BLOWING UP ACYCLIC GRAPHS AND GEOMETRICAL CONFIGURATIONS [1]

Carlos Marijuán

Abstract

Blowing up is a useful technique in algebraic and analytic geometry. In particular, it is the main tool for proving resolution of singularities. Hironaka [3] proved in 1964 that every algebraic variety over a field of characteristic zero admits a resolution of singularities which is obtained by successive blowing ups of certain regular centers. Moreover, he proves the stronger version of embedded resolution of singularities, i.e., for every (singular) subvariety X of a smooth variety Z there exists a sequence of birational morphisms

$$Z_N \to Z_{N-1} \to ..... \to Z_1 \to Z_0 = Z, \qquad (1)$$

such that π_i is the blowing up of Z_{i-1} at a regular center C_i which is transversal to the exceptional divisor E_{i-1} of $\pi_{i-1} \circ ..... \circ \pi_1$, and such that the strict transform X_N of X at Z_N is smooth and transversal (normal crossing) to the exceptional divisor E_N.

The embedded resolution of singularities was extended to analytical spaces in 1974 in [1] . Recently Villamayor [6] proved a canonical theorem of embedded resolution of singularities, i.e., a procedure to define a concrete sequence such as (1) for each variety X. This leads us to consider an algorithmic point of view (see [7], [2]).

Looking for such an algorithmic viewpoint, our initial motivation, in this paper, is to try to understand the behaviour, after blowing up π_{i+1}, of the set (or a subset) of subvarieties E_i created at Z_i by the sequence of transformations.

More precisely, we are interested in the following question: if $\pi: Z' \to Z$ is a blowing up of a smooth variety at a regular center C, and if $\mathbb{C}$ is a smooth geometrical configuration of subvarieties of Z, try to describe the transform $\mathbb{C}'$ of $\mathbb{C}$ in Z', i.e., the set of strict transforms of subvarieties in $\mathbb{C}$ not included in C and the fibers of those contained in C, as well as the irreducible components of the intersections of such objects. Here, by a smooth configuration, we mean a subset of smooth subvarieties meeting pairwise transversally, i.e., such that at each point in the intersection, both subvarieties are locally described by the vanishing of elements in a common system of parameters.

To focus this question in a combinatorial way, we associate an incidence graph to each configuration, i.e., the acyclic directed graph given by the inclusion ordering on the subvarieties of the configuration, weighted by the dimensionality of those subvarieties. Then we introduce, in a purely combinatorial way (section 2), the blowing up of a weighted acyclic digraph with center at one of its points. The main result in the paper (section 3) proves that the digraph associated to the blow up configuration $\mathbb{C}'$ is the transitivization of the blow up of the digraph associated to the configuration $\mathbb{C}$.

To complete the paper, we give a canonical procedure to "desingularize" (in some sense) a general acyclic digraph. Namely, from a given acyclic digraph we obtain another one, that we call total or geometric blow up, by blowing up successively the levels of the points of the original digraph. The geometric blow up has a nice structure which is made up from hypercubes and it is also provided of a nice weight function, as it is determined by the weight of a maximal dominant element.

1. BASIC CONCEPTS AND NOTATIONS.

A **graph** we will mean a couple (X, G) where X is a finite set and $G \subset X \times X - \{(x, x) : x \in X\}$. Elements in X and G are called **points** and **arcs** respectively. A **labeling** of X by the label set E is a bijective map $\tilde{x} : E \to X$, $\tilde{x}(e)$ being denoted by x_e for any $e \in E$.

A **path** joining the point x_1 with x_q in (X, G) of **length** $q-1$ is an injective map $s: \{1, 2, ..., q\} \to X$, $q \geq 2$, such that $(s(i), s(i+1)) \in G$, and it will be denoted alternatively as $s(1)s(2).....s(q)$ or $x_1x_2.....x_q$. If $x_1x_2.....x_q$ is a path and $(x_q, x_1) \in G$ then the sequence $x_1x_2.....x_qx_1$ is said to be a **cycle**. A graph is said to be **acyclic** if it has no cycles. Finally for a **semipath** in (X, G) we mean an injective sequence such that for any $i, 1 \leq i \leq$

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$q-1$, one has either $(s(i), s(i+1)) \in G$ or $(s(i+1), s(i)) \in G$. **Semicycles** can be defined in the same way.

A **subgraph** of (X, G) is a graph (Y, H) with $Y \subset X$ and $H \subset G$. For the **induced subgraph** on Y we will mean the graph $(Y, G/Y)$ where $G/Y = G \cap (Y \times Y)$.

Given a graph (X, G), the equivalence relationship $x \equiv y$ iff x and y are in a semipath gives rise to the partition of X into connected components $X = X_1 \cup ..... \cup X_r$. Roughly speaking the **connected components** are the induced subgraphs $(X_i, G/X_i)$. If $r = 1$ the graph is **connected.**

For a point $x \in X$ we will consider the sets $\overline{x}$ (resp. $\underline{x}$) consisting of x and those points $y \in X$ such that there exists a path from y to x (resp. x to y). A point x is said to be **dominant** if $(y, x) \in G$ for any $y \in \overline{x}$.

