lim_{A\in\mathcal{U}} \frac{\int_A g_{x^*}\, d|\nu|}{|\nu|(A)} = \lim_{A\in\mathcal{U}} \frac{x^*\nu(A)}{|\nu|(A)}.$$

That is, the net $\{\nu(A)/|\nu|(A) : A \in \mathcal{U}\}$ tends weakly to zero which contradicts our hypothesis. Q.E.D.

The previous theorem does not solve completely the problem: there exists measures for which $L^1(\nu)$ is an AL–space, so H is empty, but such that $L^1(\nu)$ is not given by a finite number of spaces of the form $L^1(|x^*\nu|)$, that is H^* is not empty. This is shown in the next example.

Example 3.17. Suppose that there exists a sequence (f_n) in $C[0,1]$ satisfying the following conditions:

a) $\|f_n\| = 1$ for every n,

b) 0 is in the weak clousure of $\{f_n : n \in \mathbb{N}\}$,

c) there exist $\varepsilon > 0, \mu \in C[0,1]^*, \|\mu\| = 1$ such that $\mu(|f_n|) \geq \varepsilon$ for every n.

Consider then the following measure

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \nu(A) = \sum_{n\in A} \frac{1}{2^n} f_n \in C[0\,1],.$$

As it is defined by an absolutely convergent series, it is countably additive and has bounded variation. Let us see that $L^1(\nu)$ is order isomorphic to AL–space. In virtue of Theorem 3.8 it suffices to verify that it satifies condition (L 2). Consider a convex combination of the functions $|f_n| = |\nu(n)|/\|\nu(n)\|$, then

$$\Big\|\sum \alpha_n |f_n|\Big\|_\infty \geq \mu(\sum \alpha_n |f_n|) \geq \varepsilon.$$

Thus $L^1(\nu)$ is an AL–space. On the other hand, let $(B_j)_1^k$ be a partition of $\mathbb{N}$. We have

$$0 \in \overline{\{f_n : n \in \mathbb{N}\}}^{\text{weak}} \subset \bigcup_{j=1}^{k} \overline{co} \left\{ \frac{\nu(A)}{|\nu|(A)} : A \subset B_j \right\},$$

so condition b) in Theorem 3.16 can not be satisfied, and so $L^1(\nu)$ is not given by a finite number of spaces of the form $|x^*\nu|$,.

The existence of a sequence of continuous functions on the interval [0,1] satisfying the required conditions follows from the next result, that, although it is known, we prove, as we have not found a precise reference for it.

Lemma 3.18. *Consider the space of continuous functions on the interval [0,1] endowed with the supremum norm. The map that associates to every function its modulus*

$$f \in C[0,1] \longmapsto |f| \in C[0,1]$$

is not weak to weak continuous at the origin.

PROOF. We will prove that there exists a weak neighborhood of the origin $\mathcal{W}$ such that, for every basic weak neighborhood of the origin $\mathcal{O}$, there exists a function f in $\mathcal{O}$ whose modulus is not in $\mathcal{W}$. Let m be the Lebesgue measure on the interval [0,1] and let $\mathcal{W} = \{g \in C[0,1] : |\int g\ dm| < 1/2\}$, which is a weak neighborhood of the origin.

Consider an arbitrary basic weak neighborhood of the origin

$$\mathcal{O} = \left\{ g \in C[0,1] : \left| \int g\ d\mu_i \right| < \varepsilon, 1 \le i \le k \right\},$$

where $\mu_1, \ldots, \mu_k$ are in $C[0,1]^*$ and $0 < \varepsilon < 1$. Each μ_i is a Radon measure on $\mathcal{M}[0,1]$, the σ–algebra of Lebesgue measurable sets in [0,1], so it can be written in the form

$$\mu_i = \tau_i + \sum_{j=1}^{\infty} a_{ij} \delta_{x_{ij}},$$

where τ_i is non atomic and (x_{ij}) are points in [0,1] where the measure takes the value a_{ij}. Let $B = \cup_{i,j}\{x_{ij}\}$, as it is countable, we have $m(B) = 0$. Consider the measure

$$A \in \mathcal{M}[0,1] \longmapsto \tau(A) = (\tau_1(A), \ldots, \tau_k(A)) \in \mathbb{R}^k\,.$$

It is a non atomic measure with values in a finite dimensional space, from the Theorem of Liapunov it follows that its range is convex. Thus, there exists a measurable set $A \in \mathcal{M}[0,1]$ such that

$$\tau(A) = \frac{1}{2} \cdot \tau([0,1]).$$

It follows that $\tau_i(A) = \tau_i(A_i^c)$ for every $1 \le i \le k$. Consider the function $f = (\chi_A - \chi_{A^c}) \cdot \chi_{B^c}$. We have $\int f \, d\mu_i = 0$ for every $1 \le i \le k$.

Consider the measure $\mu = m + |\mu_1| + \cdots + |\mu_k|$. As the function f is measurable, there exists, by the Theorem of Lusin, a compact set K in [0,1] such that the restriction of f to K is continuous and $\mu(K^c) < \varepsilon/2$. Let g be a continuous function on [0,1] which coincides with f on K and has modulus less or equal than one. Then, for every $1 \le i \le k$ we have

$$\begin{aligned}\left|\int g \, d\mu_i\right| =& \left|\int (g - f) \, d\mu_i\right| \\ \le& \int_{K^c} |f - g| \, d|\mu_i| \\ \le& 2 \cdot \mu(K^c) \\ <& \varepsilon\end{aligned}$$

So g is in $\mathcal{O}$. On the other hand

$$\int |g| \, dm \ge \int_K |g| \, dm = \int_K |f| \, dm = 1 - m(K^c) > 1 - \varepsilon/2 > 1/2.$$

So $|g|$ is not in $\mathcal{W}$. Q.E.D.

