Programa de Doctorado "Matemáticas"

PhD Dissertation

Orthogonal matrix polynomials with orthogonal differences, Rodrigues' formulas and related subjects.

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A mis padres

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Summary

The content of this thesis is part of the areas of approximation theory and special functions, in particular, of the matrix orthogonality theory. It is composed by two parts. The first part is an introduction with the corresponding preliminaries, goals, summary of the results, discussion and conclusions. The second part is formed by two published papers:

– A. J. Durán and V. Sánchez-Canales *Orthogonal matrix polynomials whose differences are also orthogonal.* J. Approx. Theory, (2014) **179**, 112–127.
– A. J. Durán and V. Sánchez-Canales *Rodrigues' Formulas for Orthogonal Matrix Polynomials Satisfying Second-Order Difference Equations.* Integral Transform. Spec. Funct. (2014) **25**, 849–863.

In these works we study four important properties of matrix orthogonal polynomials which are also eigenfunctions of a second order difference operator. We show what happens with four of the characterization properties of the classical discrete polynomials and the relations between all of them in the matrix case. We obtain new results that establish important differences between the scalar case and the matrix one: we prove that for orthogonal matrix polynomials the scalar characterizations for classical discrete families are no longer equivalent. More precisely, we prove that the equivalence between the orthogonality of the differences of orthogonal polynomials and the discrete Pearson equation for the associated weight matrix remains true for orthogonal matrix polynomials. Besides, under suitable Hermitian assumptions, they also imply that the associated orthogonal polynomials are eigenfunctions of certain second order difference operator, but the converse is, in general, not true.

We also study the question of the existence of Rodrigues' formulas for families of orthogonal matrix polynomials which are also eigenfunctions of a second order difference operator. We develop a method to find such formulas and, using it, we produce the first discrete Rodrigues' formulas in arbitrary size for two families of orthogonal matrix polynomials.

Finally, we include two original and relevant examples of families of orthogonal matrix polynomials illustrating our results.

Contents

Part I

Introduction

1.1 Preliminaries

1.1.1 Orthogonal polynomials and classical families

The first examples of orthogonal polynomials appeared at the end of the XVIII century, as a tool in the resolution of celestial mechanical problems: some difficulties in the Newton's Theory of Gravity were solved by A. M. Legendre in 1785 by introducing the first family of orthogonal polynomials, called Legendre polynomials. This family was generalized by C. G. J. Jacobi in 1859 receiving the name of Jacobi polynomials, [62]. Some years later, in 1864, C. Hermite published in [59] a new family of orthogonal polynomials, the Hermite polynomials, with the aim to generalize certain functions series to the case of several variables. Another important family are the Laguerre polynomials, introduced in 1879 by E. N. Laguerre in [65] as the solution of the differential equation of Laguerre.

Hermite, Laguerre and Jacobi polynomials are the most important families of orthogonal polynomials and they are called the classical families. Classical families have several interesting properties in common, for instance, they and only them ([4]) are eigenfunctions of a second order differential operator

$$D = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx}, \tag{1.1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree at most 2 and exactly 1, respectively.

The second order differential equation associated to the operator (1.1) has important applications in quantum physic: Hermite polynomials appear in the resolution of the one-dimensional harmonic oscillator, and Laguerre polynomials and Gegenbauer polynomials (a particular case of Jacobi polynomials) appear in the resolution of the Schrodinger equation for the hydrogen atom.

A discretization in the lattice $x(s) = s$ of the differential equation associated to the operator (1.1) is the following difference equation, known as *difference equation of hypergeometric kind*:

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0, \tag{1.2}$$

where Δ and ∇ are the usual first order difference operators

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1).$$

The families of orthogonal polynomials satisfying equations of the form (1.2) (where σ and τ are independent of n) are called the classical discrete families.

There are four of such families: Charlier, Meixner, Krawtchouk and Hahn polynomials.