Acyclic graphs have some nice properties. First, for an acyclic graph there exists at least one point x (resp. y) such that $\overline{x} = \{x\}$ (resp. $\underline{y} = \{y\}$). Such a point will be said to be a **minimal** (resp. **maximal**) in the graph. Moreover, the points in an acyclic graph (X, G) can be distributed by levels $N_0, N_1, ...$ as follows:

$N_0 = \{x \in X : x \text{ is minimal in } (X, G)\}$,

and recursively for $p \geq 1$,

$$N_p = \{x \in X - \bigcup_{i=0}^{p-1} N_i : x \text{ is minimal in } (X - \bigcup_{i=0}^{p-1} N_i,\ G/X - \bigcup_{i=0}^{p-1} N_i)\}.$$

Thus one has a partition of X, $X = \bigcup_{i=0}^{k} N_i$, k being the **height** of the graph, i.e., the greatest index such that $N_k \neq \emptyset$.

Second, we have the following characterization of acyclic graphs: A graph (X, G), with $card(X) = n$, is acyclic if, and only if, there exists a labeling of X by the label set $E = \{1, ..., n\}$ such that if $(x_i, x_j) \in G$ then $i < j$. In this case, we will say that (X, G) is **naturally ordered.**

Given two graphs $(X, G), (X', G')$, by a **graph morphism** from (X, G) to (X', G') we mean a mapping $\Delta: X \rightarrow X'$ such that for every arc $(x, y) \in G$, one has either $\Delta(x) = \Delta(y)$ or $(\Delta(x), \Delta(y)) \in G'$. The morphism is an **isomorphism** when it has an inverse, i.e., if Δ is bijective and $(x, y) \in G$ if and only if $(\Delta(x), \Delta(y)) \in G'$. The morphisms which take maximal points to maximal points will be called **dominant.**

A graph (X, G) is said to be **transitive** if for any pair of arcs (x, y), (y, z) with $x \neq z$, then (x, z) is also an arc. The graph is **antisymmetric** if it has no pair of symmetric arcs, i.e., if (x, y) and (y, x) are not both in G for $x \neq y$. A transitive graph is acyclic if and only if it is antisymmetric, and an acyclic transitive graph (X, G) has a partial ordering $(X, \leq)$ where $x < y$ iff $(x, y) \in G$. For an acyclic graph (X, G) the **transitivized** graph is the graph (X, G^t) where

$G^t = G \cup \{(x, y) : x \neq y$ and there exists a path $x_1 ..... x_q, q \geq 2$, with $x_1 = x$ and $x_q = y\}$.

Example. If $X = \{x_i : 1 \leq i \leq 12\}$ and $G = \{(x_1, x_5), (x_5, x_7), (x_7, x_9), (x_1, x_7), (x_1, x_3), (x_3, x_9), (x_3, x_{12}), (x_{11}, x_{12}), (x_2, x_6), (x_6, x_8), (x_8, x_{10}), (x_2, x_8), (x_2, x_{10}), (x_6, x_{10}), (x_2, x_4), (x_4, x_{10})\}$ then (X, G) is a naturally ordered acyclic graph with two connected components (X_1, G_1) and (X_2, G_2). The connected component (X_2, G_2) is the induced subgraph on $\overline{x}_{10} = \{x_2, x_4, x_6, x_8, x_{10}\}$ and x_{10} is the unique dominant maximal element of the graph (X, G).

The map $\Delta: X_1 \rightarrow X_2$ given by $\Delta(x_{12}) = x_4, (x_{11}) = x_2$ and $\Delta(x_i) = x_{i+1}$, for any $i \leq 9$, is a nondominant graph morphism. The transitivized graph of the induced subgraph on $\overline{x}_9 = \{x_1, x_3, x_5, x_7, x_9\}$ is $(\overline{x}_9, G/\overline{x}_9{}^t)$ with $G/\overline{x}_9{}^t = G/\overline{x}_9 \cup \{(x_1, x_9), (x_5, x_9)\}$. This graph is isomorphic to the connected component (X_2, G_2) and the restriction of the map Δ to $\overline{x}_9$ is now a dominant isomorphism between both of them.