The previous Lemma shows that the zero function is a weak cluster point of the set of functions in $C[0,1]$ which have norm one and for which $\int |f| \, dm \ge 1/2$.

As $C[0,1]$ is separable, it is separable in its weak topology, thus there exists a sequence (f_n) that is weakly dense in the above set. This is the sequence we were looking for. Q.E.D.

It should be pointed out that the map $f \in C[0,1] \longmapsto |f| \in C[0,1]$ is weak to weak sequentially continuous, thus the previous example can not be constructed from a weakly null sequence in $C[0,1]$.

We end this chapter showing that the coincidence of the sets H and H^* is related with the characterization of weak convergence in $L^1(\nu)$ seen in Theorem 1.23.

Proposition 3.19. *Consider the following conditions:*

a) *In $L^1(\nu)$ the weak convergence of bounded nets is characterized by the weak convergence (in X) of the integrals over arbitrary sets, that is, if $\sup_\alpha \|f_\alpha\|_\nu < +\infty$, then*

$$f_\alpha \overset{w}{\to} f \text{ in } L^1(\nu) \iff \int_A f_\alpha \, d\nu \overset{w}{\to} \int_A f \, d\nu \text{ in } X \text{ for every } A \in \Sigma.$$

b) *The sets H and H^* coincide.*

Then a) implies b).

PROOF. We have to prove that $H^* \subset H$. Let s be in H^*. It is given by an ultrafilter $\mathcal{U}$ in $\Sigma/|\nu|^{-1}(0)$. Consider the net $\{f_A = \chi_A/|\nu|(A) : A \in \mathcal{U}\}$. It is bounded as $\|f_A\|_\nu = \|\nu\|(A)/|\nu|(A) \leq 1$.

Let $B \in \mathcal{U}$, then

$$\begin{aligned}\lim_{A\in\mathcal{U}} \int_B f_A \, dx^*\nu &= \lim_{A\in\mathcal{U}} \frac{\int_{A\cap B} h_{x^*}\, d|\nu|}{|\nu|(A)} \\ &= \lim_{A\in\mathcal{U}, A\subset B} \frac{\int_A h_{x^*}\, d|\nu|}{|\nu|(A)} \\ &= \lim_{A\in\mathcal{U}} \frac{\int_A h_{x^*}\, d|\nu|}{|\nu|(A)} \\ &= s(h_{x^*}) = 0,\end{aligned}$$

as $s \in H^*$. For $B \notin \mathcal{U}$ we also obtain that the integral tends to zero since from a certain index on, the net is null. Thus the net $(f_A)_{\mathcal{U}}$ has the property that the integrals over arbitrary sets tend to zero weakly. From a) it follows that (f_A) is weakly null in $L^1(\nu)$. So for every h in $L^1(\nu)^*$ we have

$$s(h) = \lim_{A\in\mathcal{U}} \frac{\int_A h \, d|\nu|}{|\nu|(A)} = \lim_{A\in\mathcal{U}} \langle h, f_A\rangle = 0.$$

Thus $s(h) = 0$, and so s is in H. Q.E.D.

The next example shows the situation in which $L^1(\nu)$ is not an AL–space and the ideal $\mathcal{I}$ is not dense in $L^1(\nu)^*$, see Theorem 1.23. That is, we have $\emptyset \neq H$ and $H \neq H^*$.

Example 3.20. Let ν be the measure given in 3.17. Then $L^1(\nu)$ is an AL–space, but $H^* \neq \emptyset$. Let $1 < p < +\infty$ and let (a_n) be a positive summable sequence. Consider the measure

$$A \in \mathcal{P}(\mathbb{N}) \longmapsto \mu(A) = \sum_{n\in A} a_n e_n \in \ell^p ,$$

where (e_n) is the canonical basis of ℓ^p. It has bounded variation. From Proposition 1.22 it follows that $L^1(\mu) = \ell^p$ and the integration operator is the identity map. Thus for this measure $H \neq \emptyset$ and $H = H^*$. The measure obtained as the direct sum of the measures ν and μ, see 1.13, generates a space for which $\emptyset \neq H \neq H^*$.

CHAPTER 4: Operators with values in $L^1(\nu)$.

We study in this chapter continuous linear operators defined on an arbitrary Banach space taking values in the space $L^1(\nu)$. We will use a classical thecnique, that goes back to the study by Bartle, Dunford and Schwartz of operators defined on spaces of continuous functions [BDS]. The idea is to associate to each operator, in the above conditions, a vector measure and study the properties of the operator via de properties of the associated measure.

Let $\nu\colon \Sigma \longrightarrow X$ be a vector measure and let $L^1(\nu)$ be the space of integrable functions with respect to ν. Let Y be a Banach space. We will denote by $\mathcal{L}(Y,X)$ the space of continuous linear operators defined on Y with values in X. Let us consider a continuous linear operator:

$$T\colon Y \longrightarrow L^1(\nu) \ .$$

We will now define the measure associated to T.