These families appeared at the end of the XIX century and the beginning of the XX century. In 1905, C. V. L. Charlier introduced the Charlier polynomials in [10], where certain problems related to astronomic measurements are studied. This family is orthogonal with respect to the discrete measure ($a > 0$)

$$\omega(x) = \frac{e^{-a} a^x}{\Gamma(x+1)}, \quad x = 0, 1, 2, \ldots$$

In 1929, M. Krawtchouk discovered a new family of discrete orthogonal polynomials by following the ideas of Chebyshev (who published and improved the discrete Chebyshev polynomials in [11] and [12], respectively). This family is nowadays known as Krawtchouk polynomials and it is orthogonal with respect to ($0 < p < 1$, $n \leq N-1$)

$$\omega(x) = \frac{N! p^x (1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(x+1)}, \quad 0 \leq x \leq N+1.$$

Subsequently, in 1934, J. Meixner introduced Meixner polynomials in [73] when he was solving a problem related to generating functions (usually used in probability theory). Meixner polynomials are orthogonal with respect to the discrete measure ($c > 0$, $0 < a < 1$)

$$\omega(x) = \frac{a^x \Gamma(x+c)}{\Gamma(c)\Gamma(x+1)}, \quad x = 0, 1, 2, \ldots$$

In 1935, W. M. Hahn published in [57] the Hahn polynomials, which can be seen as a discretization of Jacobi polynomials. This familiy is orthogonal with respect to the discrete measure ($\alpha, \beta \geq -1$, $n \leq N-1$)

$$\omega(x) = \frac{\Gamma(N+\alpha-x)\Gamma(\beta+x+1)}{\Gamma(N-x)\Gamma(x+1)}, \quad 0 \leq x \leq N.$$

Lets notice that the classical discrete measures associated to the Charlier, Meixner, Krawtchouk and Hahn polynomials correspond to the discrete probability distributions of Poisson, Pascal, binomial and Pólya (or hypergeometric), respectively.

One of the most important problems in the theory of orthogonal polynomials consists of determining the common properties of the classical families. The problem for discrete families was considered for the first time in 1931 by E. H. Hildebrandt (see [60]) and in a more wide scope by Hahn (see [58]). Classifications theorems were obtained by O. E. Lancaster in [66] and P. Lesky in [67].

In our research we focus in four of these characterizations, formulated in the following theorem.

Theorem 1. *Some characterizations of classical discrete families are the following:*

1. *The classical discrete polynomials are eigenvalues of a second order difference operator of the form*

$$d(\cdot) = f_{-1}(x)\mathfrak{S}_{-1}(\cdot) + f_0(x)\mathfrak{S}_0(\cdot) + f_1(x)\mathfrak{S}_1(\cdot),$$

where $\mathfrak{S}_l$ *denotes the shift operator* $\mathfrak{S}_l(f(x)) = f(x+l)$ *and* f_i, $i = -1, 0, 1$, *are polynomials of degree not larger than 2 (independent of n) satisfying that* $\deg(\sum_{l=-1}^{1} l^k f_l) \leq k$, $k = 0, 1, 2$.

2. *The sequence of differences* $(\nabla p_n)_n$ *is again orthogonal with respect to a measure.*

3. *Every classical discrete measure* ω *satisfies a discrete Pearson equation*

$$\Delta(g_2(x)\omega(x)) = g_1(x)\omega(x),$$

where, g_2 *and* g_1 *are polynomials of degree at most 2 and exactly 1, respectively.*

4. *They can be defined by a discrete Rodrigues' formula:*

$$p_n(x) = \frac{\Delta^n\left(\omega(x)\prod_{m=0}^{n-1}\sigma(x-m)\right)}{\omega(x)}, \tag{1.3}$$

where ω *is the corresponding classical discrete weight.*

An in-depth study of characterization theorems can be found in [1,3,13, 70].

The purpose of this thesis is to show what happens with all these four properties when we consider the matrix case. We point out here that some of them are no longer equivalent for orthogonal matrix polynomials. We illustrate our results with some original and relevant examples.

1.1.2 Orthogonal matrix polynomials

The theory of orthogonal matrix polynomials started with two papers by M. G. Krein in 1949, [63,64].

A matrix polynomial can be defined by a matrix whose entries are scalar polynomials:

$$P_n(x) = \begin{pmatrix} p_{11}(x) & \cdots & p_{1N}(x) \\ \vdots & \ddots & \vdots \\ p_{N1}(x) & \cdots & p_{NN}(x) \end{pmatrix},$$

or by a polynomial which coefficients are matrices of the same size:

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0.$$

A weight matrix W is an $N \times N$ matrix of measures supported in the real line satisfying that

1. $W(A)$ is positive semidefinite for any Borel set $A \in \mathbb{R}$,
2. W has finite moments of every order $\big(\mu_n = \int x^n dW(x)\big)$,
3. $\int P(t)dW(t)P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.