A **weight map** on an acyclic graph (X, G) is an increasing map $w: X \rightarrow \mathbb{N}$, i.e., such that $w(x) < w(y)$ for any $(x, y) \in G$. The triplet (X, G, w) will be called a **weighted acyclic graph** and the weight map will be said to be **transversal** on (X, G) if for every pair of points x, y one has $w(x) + w(y) \leq w(z) + w(z')$ for any $z, z' \in G$ such that $z \in max(\overline{x} \cap \overline{y})$ and $z' \in min(\underline{x} \cap \underline{y})$ where $max(-)$ (resp. $min(-)$) stands for the set of maximals (resp. minimals) for the induced subgraph. A path $x_1 ..... x_q$ is said to be **complete** for the weight w if for any $i, 2 \leq i \leq q$, one has $w(x_i) = w(x_{i-1}) + 1$. A weighted acyclic graph (X, G, w) is said to be **completely transver-**

sal for w if all paths in (X,G) are complete for w. Two given weighted acyclic graphs (X,G,w), (Y,H,v) are **equivalent** if there exists an isomorphism of graphs $\Phi : (X,G) \rightarrow (Y,H)$ such that $w(x) - v(\Phi(x)) = k$ for k fixed and any $x \in X$.

Finally, for two given graphs (X,G), (X',G') we will denote by $(X \times X', G \times G')$ the **product graph** on $X \times X'$ where $G \times G'$ denotes the set of arcs $\{((x,x'),(y,y')) \in (X \times X') \times (X \times X') : (x = y \text{ and } (x',y') \in G') \text{ or } ((x,y) \in G \text{ and } x' = y')\}$.

Throughout the paper several graphs will be constructed from original ones. The point sets X' for such graphs will usually be denoted by $X' = \{x_\alpha : \alpha \in I'\}$ where I' is an explicitly described set . Moreover, I' will usually be a subset of sequences $i_1......i_p$ of elements of some known label set (the label set of some known graph) for various integers p. Sometimes p will be 1; thus, elements in various graphs will usually be denoted as x_i for the same subindex i, which will not be a loss of generality.

2. BLOWING UP ACYCLIC GRAPHS.

In this section we give the main concept of this paper, the blowing up at a point of an acyclic graph with transversal weight map.

Let $X = \{x_i : i \in I\}$ be a finite set of points and let (X,G,w) be a connected acyclic graph, with a dominant maximum element x_z and with transversal weight map $w: X \rightarrow \mathbb{N}$. (If the acyclic graph (X,G) does not have a dominant maximum element, we can consider the acyclic graph obtained from (X,G) connecting every point $x_i \in X$ with a new point x_z by means of the arc (x_i, x_z)).

Having fixed a point $x_c \in X$, which we shall call **blowing up center**, let us consider the following parts in the set of points X:

a) the set of points connected to the blowing up center by a path: $\overline{x}_c$,

b) the set of points connected from the blowing up center by an arc: $x_c \uparrow \; = \{x_j : (x_c, x_j) \in G\}$,

c) the set of points connected from the blowing up center by a path: $x_c^* = \underline{x}_c - \{x_c\}$,

d) the set of paths connecting $\overline{x}_c$ with $x_c \uparrow$ without passing through $\overline{x}_c \cup x_c \uparrow$, which we shall call **transversal paths** to the blowing up center, i.e., the set A_c given by

$$A_c = \{s: \{1, ...,q\} \rightarrow X : q > 2, s(1) \in \overline{x}_c, \quad s(i) \notin \overline{x}_c \cup x_c \uparrow \text{ for } 1 < i < q, s(q) \in x_c \uparrow\},$$

e) the set of points which are not joined by a path to the blowing up center through which some transversal paths pass and which we shall call **transversal points** to the blowing up center:

$B_c = \{x_t \in X - (\overline{x}_c \cup x_c \uparrow) :$ there exists a path $s \in A_c$ with $s(i) = x_t, 1 < i < q\}$.

For each transversal point $x_t \in B_c$, we will consider the set $A_c(x_t)$ of transversal paths $s \in A_c$ passing through x_t, its first term being, $s(1)$, maximal in (X,G) of the set of the first terms of the transversal paths passing through x_t and its last term being, $s(q)$, minimal in (X,G) of the set of the last terms of the transversal paths passing through x_t :

$A_c(x_t) = \{s \in A_c : s(i) = x_t \in B_c$ for any $1 < i < q$, $s(1) \in max\{s'(1) : s' \in A_c, s'(j) = x_t$ for any $1 < j < q'\}, s(q) \in min\{s'(q') : s' \in A_c, s'(k) = x_t$ for any $1 < k < q'\}\}$.

Finally, set $B_c' = \{x_t \in B_c :$ there exists a path $s \in A_c(x_t)$ with $s(2) = x_t$ and $w(s(1)) + w(s(q)) - w(x_c) - 1 \geq w(x_t)\}$.