Definition 4.1. *Let $T\colon Y \longrightarrow L^1(\nu)$ be a continuous linear operator. Let $\widetilde{T}$ be the following set function associated to the operator T:*

$$A \in \Sigma \longmapsto \widetilde{T}(A) \in \mathcal{L}(Y,X)\,,$$

where we define $\widetilde{T}(A)$ in the following way

$$y \in Y \longmapsto \widetilde{T}(A)y = \int_A Ty\, d\nu \in X\,.$$

Proposition 4.2. *$\widetilde{T}$ is a bounded finitely additive vector measure that is countably additive when considered in $\mathcal{L}(Y,X)$ the strong operator topology.*

PROOF. Let $A \in \Sigma$. $\widetilde{T}(A)$ is an operator from Y into X. As the operator T is linear and the integration with respect ν is linear, it follows that $\widetilde{T}(A)$ is also linear. Let us see that it is bounded:

$$\|\widetilde{T}(A)y\| = \| \int_A Ty\, d\nu \| \leq \|Ty\|_\nu \leq \|T\| \cdot \|y\|.$$

So $\widetilde{T}(A)$ is bounded and so $\widetilde{T}$ is well defined. $\widetilde{T}$ is finitely additive as integration with respect to ν is finitely additive. From the previous equation it follows that

$$\|\widetilde{T}(A)\| \leq \|T\| \text{ for every } A \in \Sigma.$$

So $\widetilde{T}$ is bounded. Let $(A_i)_1^\infty$ be a partition and let $y \in Y$ be fixed. Consider the measure with density $Ty \in L^1(\nu)$ with respect to ν. It is countably additive. Then

$$\widetilde{T}(\bigcup_i A_i)y = \int_{\cup_i A_i} Ty\, d\nu = \sum_i \int_{A_i} Ty\, d\nu = \sum_i \widetilde{T}(A_i)y.$$

Hence $\widetilde{T}$ is countably additive with respect to the strong operator topology. Q.E.D.

The following simple example shows that additivity of the measure $\widetilde{T}$ can not be improved, that is, it is not true, in general, that the measure $\widetilde{T}$ is countably additive when considered the uniform topology in $\mathcal{L}(Y,X)$.

Example 4.3. Consider the Lebesgue (vector) measure restricted to the interval [0,1], $m\colon \mathcal{M}[0,1] \longrightarrow \mathbb{R}$. The space of integrable functions with respect to m in the sense of Definition 1.1 coincides with the space $L^1[0,1]$. Let $Y = L^1[0,1]$ and consider the identity operator

$$f \in L^1[0,1] \longmapsto Tf = f \in L^1[0,1]\,.$$

The measure $\widetilde{T}$ takes values in the space $L_\infty[0,1]$ and it is defined as follows, for every $A \in \Sigma$

$$\widetilde{T}(A)f = \int_A f\,dm = \langle f, \chi_A \rangle$$

for each $f \in L^1[0,1]$. Thus $\widetilde{T}(A)$ can be identified with $\chi_A \in L_\infty[0,1]$. Hence the measure $\widetilde{T}$ is not countably additive in the uniform topology, as $\|\chi_A\|_\infty = 1$ for every $A \in \Sigma$ with $m(A) > 0$.

Note 4.4. It will be usefull later the following equivalent expresion for the semivariation of the measure $\widetilde{T}$:

$$1/2 \cdot \sup_{y \in B_Y} \|Ty \cdot \chi_A\|_\nu \leq \|\widetilde{T}\|(A) \leq 2 \cdot \sup_{y \in B_Y} \|Ty \cdot \chi_A\|_\nu \,.$$

To prove it it suffices to consider the equivalent expresions for the semivariation, $|||\cdot|||$, and for the norm in $L^1(\nu)$, $|||\cdot|||_\nu$, see 1.5. We will only prove one of the inequalities

$$\begin{aligned}\|\widetilde{T}\|(A) \leq& 2 \cdot \sup\{\|\widetilde{T}(B)\| : B \subset A\} = 2 \cdot \sup_{B \subset A} \sup_{y \in B_Y} \|\int_B Ty\,d\nu\| \\ =& 2 \cdot \sup_{y \in B_Y} \sup_{B \in \Sigma} \|\int_B Ty \cdot \chi_A\,d\nu\| \leq 2 \cdot \sup_{y \in B_Y} \|Ty \cdot \chi_A\|_\nu \,.\end{aligned}$$

We are interested in characterizing the operators whose associated measure is countably additive in the uniform operator topology. See the Preliminaries for the defintion of L–weakly compact set.

Theorem 4.5. *Let $T: Y \longrightarrow L^1(\nu)$ be an operator and let $\widetilde{T}: \Sigma \longrightarrow \mathcal{L}(Y,X)$ be the associated measure. The following conditions are equivalent*

a) The operator T is L–weakly compact.

b) The measure $\widetilde{T}$ is countably additive in the uniform operator topology.

c) The measure $\widetilde{T}$ is strongly additive in the uniform operator topology.

PROOF. a) $\Rightarrow$ b) We know that in $L^1(\nu)$, as it is an order continuous Banach lattice, L–weakly compact sets coincide with bounded equi–integrable sets. Thus $T(B_Y)$ is equi–integrable. Let λ be a Rybakov control measure for ν. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \Sigma$ and $\lambda(A) < \delta$ we have $\|Ty \cdot \chi_A\|_\nu < \varepsilon$ for every $y \in B_Y$. As $\|\widetilde{T}(A)y\| \leq \|Ty \cdot \chi_A\|_\nu$ it follows that if $\lambda(A) < \delta$ then $\|\widetilde{T}(A)\| \leq \varepsilon$ and b) follows as λ is countably additive.

b) $\Leftrightarrow$ c) It is a general fact, as the measure is countably additive in the strong operator topology which is a weaker Hausdorff topology.

