Large vertex transitive graphs of diameter 2

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Abstract

La cota superior de Moore $n(\Delta,2)$ para el orden de los grafos con grado máximo Δ y diámetro 2 es $n(\Delta,2)\leq\Delta^2+1$. La única cota inferior general conocida para grafos vértice simétricos es $n(\Delta,2)\geq\lfloor\frac{\Delta+2}{2}\rfloor\lceil\frac{\Delta+2}{2}\rceil$. Recientemente una construcción de grafos vértice transitivos de diámetro 2, basada en grafos de voltaje, con orden $\frac{8}{9}(\Delta+\frac{1}{2})^2$ fue obtenida en [11] para $\Delta=(3q-1)/2$ y q una potencia de un primo congruente con 1 mod 4. En este trabajo damos una construcción geométrica alternativa con la que obtenemos grafos con los mismos parámetros y, cuando q es la potencia de un primo que no es congruente con 1 modulo 4, obtenemos grafos de diametro 2 y orden $\frac{1}{2}(\Delta+1)^2$.

1 Introduction

The well-known (Δ,D)-problem asks for the largest possible number $n(\Delta,D)$ of vertices in a graph with given maximum degree Δ and diameter D. The *Moore bound*$n(\Delta,d)\leq\frac{\Delta(\Delta-1)^D-2}{\Delta-2}$ can only be reached when either $\Delta=2$ (the cycle C_{2D+1}), or $D=1$ (the complete graph $K_{\Delta+1}$) or when $D=2$ and $\Delta=1,2,3,7$ and perhaps 57 [9]. A survey of the current best know constructions of graphs with large order for given maximum degree and diameter can be found in [12]. For $D=2$, the general lower bound $n(\Delta,2)\geq\Delta^2-c_\epsilon\Delta^{7/6+\epsilon}$, where $\epsilon>0$ and c_ϵ is a constant, can be obtained from incidence graphs of projective planes, see [11] and the references therein. When the graphs are required to be vertex transitive, the only general lower bound available seems to be

$$n_{vt}(\Delta,2)\geq\lfloor\frac{\Delta+2}{2}\rfloor\lceil\frac{\Delta+2}{2}\rceil,$$

attained by the Cayley graphs $Cay(\mathbb{Z}_a\times\mathbb{Z}_b,S)$, where $a=\lfloor\frac{\Delta+2}{2}\rfloor$, $b=\lceil\frac{\Delta+2}{2}\rceil$ and $S=(\mathbb{Z}_a\times\{0\})\cup(\{0\}\times\mathbb{Z}_b)\setminus\{(0,0)\}$. Recently McKay, Miller and Siran [11] gave an infinite family of vertex transitive graphs of diameter 2 and order $\frac{8}{9}(\Delta+\frac{1}{2})^2$ when $q=(2\Delta+1)/3$ is a prime power congruent to 1 modulo 4. In view of the(unattainable) Moore bound $n(\Delta,2)\leq\Delta^2+1$ this is a remarkable result. Their construction is based on the covering graph technique. The purpose of this note is to provide a geometric construction of vertex symmetric graphs with the same parameters. The construction also provides the following lower bound for vertex symmetric graphs of diameter 2:

$$n_{vt}(\Delta,2)\geq\frac{1}{2}(\Delta^2+1),$$

when $q=(\Delta+1)/2$ is a prime power.