2.1 DEFINITION. Let (X,G,w) be a connected acyclic graph with dominant maximum element and with transversal weight map w, x_c be a point of G and B_c, B_c' the sets introduced before. We define the **blow up** of (X,G,w) with center at x_c and relative to B_c to be the graph (X_c, G_c, w_c) where

$X_c = (X - \overline{x}_c) \cup \{x_{ij} : x_i \in \overline{x}_c, x_j \in x_c \uparrow\} \cup \{x_{ij}: (x_i,x_j) \in G, x_i \in \overline{x}_c - \{x_c\}, x_j \in x_c^* - x_c \uparrow\} \cup T_c$,
with $T_c = \{x_{ht} : x_t \in B_c', s \in A_c(x_t), s(1) = x_h\}$,
and
$G_c = (G - G/\overline{x}_c - \{(x_i,x_j) \in G : x_i \in \overline{x}_c, x_j \in x_c^*\}$
$-\{(x_h,x_t) \in G : x_t \in B_c, s \in A_c(x_t), s(1) = x_h\})$
$\cup(G/\overline{x}_c \times G/x_c \uparrow)$
$\cup\{(x_{ij},x_j) : (x_i,x_j) \in G, x_i \in \overline{x}_c, x_j \in x_c^*\}$
$\cup\{(x_{ht},x_{hk}),(x_{ht},x_t) : x_t \in B_c', s \in A_c(x_t),$
$s(1) = x_h, s(2) = x_t, s(q) = x_k\}$
$\cup\{(x_{hk},x_t) : x_t \in B_c - B_c', s \in A_c(x_t),$
$s(1) = x_h, s(2) = x_t, s(q) = x_k\}$
$\cup\{(x_{ht_i},x_{ht_j}) : x_{t_i}, x_{t_j} \in B_c', s \in A_c(x_{t_i}),$
$s' \in A_c(x_{t_j}), s(1) = s'(1) = x_h, (x_{t_i},x_{t_j}) \in G\}$.

The weight map $w_c: X_c \rightarrow \mathbb{N}$ is defined by:
$w_c(x_j) = w(x_j)$, if $x_j \in X_c \cap X$,
$w_c(x_{ij}) = w(x_i) + w(x_j) - w(x_c) - 1$, if $x_{ij} \in X_c - X - T_c$, $w_c(x_{ht}) = w(x_t) - 1$, if $x_{ht} \in T_c$.

Notice that the new graph is defined by preserving some labels i and adding new labels ij, ht, etc ... to the labeling set. In other words, one has that the point set X_c can be obtained from X by replacing the points of $\overline{x}_c$ with those of $\overline{x}_c \times x_c \uparrow$, adding a point x_{ij} for each arc (x_i, x_j) connecting $x_i \in \overline{x}_c - \{x_c\}$ with $x_j \in x_c^* - x_c \uparrow$; and also adding the points of the set T_c. The induced subgraph by (X, G) on $X - \overline{x}_c$ is preserved and the induced subgraph on $\overline{x}_c$ is replaced by $G/\overline{x}_c \times G/x_c \uparrow$ which is connected to $G/x_c \uparrow$ from $\{x_c\} \times G/x_c \uparrow$. The points $(x_i, x_j) \in \overline{x}_c \times x_c \uparrow$ are denoted by x_{ij} and then the arcs $(x_i, x_j) \in G$ connecting $\overline{x}_c$ with x_c^* becoming arcs $(x_{ij}, x_j) \in G_c$. The arcs $(x_h, x_t) \in G$ disappear and each point $x_{ht} \in T_c$, associated to the transversal path s passing through $x_t \in B'_c$, with $s(1) = x_h$, $s(2) = x_t$, $s(q) = x_k$, is linked to the subgraph $G/\overline{x}_c \times G/x_c \uparrow$ in the point x_{hk} and to the induced subgraph on $X - \overline{x}_c$ in the point x_t. If, on the contrary, $s(2) = x_t \in B_c - B'_c$, then the arc (x_h, x_t) of G is replaced by the arc (x_{hk}, x_t) in G_c; moreover, some of the points $x_{ht} \in T_c$ can be connected to each other.

2.2 PROPOSITION. The blow up graph (X_c, G_c, w_c) is a connected acyclic graph with dominant maximum element and with transversal weight map. The sets B'_c and T_c are empty if and only if for each transversal point $x_t = s(2)$, with $s \in A_c(x_t)$, one has $w(s(1)) + w(s(q)) = w(x_c) + w(x_t)$.

2.3 DEFINITION. Under the conditions of definition 2.1, the **blowing up** of a connected acyclic graph (X, G, w) at a point x_c is defined to be the morphism $\pi_c: (X_c, G_c, w_c) \to (X, G, w)$ given by

$\pi_c(x_j) = x_j$, if $x_j \in X_c \cap X$;
$\pi_c(x_{ij}) = x_i$, if $x_{ij} \in X_c - X - T_c$ and
$\pi_c(x_{ht}) = x_h$, if $x_{ht} \in T_c$.

It is clear that if $(x, y) \in G_c$, then either $\pi_c(x) = \pi_c(y)$ or $(\pi_c(x), \pi_c(y)) \in G$ since for the arcs $(x_{ht}, x_t) \in G_c$, with $x_{ht} \in T_c$, one has $x_h = s(1)$ and $x_t = s(2)$, and for the other arcs it is obvious. In consequence, π_c is a dominant morphism of acyclic graphs.