c) $\Rightarrow$ a) Suppose that T is not L–weakly compact. Then there exists a sequence (g_n) in $L^1(\nu)$ of positive pairwise disjoint functions such that for every n there exists $y_n \in B_Y$ such that $0 \leq g_n \leq |Ty_n|$, but (g_n) does not converge to zero in $L^1(\nu)$. Thus we can assume that $\|g_n\|_\nu \geq \varepsilon > 0$ for every n and some $\varepsilon > 0$. Let $A_n = \{\omega \in \Omega : g_n(\omega) > 0\}$. They are measurable sets and we can assume that they are pairwise disjoint. Then, by using the equivalent expresion given for the semivariation of the measure $\widetilde{T}$, we have

$$2 \cdot \|\widetilde{T}\|(A_n) \geq \|Ty_n \cdot \chi_{A_n}\|_\nu \geq \|g_n\|_\nu \geq \varepsilon,$$

for every n. Hence $\widetilde{T}$ is not strongly additive (see [DU, Corollary I.1.18]). Q.E.D.

In order to characterize the operators whose associated measure has bounded variation we have to consider a more restrictive class of operators. Recall that an operator defined on a Banach space and with values in a Banach lattice is *order bounded* if it maps norm bounded sets into order bounded sets.

From now on we are going to use the condition of σ–finiteness of the variation of the measure ν. Although it a restriction we recall that it is equivalent to the localizability of $|\nu|$, 1.13.

Theorem 4.6. *Let ν be a measure with σ–finite variation. Let $T\colon Y \longrightarrow L^1(\nu)$ be an operator and $\widetilde{T}\colon \Sigma \longrightarrow \mathcal{L}(Y,X)$ the associated measure. The following conditions are equivalent:*

a) *The measure $\widetilde{T}$ has bounded variation.*

b) *The operator T factorizes in the following way: $T = i \circ S$ where the operator $i\colon L^1(|\nu|) \longrightarrow L^1(\nu)$ is the natural inclusion and $S\colon Y \longrightarrow L^1(|\nu|)$ is an order bounded operator.*

In the previous conditions, the operator T is integral.

PROOF. a) $\Rightarrow$ b) The measure $\widetilde{T}$ has bounded variation, so it is strongly additive. From the previous theorem it follows that it is countably additive in the uniform operator topology. Let $y \in Y$, consider the measure with density Ty with respect ν. Let (A_i) be a partition of a set $A \in \Sigma$. Then

$$\sum_i \| \int_{A_i} Ty \, d\nu \| = \sum_i \|\widetilde{T}(A_i)y\| \leq \sum_i \|\widetilde{T}(A_i)\| \cdot \|y\| \leq |\widetilde{T}|(A) \cdot \|y\|. \quad (1)$$

We deduce that the measure with density Ty with respect to ν has bounded variation and so by Theorem 1.12 it follows that Ty is in $L^1(|\nu|)$. So $T(Y)$ is in $L^1(|\nu|)$. From inequality (1) and Theorem 1.12 again it follows that, for every $A \in \Sigma$, we have

$$\int_A |Ty| \, d|\nu| \leq |\widetilde{T}|(A) \cdot \|y\|. \quad (2)$$

Consider $A = \Omega$. Then by (2) if follows that

$$\|Ty\|_{L^1(|\nu|)} \leq |\widetilde{T}|(\Omega) \cdot \|y\|,$$

So the operator T factorizes through $L^1(|\nu|)$.

The measure $\widetilde{T}$ has bounded variation so it is strongly additive and thus countably additive. As it is absolutely continuous with respect to $|\nu|$, which is σ–finite, it follows from the Radon–Nikodym Theorem that there exists a function $g \in L^1(|\nu|)$ such that

$$|\widetilde{T}|(A) = \int_A g\, d|\nu| \qquad \text{for every } \; A \in \Sigma. \tag{3}$$

Let $y \in B_Y$, it follows from inequality (2) and equality (3) that

$$\int_A |Ty|\, d|\nu| \le |\widetilde{T}|(A) = \int_A g\, d|\nu| \qquad \text{for every } \; A \in \Sigma.$$

So $|Ty| \le g$ almost everywhere with respect to $|\nu|$, for every $y \in B_Y$.

Hence, the operator $S\colon Y \longrightarrow L^1(|\nu|)$ defined by $Sy = Ty$ is well defined, linear, continuous and order bounded.

b) $\Rightarrow$ a) As $S(B_Y)$ is order bounded in $L^1(|\nu|)$, there exists a function $g \in L^1(|\nu|)$ such that $|Sy| \le g$ for every $y \in B_Y$. So, for $y \in B_Y$ and $A \in \Sigma$ we have

$$\|\widetilde{T}(A)y\| = \| \int_A Ty\, d\nu \| \le \int_A |Ty|\, d|\nu| = \int_A |Sy|\, d|\nu| \le \int_A g\, d|\nu|.$$

Taking supremum over $y \in B_Y$ it follows that $\|\widetilde{T}(A)\| \le \int_A g\, d|\nu|$. Hence the measure $\widetilde{T}$ has bounded variation.

A theorem of Grothendieck characterizes integral operators $T\colon Y \longrightarrow L^1(\mu)$ for μ a positive measure, as those that are order bounded (see [DU, p. 258]). Thus, condition b) implies that S is integral and from the ideal property of integral operators, it follows that T is integral. Q.E.D.

Under what conditions every integral operator has an associated measure with bounded variation? The next proposition shows that the fact that every onedimensional (so integral) operator factorizes through $L^1(|\nu|)$ characterizes AL–spaces (see Proposition 3.1).

