Catalan Matroids

Joseph E. Bonin†, Anna de Mier‡, and Marc Noy‡

†Department of Mathematics
The George Washington University
Washington, D.C., USA
e-mail: jbonin@gwu.edu,

‡Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya.
e-mail:{demier,noy}@ma2.upc.es

Abstract

This talk will connect two combinatorial topics: lattice paths and matroids. Particular attention will be given to Catalan paths and the corresponding Catalan matroids, as well as to several natural generalizations. In particular, we will show that the Tutte polynomial of the n-th Catalan matroid is a two-variable polynomial counterpart of the n-th Catalan number, and we will show that the coefficients of the Tutte polynomial give a refined enumeration of Catalan paths according to two basic lattice path parameters. In addition to establishing connections between these fields and giving lattice path interpretations of some of the most important invariants of matroids, this work also leads to new and attractive theorems about lattice paths.

All of the background in matroid theory that is needed to understand this talk will be provided.

There are many well-known connections between matroid theory and enumeration (see, for example, [1, 3]). For instance, the chromatic and flow polynomials of graphs are special evaluations of the Tutte polynomial; the general setting for studying Tutte polynomials is matroid theory. Likewise, from the Tutte polynomials of certain matroids, one can obtain the number of acyclic orientations of a graph, the number of regions of an arrangement of hyperplanes, and the weight enumerator of a linear code. The study of lattice paths is a branch of enumerative combinatorics for which connections with matroid theory have only recently been discovered; these connections will be the focus of this talk.

Catalan paths are the lattice paths that go from $(0,0)$ to (n,n), that use East and North steps of unit length, and that do not go above the line $y = x$; such paths are enumerated by the Catalan numbers 1, 1, 2, 5, 14, 42, 132, 429, ..., which are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The k-Catalan numbers are a natural and well-studied generalization of the Catalan numbers given by

$$C_n^k = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

This sequence enumerates the k-Catalan paths, that is, the lattice paths that go from

$(0,0)$ to (kn, n), that use East and North steps, and that do not go above the line $y = x/k$. We show that there is a natural bijection between the Catalan paths from $(0,0)$ to (n,n) and the bases of a certain rank-n matroid on $2n$ elements, the n-th Catalan matroid. We also develop a counterpart for k-Catalan paths. This is further generalized, as we describe next. Assume we have two lattice paths, P_1 and P_2, that use East and North steps, that go from $(0,0)$ to (m,n), and such that P_2 never goes above P_1. Let $\mathcal{P}$ be the collection of lattice paths that use East and North steps, that go from $(0,0)$ to (m,n), and that go neither above P_1 nor below P_2. There is a natural bijection between $\mathcal{P}$ and the bases of a certain matroid.

As is true of the other connections between enumeration and matroid theory cited above, the Tutte polynomial plays a central role in this work. While computing Tutte polynomials is very hard in general, certain structural features of the Catalan matroids and some of their generalizations make these Tutte polynomials relatively easy to compute. The coefficients in these polynomials have simple descriptions in terms of lattice paths. Among the results we will discuss in the talk is the following theorem for k-Catalan matroids.

Theorem 1 *The coefficient of $x^i y^j$ in the Tutte polynomial of the k-Catalan matroid M_n^k is the number of lattice paths that go from $(0,0)$ to (kn, n), use East and North steps, do not go above the line $y = x/k$, have exactly j East steps on the x-axis, and return to the line $y = x/k$ exactly i times.*

This lattice path description of the coefficients of the Tutte polynomials allows us to obtain the following generating function for the sequence of Tutte polynomials for the k-Catalan matroids $M_0^k, M_1^k, M_2^k, \ldots$

Theorem 2 *Let*

$$C = C(z) = \sum_{n\geq 0} \frac{1}{kn+1}\binom{(k+1)n}{n} z^n$$

be the generating function for the k-Catalan numbers. The generating function

$$\sum_{n\geq 0} t(M_n^k; x, y) z^n$$

for the Tutte polynomials of the k-Catalan matroids is

$$1 + \left(\frac{xzy^k}{1 - z\sum_{j=1}^{k} y^j C^{k-j+1}}\right)\frac{1}{1 - xzC^k}.$$

A close examination of the Tutte polynomials of these matroids has led to new theorems about lattice paths, among which is the following theorem. The polygonal line $\widehat{B}_n^k$ that the following result refers to is the lattice path that consists of the following sequence of steps: $k-1$ North steps, $n-1$ strings, each of which consists of k East steps followed by k North steps, and a final string of $k-1$ East steps.

Theorem 3 *The number of paths that go from $(0,0)$ to $(nk-1, nk-1)$, that use East and North steps, and that do not go above the polygonal line $\widehat{B}_n^k$ is $k\,C_{kn-1}$.*

The formula in this theorem is striking: k times every k-th Catalan number, or, more precisely, k times the number of Catalan paths with the same endpoints. Equally notable is the following fact. If the bounding polygonal line $\widehat{B}_n^k$ is altered slightly, with k (rather than $k-1$) initial North steps and final East steps, the number of lattice paths that result is not so easy to count; the only known formulas are in terms of determinants.

This direction for developing the subject hints at yet another connection between enumeration and generalizations of the Catalan matroids: we establish a connection between certain matroids and the following problem of current interest in enumeration [2].

> THE $(k+l,l)$ TENNIS BALL PROBLEM.
> Assume we have a sequence of distinct balls $b_1, b_2, \ldots, b_{(k+l)n}$. At stage 1, the first $k+l$ balls $b_1, b_2, \ldots, b_{k+l}$ are put in bin A and then l balls are moved from bin A to bin B. At stage i, the next $k+l$ balls $b_{(i-1)(k+l)+1}, b_{(i-1)(k+l)+2}, \ldots, b_{i(k+l)}$ are put in bin A and then some set of l balls from bin A are moved to bin B. (Thus, any balls that remain in bin A at the end of the $(i-1)$-st stage are among those that can go into bin B at stage i.) How many different sets of nl balls can be in bin B after n iterations?

The rich interplay of ideas from the study of lattice paths and the theory of matroids that will be outlined in this talk suggest that these newly discovered enumerative aspects of matroid theory are ripe for further exploration and many interesting results remain to be discovered.

References

[1] T. Brylawski and J. G. Oxley. *The Tutte polynomials and its applications.* Matroid Applications (N. White, ed.). Cambridge University Press. pp.123-225. 1992.

[2] D. D. Merlini, R. Sprugnoli, and C. Verri. *The tennis ball problem.* J. Combin. Theory, Ser. A. To appear.

[3] D. J. A. Welsh. *Complexity: Knots, Colourings, and Counting.* Cambridge University Press. 1993.