We will call **exceptional divisor** to the maximal point x_{cz} of the induced subgraph on $X_c - X$ and each point x_{iz}, with $x_i \in \overline{x}_c - x_c$, will be said to be an **exceptional fiber** of x_i in the exceptional divisor; so that the induced subgraph on $\{x_{iz}: x_i \in \overline{x}_c\}$ will be refered to as the **exceptional subgraph** of the blow up graph. The points $x_j \in X_c \cap X$, preserving the label by the blowing up, shall be called **strict transforms** (of the points $x_j \in X - \overline{x}_c$). The points x_{cj} with $x_j \in x_c \uparrow$ shall be called the **cuts** of the exceptional divisor x_{cz} with the strict transform x_j and the points x_{ij}, with $x_i \in \overline{x}_c - x_c$, $x_j \in x_c \uparrow$, the **cuts** of the exceptional fiber x_{iz} with the strict transform x_j. Finally, the points $x_{ht} \in T_c$ will be called **transversal cuts** of the exceptional fiber x_{hz} with the strict transform x_t.

2.4 THEOREM. If two given weighted acyclic graphs $(X, G, w), (Y, H, v)$ are equivalent by means of an isomorphism Φ and if $\pi_c: (X_c, G_c, w_c) \to (X, G, w)$, $\pi'_c: (Y_c, H_c, v_c) \to (Y, H, v)$ are their blowing up morphisms with center at x_c, $y_c = \Phi(x_c)$, respectively, then the blow up acyclic graphs (X_c, G_c, w_c) , (Y_c, H_c, v_c) are equivalent for an isomorphism Δ such that $\pi'_c \circ \Phi = \Delta \circ \pi_c$ and $w_c(x) - v_c(\Delta(x)) = w(y) - v(\Phi(y)) = k$, where $y = \pi_c(x)$ and k is a constant integer.

2.5 REMARK. 1) If the blowing up center is a minimal point x_o of the weighted acyclic graph (X, G, w), then there is no transversal point and so A_o, B_o and T_o are empty sets. In this case, we will denote the blow up graph with center in x_o by (X_o, G_o, w_o) where:

$$\begin{aligned} X_o &= (X - \{x_o\}) \cup \{x_{oj} : x_j \in x_o \uparrow\} \text{ and} \\ G_o &= G - \{(x_o, x_j) : x_j \in x_o \uparrow\} \\ &\quad \cup \{(x_{oj}, x_{ok}) : (x_j, x_k) \in G/x_o \uparrow\} \\ &\quad \cup \{(x_{oj}, x_j) : x_j \in x_o \uparrow\}. \end{aligned}$$

The transversal weight map $w_o: X_o \to \mathbb{N}$ is given by $w_o(x_j) = x_j$ if $x_j \in X_o \cap X = X - \{x_o\}$ and by $w_o(x_{oj}) = w(x_j) - 1$ if $x_{oj} \in X_o - X$. The π_o-blowing up of (X, G) is the acyclic graph morphism $\pi_o: (X_o, G_o, w_o) \to (X, G, w)$ given by $\pi_o(x_j) = x_j$ if $x_j \in X_o \cap X$ and by $\pi_o(x_{oj}) = x_o$ otherwise.

2) Acyclic transitive graphs (that represent partial orderings) are especially interesting from the point of view of blowing ups. If (X^t, G^t, w^t) is a graph of this kind, under the conditions of definition 2.1 one has, in general, that its blow up graph (X^t_c, G^t_c, w^t_c) is not transitive. More precisely, in (X^t, G^t, w^t) we have $x_c \uparrow = x_c^*$ and so the point set of the blow up graph is

$$X_c^t = (X^t - \overline{x}_c) \cup \{x_{ij} : x_i \in \overline{x}_c, x_j \in x_c^*\} \cup T_c.$$

Now, the arcs of the set $\{(x_{ij}, x_j) : (x_i, x_j) \in G^t, x_i \in \overline{x}_c, x_j \in x_c^*\}$ are redundant by transitivity over the arcs $(x_{ij}, x_{cj}) \in G^t/\overline{x}_c \times G^t/x_c^*$ and (x_{cj}, x_j). For every path $x_{t_1}x_{t_2}.....x_{t_{q-1}}x_{t_q} \in A_c(x_{t_2})$, with $x_{t_1} = x_h$ and $x_{t_q} = x_k$, since $w_c^t(x_{hk}) < w_c^t(x_k)$, there exists i, with $2 \leq i \leq q-1$, such that $w_c^t(x_{t_{i-1}}) \leq w_c^t(x_{hk}) < w_c^t(x_{t_i})$, i. e. $x_{t_{i-1}} \in B_c'$ and $x_{t_i} \in B_c - B_c'$. Also the arcs (x_{hk}, x_{t_j}) are redundant for $j > i$ and the arcs (x_{ht_j}, x_{hk}) are redundant for $j < i - 1$. Thus, one has, on the one hand, the arc (x_{hk}, x_{t_i}) and, on the other, the path $x_{ht_2}.....x_{ht_{i-1}}$ connected to the product graph $G^t/\overline{x}_c \times G^t/x_c^*$ by the arc $(x_{ht_{i-1}}, x_{hk})$ and to the path $x_{t_2}.....x_{t_{i-1}}$ by the arcs (x_{ht_j}, x_{t_j}), $2 \leq j \leq i-1$.