Proposition 4.7. *Let ν be a measure with σ–finite variation. Suppose that there exists a Banach space $Y \neq \{0\}$ such that every onedimensional operator $T: Y \longrightarrow L^1(\nu)$ factorizes through $L^1(|\nu|)$. Then $L^1(\nu)$ is order ismorphic to $L^1(|\nu|)$.*

PROOF. Let $f \in L^1(\nu)$ and $y^* \in Y^*$, non null. Consider the onedimensional operator $T: Y \longrightarrow L^1(\nu)$ defined by $Ty = y^*(y) \cdot f$. By hypothesis T factorizes through $L^1(|\nu|)$. So $f \in L^1(|\nu|)$. It follows that the natural inclusion of $L^1(|\nu|)$ into $L^1(\nu)$ is sujective and so is an order preserving isomorphism. Q.E.D.

We consider now the problem of finding conditions in order to guarantee that the measure $\widetilde{T}: \Sigma \longrightarrow \mathcal{L}(Y, X)$ associated to the operator $T: Y \longrightarrow L^1(\nu)$, has a Radon–Nikodym derivative with respect to the measure $|\nu|$. That is, there exists a function $F: \Omega \longrightarrow \mathcal{L}(Y, X)$ Bochner integrable with respect to $|\nu|$ such that $\widetilde{T}(A) = \int_A F(\omega)\, d|\nu|(\omega)$ for every $A \in \Sigma$.

Theorem 4.8. *Let ν be a measure with σ–finite variation. Let $T: Y \longrightarrow L^1(\nu)$ be an operator and $\widetilde{T}: \Sigma \longrightarrow \mathcal{L}(Y, X)$ its associated measure. Consider the following conditions:*

a) The measure $\widetilde{T}$ has a Bochner integrable derivative with respect to the variation of ν.

b) The operator T factorizes in the following way: $T = i \circ S$ where the operator $i: L^1(|\nu|) \longrightarrow L^1(\nu)$ is the natural inclusion and $S: Y \longrightarrow L^1(|\nu|)$ is a nuclear operator.

c) The operator T is nuclear.

Then a) implies b) and b) implies c).

PROOF. a) $\Rightarrow$ b) Let $F: \Omega \longrightarrow \mathcal{L}(Y, X)$ be a function Bochner integrable with respect to $|\nu|$ such that $\widetilde{T}(A) = \int_A F(\omega)\, d|\nu|(\omega)$ for every $A \in \Sigma$. The measure $\widetilde{T}$

has bounded variation and so by Theorem 4.5 the operator T factorizes through $L^1(|\nu|)$ as $T = i \circ S$ where $S\colon Y \longrightarrow L^1(|\nu|)$ is an order bounded operator. A theorem of Grothendieck characterizes nuclear operators with values in $L^1(\mu)$, for a positive measure μ, as those that are order bounded and for which the image of the unit ball is equimeasurable (see [DU, p. 258]). A set K in $L^1(\mu)$ is *equimeasurable* if for every $\varepsilon > 0$ there exists a set A with $\mu(A) < \varepsilon$ such that $\{f \cdot \chi_{\Omega\setminus A} : f \in K\}$ is relatively compact in $L_\infty(\mu)$. We just have to prove this last condition for $S(B_Y)$.

Let $x^* \in B_{X^*}$ such that the measure $\lambda = |x^*\nu|$ is a Rybakov control measure for ν. Let $y \in B_Y$ and $A \in \Sigma$. On the one hand

$$\langle x^*, \widetilde{T}(A)y\rangle = \left\langle x^*, \int_A F(\omega)y\, d|\nu|(\omega)\right\rangle = \int_A \langle x^*, F(\omega)y\rangle\, d|\nu|(\omega),$$

on the other hand

$$\langle x^*, \widetilde{T}(A)y\rangle = \left\langle x^*, \int_A Sy\, d\nu\right\rangle = \int_A Sy(\omega)\, dx^*\nu(\omega).$$

It follows that

$$\int_A Sy(\omega)\, dx^*\nu(\omega) = \int_A \langle x^*, F(\omega)y\rangle\, d|\nu|(\omega)$$

As the measure $|\nu|$ is σ–finite and absolutely continuous with respect to λ, there exists a function h locally integrable with respect to λ such that $|\nu|(A) = \int_A h\, d\lambda$ (both members infinite in case one of them is). Then we have

$$\int_A Sy(\omega)\, dx^*\nu(\omega) = \int_A \langle x^*, F(\omega)y\rangle h(\omega)\, d\lambda(\omega). \tag{4}$$

Consider the Hahn decomposition of the measure $x^*\nu$, we can find a measurable function g with $|g| = 1$ such that equation (4) can be written as

$$\int_A Sy(\omega)\, d\lambda(\omega) = \int_A \langle x^*, F(\omega)y\rangle h(\omega)g(\omega)\, d\lambda(\omega).$$

Consider the function $F' = h \cdot g \cdot F$. Then we have that $F' \in L^1(\lambda, \mathcal{L}(Y,X))$ and

$$\int_A Sy(\omega)\, d\lambda(\omega) = \int_A \langle x^*, F'(\omega)y\rangle\, d\lambda(\omega).$$

As the previous expression holds for every $A \in \Sigma$ we deduce that $Sy(\omega) = \langle x^*, F'(\omega)y\rangle$ almost everywhere with respect to λ, for every $y \in B_Y$.