Condition 3 is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W.

All the examples considered in this memory are discrete weight matrices of the form

$$W = \sum_{x\in\mathbb{N}} W(x)\delta_x.$$

For a discrete weight matrix $W = \sum_{x\in X} W(x)\delta_x$ supported in a countable set X of real numbers, the Hermitian sesquilinear form defined by $\langle P, Q\rangle = \int PdWQ^*$ takes the form

$$\langle P, Q\rangle = \sum_x P(x)W(x)Q^*(x).$$

We say that a weight matrix reduces to scalar measures if there exist a nonsingular matrix T (independent of x) and a diagonal matrix weight $D(x)$ such that $W(x) = TD(x)T^*$. The study of this kind of weight matrices belongs to the scalar theory more than to the matrix one. Because of this reason, we are only interested in weight matrices that do not reduce to scalar weights.

A sequence of matrix polynomials $(P_n)_n$ will be orthogonal with respect to W if it satisfies that every P_n, $n \geq 0$, has degree exactly n and non singular leading coefficient, and $\int P_n dW P_m^* = \Gamma_n \delta_{n,m}$, where Γ_n, $n \geq 0$, is a positive definite matrix. When $\Gamma_n = I$, we say that the polynomials $(P_n)_n$ are orthonormal.

There are several reasons that point out the importance of matrix orthogonality within the field of approximation theory.

On the one hand, orthogonal matrix polynomials are useful in the resolution of scalar theory problems, because scalar orthogonality has a nontrivial representation as matrix orthogonality. At the end of the eighties it was asked whether a sequence of scalar polynomials $(p_n)_n$ satisfying certain higher order recurrence relation enjoys some kind of canonical orthogonality

(Favard's Theorem answers a particular case of the problem, when the recurrence relation has order three). The solution of that problem, which provides an extension to higher order recurrence relations of the well-known Favard's Theorem, was found by A. J. Durán in 1993 (see [19] and [20]) and shows the following connection between scalar polynomials satisfying higher order recurrence relations and orthogonal matrix polynomials: If $(p_n)_n$ is a sequence of scalar polynomials satisfying certain higher order recurrence relation, then they can be splited up in some way to form a sequence of matrix polynomials $(P_n)_n$ which is orthogonal on the real line with respect to a weigh matrix. Reciprocally, given a sequence $(P_n)_n$ of orthogonal matrix polynomials with respect to a weight matrix, a sequence of scalar polynomials satisfying certain higher order recurrence relation can be constructed from the polynomials on the rows of each P_n (see also [50]).

On the other hand, some specificities of the matrix structure, like the existence of singular matrices without inverse, and the non-commutativity of the matrix product, introduce important changes that demand the develop of new techniques and original ideas to deal with the resolution of matrix problems. These differences become specially important in the looking for families of orthogonal matrix polynomials satisfying second order differential or difference equations.

There are also several connections between matrix orthogonal polynomials and other researching areas. For instance, the relation with probability theory showed in [53], with random matrices showed in [61] or with quantum physics in [34]. We can find more applications of matrix orthogonality theory in [76].

In the last twenty years, a systematic study of matrix orthogonality has been developed, turning orthogonal matrix polynomials into one of the main areas of study in the theory of orthogonal polynomials. For instance, asymptotic properties are studied in [15,16,23,44,71,72], algebraic aspects, properties of the zeros and Gaussian quadrature formulas in [17,21,39,45,47,76], and density problems and problems regarding matrix moments in [40–42,68,69].

Recurrence formulas and an extension of Favard's Theorem can be found in [19,20,50]. In that way, an analogous of the three terms recurrence relation takes, in the matrix case, the form

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x),$$

(where we take $P_{-1} = 0$). In that formula the coefficients B_n have to be hermitian, and since each of the polynomials P_n has nonsingular leading coefficient, it follows that the coefficients A_n have to be nonsingular. This equivalence is, in fact, a consequence of the symmetry of the operator of multiplication by x with respect to the hermitian sesquilinear form defined in the linear space of matrix polynomials by a weight matrix supported on the real line.