On the contrary, the transitivity fails on $G^t/\overline{x}_c \times G^t/x_c^*$, on the connection of this with G^t/x_c^* and on the connection, between them, of the paths $x_{ht_2}.....x_{ht_{i-1}}$ and $x_{t_2}.....x_{t_{i-1}}$.

3. GRAPHIC REPRESENTATION OF THE BLOWING UP FOR A GEOMETRIC CONFIGURATION.

Throughout this section, for a **variety** we will mean a connected and smooth algebraic or complex analytic variety, and all the **subvarieties** considered will be connected and smooth. Two subvarieties are said to be **transversal** if at each point of their intersection there exists a system of local coordinates $x_1, ..., x_n$ such that both varieties are locally defined by $x_i = 0, i \in A$ and $x_i = 0, i \in B$, A and B being non empty subsets of $\{1, 2, ..., n\}$, $A \not\subset B$ and $B \not\subset A$.

Let us consider a smooth **geometric configuration** $\mathbb{C}$, i.e., a set consisting of a variety Z and a finite subset of subvarieties such that

(1) for each couple of subvarieties in $\mathbb{C}$, either there is an inclusion, their intersection is empty or they are transversal

(2) each component of the intersection of two subvarieties in $\mathbb{C}$, is a subvariety in $\mathbb{C}$.

If $C \subset Z$ is a subvariety of dimension r and Z' the blow up of Z with center C, then the points in the inverse image by the blowing up morphism $p_C: Z' \to Z$, of a point $P \in C$ correspond biunivocally to the sets of $(r+1)$-vector subspaces of the tangent space to the ambient variety, $T_P Z$, containing the tangent space to the blowing up center, $T_P C$. That is to say, to each point $P \in C$ corresponds in Z' the fiber

$F_P = \{$ Vector subspaces of $T_P Z$ generated by $T_P C$ and a vector $v \notin T_P C\}$.

On the other hand, the inverse image of the blowing up center C is the exceptional divisor $E_{cz} = \bigcup_{P \in C} F_P$. One has $codim_Z F_P = r + 1$, $codim_Z E_{cz} = 1$ and a bijection between Z' and $(Z - C) \cup E_{cz}$.

If the subvariety J contains the blowing up center C, we will denote the strict transform of J in Z' by J', i.e. J' is the closure in Z' of the inverse image of $J - C$ by the blowing up morphism, i.e., $\overline{p_C^{-1}(J - C)} = J'$. The intersection of the exceptional divisor E_{cz} with the strict transforms J' will be denoted by CJ'. One has $dimJ' = dimJ$ and $dimCJ' = dimJ - 1$.

If I is a subvariety contained in C, we will denote the fiber of I in the exceptional divisor E_{cz} by $F_{iz} = \bigcup_{P \in I} F_P$. Let us denote the cuts of each fiber F_{iz} with each strict transform J' of a variety J as above by IJ'. One has $dimF_{iz} = dimI + dimZ - dimC - 1$ and $dimIJ' = dimI + dimJ - dimC - 1$.

If T represents a transversal subvariety to the blowing up center C in a subvariety H, then each vector tangent to T in a point $P \in H$ has its corresponding vector subspace $V \in F_P$ and the strict transform T' is a subvariety of Z' for which the inverse image of P by the blowing up morphism is the collection of subspaces $V \in F_P$ corresponding to the vectors $v \in T_P T$, so that $\overline{p_C^{-1}(T - C)} = T'$. The intersection of the strict transform T' with the exceptional divisor E_{cz} will be denoted by HT'. One has $dimT' = dimT$ and $dimHT' = dimT - 1$.

If L is a disjoint subvariety with C, we will denote its corresponding transform in Z' by L'. One has $dimL' = dimL$.

Let $\mathbb{C}'$ be the **blow up configuration**, i.e., the configuration of the subvarieties of Z' consisting of the objects Z', E_{cz}, J' and CJ' for $J \supset C, F_{iz}$ and IJ' for $I \subset C, T'$ and HT' for T transversal to C, and L' for $L \cap C = \emptyset$, and J, I, T, L in the configuration $\mathbb{C}$. Note that all the above objects are (connected and smooth) subvarieties of Z'. Then the blowing up morphism gives rise to the map $p_c: \mathbb{C}' \to \mathbb{C}$ given, with notations as above, by $p_c(Z') = Z, p_c(E_{cz}) = C, p_c(J') = J, p_c(CJ') =$

$C, p_c(F_{iz}) = p_c(IJ') = I, p_c(T') = T, p_c(HT') = H$ and $p(L') = L$.