2 The construction

Let $q\neq 2$ be a prime power and denote by $\mathbb{F}_q$ the Galois field of order q. Let $L_0,L_1,\ldots,L_q$ be the parallel classes of lines in the affine plane $A(2,q)$. Assume that the plane is coordinatised in such a way that L_0 consists of the lines with equation $x=c$ for each $c\in\mathbb{F}_q$. Consider the incidence graph B_q of the points in $A(2,q)$ and the set $L\setminus L_0$ of all lines in $A(2,q)$ except the ones in the parallel class L_0. We denote the line of equation $y=mx+b$ in $L\setminus L_0$ by $[m,b]$. The graph B_q is bipartite, q-regular, has $2q^2$ vertices and it is vertex transitive. Indeed, the set of translations of $A(2,q)$ is a group of automorphisms of the graph which acts regularly on the set of points of $A(2,q)$, and the map ϕ which exchanges the point (a,b) with the line $[a,-b]$ with equation $y=ax-b$ is an automorphim of B_q which exchanges stable sets. Every two points not in a line of L_0 determine a unique line in $L\setminus L_0$ and thus they are at distance 2 in B_q. Similarly, two lines not in the same parallel class of $L\setminus L_0$ intersect in a unique point and so they are at distance 2 in the graph. The construction is completed by inserting apropriate copies of graphs of diameter 2 in the set of points of each line of L_0, and in the set of lines in each parallel class of lines except L_0. Since we require our graphs to be vertex transitive, additional care must be taken in the way to insert these graphs.

Let S_1, S_2 be subsets of $\mathbb{F}_q$ satisfying the following three conditions:

(i) There is $\alpha \in \mathbb{F}_q$ such that $\alpha S_1 = S_2$ and $\alpha S_2 = S_1$.

(ii) $S_1 \cup S_2$ cover all non zero elements of $\mathbb{F}_q$.

(iii) $S_i = -S_i$, $i = 1, 2$.

Let G_i be the Cayley graph on the additive group of $\mathbb{F}_q$ with generating set S_i, $i = 1, 2$. Then, by condition (i), the two graphs are isomorphic. Actually, if we denote still by α the map $\alpha(x) = \alpha x$, then $\alpha(G_1) = G_2$. Moreover, by (ii), we have $|S_1| = |S_2| \geq (q-1)/2$. Then both G_1 and G_2 have diameter at most 2. Consider the graph $B_q(G_1, G_2)$ constructed from B_q by adding copies of G_1 and G_2 as follows.

a) For each line in L_0, we embed a copy of G_1 in its set of points. More precisely, if S_1 is the generating set of the Cayley graph G_1, for each line in L_0 with equation $x = c$ the neighborhood of vertex (c, y) in the inserted copy of G_1 is the set of points $(c, y) + (\{0\} \times S_1)$.

b) Similarly, to each parallel class L_i, $i \neq 0$, we embed a copy of G_2 in such a way that the neighborhood of the line $[a, b]$ is the set of lines $[a, b] + (\{0\} \times S_2)$ in the same parallel class.

Proposition 1 *The graph $B_q(G_1, G_2)$ is vertex transitive and has diameter* 2.

Proof. Let us first show that $B_q(G_1, G_2)$ is vertex transitive. Note that the translations in the affine plane $A(2, q)$ still act as automorphisms of $B_q(G_1, G_2)$. Indeed, a transllation sends each line $[m, b]$ to a parallel one of the form $[m, b + x_m]$ for some $x_m \in \mathbb{F}_q$, so that it acts as a transllation of the induced graph G_2 in the parallel class, i.e. an automorphism of G_2. Similarly, a transllation sends the set of points of a line in L_0 to the points of another line in L_0 and thus an induced copy of G_1 to another one. Let us denote by α' the bijection on the vertex set of $B_q(G_1, G_2)$ defined as $\alpha'(x, y) = (x, \alpha y)$ in the set of points and $\alpha'[x, y] = [\alpha x, \alpha y]$ on the set of lines, where $\alpha \in \mathbb{F}_q$ satisfies $\alpha S_1 = S_2$ (and $\alpha S_2 = S_1$). Clearly α' preserves the incidence relations in $A(2, q)$ and it is therefore an automorphism of the graph B_q. Moreover, α' exchanges the copies of G_1 with the copies of G_2. Hence, $\alpha'\phi$ is an automorphism of $B_q(G_1, G_2)$ which exchanges points with lines. Therefore, the group of automorphims generated by the transllations and $\alpha'\phi$ acts transitively on the set of vertices of the graph.