Suppose that F' is a simple function, that is, $F'(\omega) = \sum_1^n T_i \cdot \chi_{A_i}(\omega)$, for $T_i \in \mathcal{L}(Y,X)$. Then $Sy(\omega) = \sum_1^n \langle x^*, T_i y\rangle \cdot \chi_{A_i}(\omega)$ for every $y \in B_Y$. Thus the set $\{Sy : y \in B_Y\}$ is included in the set of linear combinations of the functions $\{\chi_{A_i} : 1 \leq\ i \leq n\}$ with bounded coeficients $|\langle x^*, T_i y\rangle| \leq \|T_i\|$ (as $\|y\| \leq 1$ and $\|x^*\| \leq 1$). Thus, $S(B_Y)$ is a compact in $L_\infty(\lambda)$.

In the general case, let G_n be a sequence of simple functions that converge almost everywhere with respect to λ to F. By Egoroff's Theorem, given $\varepsilon > 0$ we can find $A \in \Sigma$ with $\lambda(A) < \varepsilon$ such that in $\Omega \setminus A$ the convergence is uniform. Thus, given $\delta > 0$ we can find n such that $\|F'(\omega) - G_n(\omega)\|_{\mathcal{L}(Y,X)} < \delta$ for every $\omega \in \Omega \setminus A$. As $\|x^*\| \leq 1$, for every $y \in B_Y$ we have

$$|\langle x^*, F'(\omega)y\rangle - \langle x^*, G_n(\omega)y\rangle| < \delta \qquad \text{for every } \ \omega \in \Omega \setminus A.$$

We have seen that $Sy(\omega) = \langle x^*, F'(\omega)y\rangle$ almost everywhere with respect to λ, for each $y \in B_Y$. We deduce then that

$$|Sy(\omega) - \langle x^*, G_n(\omega)y\rangle| < \delta \qquad \text{for almost every } \ \omega \in \Omega \setminus A.$$

Thus $\|Sy \cdot \chi_{\Omega\setminus A} - \langle x^*, G_n y\rangle \cdot \chi_{\Omega\setminus A}\|_\infty < \delta$ for every $y \in B_Y$.

Let $S_n y = \langle x^*, G_n y\rangle \cdot \chi_{\Omega\setminus A}$ for every $y \in Y$. As G_n is a simple function, following what we have seen for F' simple, we have that the set $S_n(B_Y) = \{\langle x^*, G_n y\rangle \cdot \chi_{\Omega\setminus A} : y \in B_Y\}$ is compact in $L_\infty(\lambda)$. Hence for every $\delta > 0$ there exists a compact set with distance less than δ to the set $\{Sy \cdot \chi_{\Omega\setminus A} : y \in B_Y\}$. It follows that this last set is relatively compact in $L_\infty(\lambda)$.

This procedure can be done for every $\varepsilon > 0$, so $S(B_Y)$ is equimeasurable in $L^1(\lambda)$. As the measures λ and $|\nu|$ have the same null sets, it follows that $S(B_Y)$ is equimeasurable in $L^1(|\nu|)$ and so the operator T is nuclear.

b) $\Rightarrow$ c) It follows from de ideal property of nuclear operators. Q.E.D.

Proposition 4.7 shows that a less restrictive condition than c) $\Rightarrow$ b) in the previous theorem for every operator with values in $L^1(\nu)$ implies that $L^1(\nu)$ is an AL–space.

We are interested in the implication b) $\Rightarrow$ a) in Theorem 4.8. The next theorem shows that it is related to the existence of a derivative of the measure ν with respect to its variation $|\nu|$ which is Pettis integrable and strongly measurable (pointwise limit almost everywhere with respect to $|\nu|$ of a sequence of simple functions).

Theorem 4.9. *Let ν be a measure with σ–finite variation. The we have:*

1. *If the measure ν has a strongly measurable and Pettis integrable density with respect to its variation, then every operator $T: Y \longrightarrow L^1(\nu)$ that satisfies condition b) in Theorem 4.8 satisfies condition a).*
2. *If there exists a Banach space $Y \neq \{0\}$ such that every operator $T: Y \longrightarrow L^1(\nu)$ satisfying condition b) in Theorem 4.8 also satisfies condition a), then the measure ν has a Pettis integrable and strongly measurable density with respect to its variation.*

PROOF. 1. Let $G: \Omega \longrightarrow X$ be a strongly measurable function that is Pettis integrable with respect to $|\nu|$ such that $\nu(A) = \int_A G(\omega)\, d|\nu|$ for every $A \in \Sigma$. Clearly $\|G(\omega)\| = 1$ almost everywhere with respect to $|\nu|$. For $f \in L^1(\nu)$ and

$A \in \Sigma$, we have

$$\int_A f(\omega)\, d\nu(\omega) = \int_A f(\omega) G(\omega)\, d|\nu|(\omega).$$

Let $T: Y \longrightarrow L^1(\nu)$ be an operator that can be factorized as $T = i \circ S$ where $S: Y \longrightarrow L^1(|\nu|)$ is nuclear. The operator S can be written as $S = \sum y_n^* \otimes f_n$ where $y_n^* \in Y^*$, $f_n \in L^1(|\nu|)$ and $\sum \|y_n^*\| \cdot \|f_n\|_1$ is finite. Let $y \in Y$. As the serie $\sum y_n^*(y) f_n$ converges in $L^1(\nu)$, we have

$$\widetilde{T}(A)y = \int_A Sy\, d\nu = \int_A (\sum y_n^*(y) f_n)\, d\nu = \sum y_n^*(y) \int_A f_n\, d\nu.$$

We deduce that $\widetilde{T}(A) = \sum y_n^* \otimes \int_A f_n\, d\nu$, for every $A \in \Sigma$, where the convergence of the series is absolute as

$$\sum \|y_n^* \otimes \int_A f_n\, d\nu\| \leq \sum \|y_n^*\| \cdot \|f_n\|_\nu \leq \sum \|y_n^*\| \cdot \|f_n\|_1.$$