We can associate a weighted directed graph (X, G, w) to each geometric configuration $\mathbb{C}$ where points in X represent the subvarieties in $\mathbb{C}$, the arcs represent the inclusion relations between them, so that $(H, K) \in G$ iff $H \subset K$, and the weights are the dimensions of the corresponding subvarieties. Thus in the above situation, both the smooth geometric configuration $\mathbb{C}$ formed by the variety Z and their subvarieties, and its blow up $\mathbb{C}'$ have associated weighted graphs (X, G, w) and (X', G', w') which clearly are connected transitive acyclic and have only one maximal element (Z and Z') which dominates the points in its respective graph. Note that the weight maps w and w' are transversal because of the condition (1) of transversality in the geometric case. We can associate a dominant graph morphism $\tilde{p}_c: (X', G') \to (X, G)$ to the map $p_c: \mathbb{C}' \to \mathbb{C}$ in a natural way.

We will prove that the concept of blowing up for a weighted acyclic graph given in the preceding section supports the blowing up of the associated geometric configuration; more precisely, we are going to prove that the blow up of the graph (X, G, w), associated to the geometric configuration $\mathbb{C}$, at the blowing up center C, is an acyclic graph (X_c, G_c, w_c) whose transitivized is the graph (X', G', w') associated to the configuration $\mathbb{C}'$.

3.1 LEMMA. If $T_2 \subset T_3 \subset ..... \subset T_{q-1}$ is a chain in the geometric configuration $\mathbb{C}$, of transversal subvarieties to the blowing up center C, $T_1 \subset T_2$ a component of the intersection of T_{q-1} and C, and T_q a minimal subvariety of those that contain T_{q-1} and C, then for every i, with $2 \leq i \leq q-1$, and with notations as above, one has:

a) $T_1T'_i \subseteq T_1T'_q$.

b) $dimT_1T'_i = dimT_1T'_q$ if and only if $T_1T'_i = T_1T'_q \subset T'_i$. And, in this case, $dimT_1 + dimT_q = dimC + dimT_i$.

c) $dimT_1T'_i < dimT_1T'_q$ if and only if $T_1T'_i = T_1T'_q \cap T'_i$. And, in this case, $dimT_1 + dimT_q > dimC + dimT_i$.

d) either $T_1T'_q \subset T'_i$ for any $i \geq 2$, or there exists an i, with $2 < i \leq q$, such that $T_1T'_q \subset T'_j$ for $i \leq j \leq q$ and $T_1T'_j = T_1T'_q \subset T'_j$ for $2 \leq j < i$.

e) If $T_1 \subset T_2 \subset ..... \subset T_q$ is a complete chain (i.e., if $\dim T_{i+1} = \dim T_i + 1, i = 1, ..., q-1$) and i is the smaller integer for which $T_1T'_q \subset T'_i$, then $dimT_q = dimC + i - 1$.

3.2 THEOREM. Let $\mathbb{C}$ be a geometric configuration of subvarieties of Z and let (X, G, w) be the associated graph labeled by the own configuration $\mathbb{C}$ by means of the bijection $\tilde{x}: \mathbb{C} \to X$ which takes each subvariety H in a point of X denoted by x_h, the weight being given by $w(x_h) = dimH$. Let $\mathbb{C}'$ be the blow up configuration of $\mathbb{C}$ with center at $C \in \mathbb{C}$ and (X'_c, G'_c, w'_c) its associated weighted graph. The map $p_c: \mathbf{C}' \to \mathbf{C}$ induces a dominant acyclic transitive graph morphism $\sigma_c: (X'_c, G'_c, w'_c) \to (X, G, w)$ given by $\sigma_c(K') = x_h$ where $H = p_c(K')$ for $K' \in \mathbb{C}'$. Let (X_c, G_c, w_c) be the weight acyclic transitive blow up graph of (X, G, w) on the blowing up center x_c and let $\pi_c: (X_c, G_c, w_c) \to (X, G, w)$ be the corresponding blowing up.

Then, the weighted graph (X'_c, G'_c, w'_c) is isomorphic to the transitivized of (X_c, G_c, w_c) by an isomorphism Δ_c preserving the weights and such that $\sigma_c = \pi_c \circ \Delta_c$.

4. GEOMETRIC MODIFICATION FOR ACYCLIC GRAPHS.

If (X, G, w) is a connected acyclic graph with dominant maximum element and with transversal weight map, its blow up (X_c, G_c, w_c) is a graph of the same kind, so it can be blown up again at any of its points. If the graph (X, G, w) is labeled by the label set $E = \{1, 2, ..., n\}, n$ being the $Card(X)$, so that the graph is **naturally ordered by levels**, i.e., if $x_i \in N_p, x_j \in N_q$ and $p < q$, then $i < j$, we can blow up successively with center on x_i, from $i = 1$ to $i = n$.