Finally, let us show that $(0, 0)$ has eccentricity 2. All points not in the line of L_0 incident to $(0, 0)$ are at distance 2 from $(0, 0)$ in the subgraph B_q, and the points in this line are also at distance at most 2 in the copy of G_1 embedded in it. On the other hand, $(0, 0)$ has the lines $[m, 0]$, $m \in \mathbb{F}_q$ at distance 1 and the lines $[m, u], u \in S_2$ and $[m, v]$, $v \in S_1$ at distance 2. Since $S_1 \cup S_2 = \mathbb{F}_q^*$, all lines are at distance at most 2 from $(0, 0)$. Therefore, $B_q(G_1, G_2)$ has diameter 2. □

The above construction provides instances of large graphs with diameter 2 by appropriate choices of sets S_1, S_2 satisfying conditions (i)-(iii) above. Part (ii) of the following theorem is proved in [11] using a different construction.

Theorem 1 *Let q be a prime power, $q \neq 2$. Then*

a) $n_{vt}(\Delta, 2) \geq \frac{1}{2}(\Delta + 1)^2$ *for* $\Delta = 2q - 1$.

b) $n_{vt}(\Delta, 2) \geq \frac{8}{9}(\Delta + \frac{1}{2})^2$ *for* $\Delta = (3q - 1)/2$ *and* $q \equiv 1 \pmod 4$.

c) $n_{vt}(6, 2) \geq 32$.

Proof. A trivial choice for the sets S_1, S_2 involved in the construction of $B_q(G_1, G_2)$ is $S_1 = S_2 = \mathbb{F}_q^*$. Then, both G_1 and G_2 are complete graphs and $B_q(G_1, G_2)$ has degree $\Delta = 2q - 1$ and order $n = \frac{1}{2}(\Delta + 1)^2$. This proves (i). When q is a prime power congruent to 1 mod 4 then a better choice for S_1 and S_2 with cardinality $(q-1)/2$ (the minimum possible) can be found. Let η be a primitive root of $\mathbb{F}_q$, $S_1 = \{\eta, \eta^3, \ldots, \eta^{(q-1)/2}\}$ and $S_2 = \eta S_1$. Since $(q-1)/2$ is even, we have $S_1 = -S_1$. Moreover, $\eta S_2 = S_1$ and $S_1 \cup S_2 = \mathbb{F}_q^*$. The resulting graph $B_q(G_1, G_2)$ has degree $\Delta = (3q - 1)/2$ and order $n = \frac{8}{9}(\Delta + \frac{1}{2})^2$. This proves (ii).

Finally, for $q = 4$, we may choose $S_1 = \{1, \eta\}$ and $S_2 = \{1, \eta^2\}$ which results in a graph of degree 6 and order 32. This proves (iii). □

Note that, if we omit condition (i) in the choice of sets S_1 and S_2 then we still obtain a graph of maximum degree $q+\max\{|S_1|,|S_2|\}$ and diameter 2 which may be not vertex transitive. Again, the best possible choice of S_1 and S_2 provides graphs with the same parameters as the ones obtained in [11].

When $q \equiv 0 \pmod 4$ we may choose $S_1 = \{1,\eta,\cdots,\eta^{q/2-1}\}$ and $S_2 = \{1,\eta^{q/2},\cdots,\eta^{q-1}\}$. We have $S_i = -S_i$, $i = 1,2$, and $S_1 \cup S_2 = \mathbb{F}_q^*$. The resulting graph $B_q(G_1,G_2)$ have diameter 2 and order $n = \frac{8}{9}\Delta^2$. When $q \equiv -1 \pmod 4$, a possible choice is $S_1 = \{1,\eta,\cdots,\eta^{(q-3)/4}\} \cup \eta^{(q-1)/2}\{1,\eta,\cdots,\eta^{(q-3)/4}\}$ and $S_2 = \{1,\eta^{(q+1)/4},\dots,\eta^{(q-3)/2}\} \cup \eta^{(q-1)/2}\{1,\eta^{(q+1)/4},\dots,\eta^{(q-3)/2}\}$, which gives rise to a graph with diameter 2 and degree $n = \frac{8}{9}(\Delta^2 - 1/2)$.

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