Consider the following function:

$$\omega \in \Omega \longmapsto F_n = y_n^* \otimes f_n(\omega) \cdot G(\omega) \in \mathcal{L}(Y, X)\,.$$

As f_n is measurable and G is strongly measurable it follows that F_n is strongly measurable. It is integrable with respect to $|\nu|$ as

$$\begin{aligned}
\int \|F_n(\omega)\|\, d|\nu|(\omega) &= \int \|y_n^* \otimes f_n(\omega) \cdot G(\omega)\|\, d|\nu|(\omega) \\
&= \int \|y_n^*\| \cdot |f_n(\omega)| \cdot \|G(\omega)\|\, d|\nu|(\omega) \\
&= \|y_n^*\| \int |f_n|\, d|\nu| \\
&= \|y_n^*\| \cdot \|f_n\|_1.
\end{aligned}$$

We define the function $\omega \in \Omega \longmapsto F(\omega) = \sum F_n(\omega) \in \mathcal{L}(Y, X)$. It is well defined and strongly measurable. It is integrable with respect to $|\nu|$ as

$$\int \|F(\omega)\|\, d|\nu|(\omega) \leq \sum \int \|F_n(\omega)\|\, |\nu|(\omega) = \sum \|y_n^*\| \cdot \|f_n\|_1 < +\infty.$$

Let $A \in \Sigma$, then

$$\begin{aligned}\int_A F(\omega)\, d|\nu|(\omega) &= \sum \int_A y_n^* \otimes f_n(\omega) \cdot G(\omega)\, d|\nu|(\omega)\\ &= \sum y_n^* \otimes \int_A f_n(\omega) \cdot G(\omega)\, d|\nu|(\omega)\\ &= \sum y_n^* \otimes \int_A f_n\, d\nu\\ &= \widetilde{T}(A).\end{aligned}$$

So F is Bochner integrable and is the Radon–Nikodym derivative of the measure $\widetilde{T}$ with respect to $|\nu|$.

2. As $|\nu|$ is σ–finite, there exists a partition (B_n) of Ω such that $|\nu|(B_n) < +\infty$ for every n. Consider $y_0 \in Y$ with norm one and a functional $y^* \in B_{Y^*}$ such that $y^*(y_0) = 1$. Consider the operator $y \in Y \longmapsto T_n(y) = y^*(y)\chi_{B_n} \in L^1(\nu)$, that is $T_n = y^* \otimes \chi_{B_n}$. It factorizes through $L^1(|\nu|)$ as

$$\int |T_n(y)|\, d|\nu| = |y^*(y)| \cdot |\nu|(B_n) \leq |\nu|(B_n) \cdot \|y\|.$$

Let us define $S_n y = T_n y \in L^1(|\nu|)$. Then $S_n\colon Y \longrightarrow L^1(|\nu|)$ is nuclear and we have the fatorization $T_n = i \circ S_n$. By hypothesis, the associated measure $\widetilde{T}_n$ has a derivative which is Bochner integrable with respect to $|\nu|$. That is, there exists a function $F_n \in L^1(|\nu|, \mathcal{L}(Y, X))$ such that

$$\widetilde{T}_n(A) = \int_A F_n(\omega)\, d|\nu|(\omega) \quad \text{para todo} \quad A \in \Sigma.$$

Fix $A \in \Sigma$. Then, on the one hand we have the equality

$$\widetilde{T}_n(A) y_0 = \int_A F_n(\omega) y_0\, d|\nu|(\omega).$$

On the other hand, by definition of T_n, it follows that

$$\widetilde{T}_n(A) y_0 = \int_A T_n y_0\, d\nu = \int_A y^*(y_0) \cdot \chi_{B_n}\, d\nu = \nu(A \cap B_n).$$

We deduce that the measure ν satisfies

$$\nu(A \cap B_n) = \int_A F_n(\omega) y_0 \, d|\nu|(\omega)$$

where the function $F_n y_0$ is in $L^1(|\nu|, X)$ and it is null outside of B_n.

Let $x^* \in X^*$. We have

$$|x^*\nu|(\Omega) = \sum |x^*\nu|(B_n) = \sum \int |\langle x^*, F_n(\omega) y_0 \rangle| \, d|\nu|(\omega).$$

Thus the series $\sum \langle x^*, F_n y_0 \rangle$ converges absolutely in $L^1(|\nu|)$ and so the function $\sum \langle x^*, F_n y_0 \rangle$ is in $L^1(|\nu|)$. This holds for every $x^* \in X^*$, thus the strongly measurable function $\sum F_n y_0$ is scalarly integrable.

Let $A \in \Sigma$ and $x^* \in X^*$, then

$$\begin{aligned}
\left\langle x^*, \int_A (\sum F_n(\omega) y_0) \, d|\nu|(\omega) \right\rangle &= \int_A \langle x^*, \sum F_n(\omega) y_0 \rangle \, d|\nu|(\omega) \\
&= \int_A \left(\sum \langle x^*, F_n(\omega) y_0 \rangle \right) \, d|\nu|(\omega) \\
&= \sum \int_A \langle x^*, F_n(\omega) y_0 \rangle \, d|\nu|(\omega) \\
&= \sum \left\langle x^*, \int_A F_n(\omega) y_0 \, d|\nu|(\omega) \right\rangle \\
&= \sum x^* \nu(A \cap B_n) \\
&= x^* \nu(A).
\end{aligned}$$

So $\int_A (\sum F_n(\omega) y_0) \, d|\nu|(\omega) = \nu(A) \in X$. It follows that the function $\sum F_n y_0$ is Pettis integrable. Hence ν has a derivative which is strongly measurable and Pettis integrable with respect to its variation. Q.E.D.