4.1 DEFINITION. With conditions as above, if the blow up of (X, G, w) with center on x_1 is denoted (X_1, G_1, w_1) and if, for every $i = 2, .., n$, the blow up of $(X_{i-1}, G_{i-1}, w_{i-1})$ with center on x_i is denoted (X_i, G_i, w_i), then the graph (X_n, G_n, w_n) will be called **total or geometric blow up** of (X, G, w). Each blowing up on x_i has associated the graph morphism $\pi_i: (X_i, G_i, w_i) \to (X_{i-1}, G_{i-1}, w_{i-1})$ defined at 2.3; then the morphism $\pi: (X_n, G_n, w_n) \to (X, G, w)$ will be called **geometric modification** of (X, G, w), where $\pi = \pi_1 \circ \pi_2 \circ ..... \circ \pi_n$. We will also consider, for every i, with $1 \leq i \leq n$, the **partial geometric modification** $\pi_1 \circ \pi_2 \circ ..... \circ \pi_i: (X_i, G_i, w_i) \to (X, G, w)$

and we will say that (X_i, G_i, w_i) is the **i$^{\text{th}}$ partial geometric blow up** of (X, G, w).

These concepts are relative to the ordering of the elements of X. If $x_1, \ldots, x_q$ are consecutive points of level N_0 and $(x_{s(1)}, \ldots, x_{s(q)})$ is a permutation of these points, the q^{th} partial geometric blow up (X_q, G_q, w_q) follows also as the successive blowing ups of the points $x_{s(1)}, \ldots, x_{s(q)}$. Note that, in these blowing ups there is no transversality since B_c is empty.

Once a blowing up with center on a point x_c, with $1 \leq c \leq q$, has been realized the induced subgraph by the graph (X_c, G_c, w_c) on the point set $\{x_{cj} : x_j \in x_c \uparrow\}$ is a transversal structure at the next blowing up of the graph (X_q, G_q, w_q) with center on a point x_{j_1} such that $(x_c, x_{j_1}) \in G$. In this transversal structure, every transversal path $x_{cj_1} \ldots x_{cj_k}$ is connected to the path $x_{j_1} \ldots x_{j_k}$ by the arcs (x_{cj_i}, x_{j_i}), for $1 \leq i \leq k$, and $w(x_{cj_i}) = w(x_{j_i}) - 1$, and then, for every transversal point x_{cj_i} of this path, one has $x_{cj_i} \notin B'_c$. Moreover, for each semicycle formed by the paths $x_{cj_i} x_{cj_{i+1}} x_{j_{i+1}}$ and $x_{cj_i} x_{j_i} x_{j_{i+1}}$, one has $w(x_{cj_{i+1}}) + w(x_{j_i}) = w(x_{cj_i}) + w(x_{j_{i+1}})$. Therefore, at the blowing up of (X_q, G_q, w_q) with center at a point $x_{j_1} \in N_1$, such that $(x_c, x_{j_1}) \in G$, there exist transversal points but both B'_c and T_c are empty.

If $x_{r+1}, \ldots, x_u$ are consecutive points of level N_p and $(x_{s(r+1)}, \ldots, x_{s(u)})$ is a permutation of these points, then, the u^{th} partial geometric blow up (X_u, G_u, w_u) also results from the successive blowing ups at the points $x_1, \ldots, x_r, x_{s(r+1)}, \ldots, x_{s(u)}$. In consequence, if X is classified by levels, $X = \bigcup_{p=0}^{k} N_p$, with $k = height(X, G)$, the geometric blow up (X_n, G_n, w_n) also results from the successive blowing ups at the points $x_{s(1)}, \ldots, x_{s(n)}$, $(x_{s(1)}, \ldots, x_{s(n)})$ being a permutation of $x_1, \ldots, x_n$ preserving the order of the levels, i.e., if $x_{s(i)} \in N_p, x_{s(j)} \in N_q$ and $p < q$, then $s(i) < s(j)$. In all steps the transversal sets B'_c and T_c are empty and so, after proposition 2.2, we can state:

4.2 THEOREM. If (X, G, w), with $Card(X) = n$, is a connected acyclic graph naturally ordered by levels with dominant maximum element x_z and with transversal weight map, then its geometric blow up (X_n, G_n, w_n) is a connected weighted graph with dominant maximum element x_z and with completely transversal weight map. The weight $w(x_z) = w_n(x_z)$ of the dominant maximum point determines the weight map w_n as follows: if $x \in X_n$, then $w_n(x)$ is equal to $w(x_z)$ minus the length of any path connecting x with x_z. The graphic structure (X_n, G_n, w_n) is, therefore, independent of the weight map w given for the original graph (X, G, w).

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