Let us see some applications of the previous results. The problem of relating the existence of a subspace isomorphic to ℓ_∞ in the space $\mathcal{L}(Y, X)$ with the coincidence of $\mathcal{L}(Y, X)$ with an ideal of operators in $\mathcal{L}(Y, X)$, has been considered by several authors (see for example [K], [To]).

Theorem 4.10. *Let E be an order continuous Banach lattice with weak unit and let Y be a Banach space. If $\mathcal{L}(Y,E)$ contains no subspace isomorphic to ℓ_∞, then every operator from Y into E is L–weakly compact.*

PROOF. As E is an order continuous Banach lattice with weak unit, by Theorem 1.15 there exists an E–valued measure such that $E \equiv L^1(\nu)$. Let $T\colon Y \longrightarrow E \equiv L^1(\nu)$ be a continuous linear operator. The associated measure $\widetilde{T}$ is bounded and takes values in $\mathcal{L}(Y,E)$. As this space does not contain subspaces isomorphic to ℓ_∞, it follows from a theorem of Diestel and Faires [DU, Theorem I.4.2] that the measure $\widetilde{T}$ is strongly measurable, and so by Theorem 4.5 the operator T is L–weakly compact. Q.E.D.

The converse to the previous result is not true as the following example shows.

Example 4.11. Let $E = L^1[0,1]$ and $Y = \ell^2$. Every operator from ℓ^2 into $L^1[0,1]$ is weakly compact. In $L^1[0,1]$ relatively weakly compact sets and L–weakly compact sets coincide due to the Dunford–Pettis Theorem. Thus every operator from ℓ^2 into $L^1[0,1]$ is L–weakly compact. On the other hand the space $L^1[0,1]$ has a subspace isomorphic to ℓ^2, thus the space $\mathcal{L}(\ell^2, L^1[0,1])$ contains a subspace isomorphic to the space $\mathcal{L}(\ell^2,\ell^2)$ and this space contains a subspace isomorphic to ℓ_∞.

In [K, Theorem 6] it is proven, among other results, that the equivalence between the conditions "every operator from E into F is compact" and "$\mathcal{L}(E,F)$ does not contain a copy of ℓ_∞" holds when F is an arbitrary Banach space and E is a Banach space with an unconditional finitedimensional decomposition of the identity. We prove a similar result without restrictions on the initial space and the range space is an order continuous and atomic Banach lattice, see Preliminaries.

Theorem 4.12. *Let F be an order continuous and atomic Banach lattice and let Y be a Banach space. The following conditions are equivalent:*

a) Every operator from Y into F is compact.

b) $\mathcal{L}(Y, F)$ does not contain a subspace isomorphic to ℓ_∞.

PROOF. a) $\Rightarrow$ b) Is a general fact, independently of the spaces F and Y, its proof can be gleaned in [K, Theorem 6].

b) $\Rightarrow$ a) Let us consider first the case in which the Banach lattice has a weak unit. In this case by Theorem 4.10 every operator $T: Y \longrightarrow F$ is L–weakly compact. We know that in atomic order continuous Banach lattices relatively compact sets and L–weakly compact sets coincide, see Preliminaries. It follows that L–weakly compact operators coincide with compact operators. Hence every operator is compact.

In the general case, let us assume that there exists an operator $T: Y \longrightarrow F$ that is non compact. Then there exists a sequence (x_n) in $T(B_Y)$ and there exists $\varepsilon > 0$ such that $\|x_n - x_m\| > \varepsilon$ for $n \neq m$.

Let (z_α) be the family of atoms of F and denote by P_{z_α} the projection associated with z_α (see [LT vol. II, p. 8]). As F is order continuous, every element x in F is disjoint from the atoms of F but for at most a countable number of them. This follows from the fact that for every $\varepsilon > 0$ there exists at most a finite number of atoms z_α such that $\|P_{z_\alpha}(x)\| \geq \varepsilon$. Assume this last assertion is not true. Then there would exist $\varepsilon > 0$ and a sequence of atoms (z_i) such that $\|P_{z_i}(x)\| \geq \varepsilon$. Consider the sequence $h_k = P_{\sup\{z_1,\ldots,z_k\}}(x)$. It is increasing, order bounded by $|x|$ but it is not convergent as $\|h_k - h_{k-1}\| = \|P_{z_k}(x)\| \geq \varepsilon$. This contradicts the order continuity of F.

So there exists a countable family of atoms such that every element x_n is disjoint from all atoms outside the family. Let Z be the space generated by this family of atoms. As Z is a band in the order continuous Banach lattice F, it is complemented by a norm one projection $P\colon F \longrightarrow Z$, satisfying $\|P(x_n) - P(x_m)\| = \|x_n - x_m\| \geq \varepsilon$ for every $n \neq m$. Then the operator $P \circ T$ is a non compact operator from Y into Z.

As $\mathcal{L}(Y, Z)$ is isometrically embedded into $\mathcal{L}(Y, F)$, from our hypothesis it folllows that $\mathcal{L}(Y, Z)$ does not contain a subspace isomorphic to ℓ_∞. But Z is an atomic order continuous Banach lattice with weak unit (as it is separable). It follows, as we have seen in the previous case, that every operator is compact. This contradiction establishes the claim. Q.E.D.

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