A very attractive subject in the field of orthogonal matrix polynomials are the examples of families satisfying second order differential equations. Related results can be found in [5–9, 22, 24–27, 29–38, 51, 54–56, 74, 75]. A parallel and recently opened field is formed by matrix families satisfying second order difference equations (see [2, 28, 46, 48, 49]). New and innovator methods have been developed to find these families, but some other will be needed to get classifications theorem like those of Bochner and Lancaster for the classical and classical discrete families. We go in depth in this subject in the following section.

More details about matrix orthogonal polynomials theory can be found in [14] and [43], and references therein.

1.1.3 Second order difference operators

In the last years a big amount of papers has been devoted to introduce and study families of orthogonal matrix polynomials satisfying second (and higher) order differential equations of the form

$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0 = \Lambda_n P_n, \quad n \geq 0, \tag{1.4}$$

where F_2, F_1 and F_0 are matrix polynomials (which do not depend on n) of degree less than or equal to 2, 1 and 0, respectively. With the paper [28], Antonio Durán opened in 2012 the issue of orthogonal matrix polynomials satisfying difference equations of the form

$$P_n(x-1)F_{-1}(x) + P_n(x)F_0(x) + P_n(x+1)F_1(x) = \Lambda_n P_n(x), \quad n \geq 0, \tag{1.5}$$

where F_i, $i = -1, 0, 1$, are matrix polynomials of degree not larger than 2 (independent of n) satisfying that $\deg(\sum_{l=1}^{1} l^k F_l) \leq k$, $k = 0, 1, 2$. In [28], A. J. Durán introduces the first non-trivial examples of weight matrices whose orthogonal polynomials satisfy second order difference equations (by non-trivial we mean that the weight matrix is not a diagonal matrix with discrete classical measures on the diagonal and the difference operator is not a diagonal matrix with the associated difference operators on its diagonal, or some other examples what can be reduced to that situation). It is the initial motivation for this thesis, exploring new phenomena, and looking for new examples of discrete families what could contribute in the elaboration of a general classification theorem.

New examples of orthogonal matrix polynomials $(P_n)_n$ satisfying second order difference equations like (1.5) appeared in the last 2 years [2, 48, 49].

Equations (1.4) and (1.5) are equivalent to looking for second order differential and difference operators of the form

$$D(\cdot) = \frac{d^2}{dx^2}(\cdot)F_2(x) + \frac{d}{dx}(\cdot)F_1(x) + (\cdot)F_0, \tag{1.6}$$

and

$$D(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1}(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_1(\cdot)F_1(x), \tag{1.7}$$

respectively, having the polynomials $(P_n)_n$ as eigenfunctions, that is, $D(P_n) = \Lambda_n P_n$.

Let us notice that the difference operator acts on the right, and the eigenvalues Λ_n multiply on the left. The main reason for this election is that it keeps the left-linearity of the operator: A right difference operator D is left-linear, but not right-linear, that is, for a matrix function P and a constant matrix A, we have

$$D(AP) = AD(P), \quad D(PA) \neq D(P)A \text{ (in general)}.$$

It can be developed an analogous theory considering second order difference operators acting on the left, with right hand side eigenvalues, but it would be convenient to use the family $(P_n^*)_n$ as well as the inner product $(P, Q)_W = \int_{\mathbb{R}} Q^*(t)dW(t)P(t)$.

Nevertheless, to consider right hand side eigenvalues and difference operators acting also on the right lacks in interest, at first, because the considered inner product is not right linear, it means that $D(P_n)$ lose in general the orthogonality when we take the equation $D(P_n) = P_n\Lambda_n$. And secondly, because the weight matrix associated to the solution of this kind of equation can be reduced to scalar weights. On the contrary, an equation like (1.5) does generate non trivial examples.

Families of orthogonal matrix polynomials satisfying second order differential or difference equations are among those that are likely to play in the case of matrix orthogonality the role of the classical or classical discrete families in the case of scalar orthogonality. From the theoretical and applied point of view, this is the reason why their recent discovery and study have raised such an expectation. Very important differences have been found between matrix and scalar case. The first difference is related to the difficulty to find these kind of examples and to solve the associated second order differential (or difference) matrix equations. It has imposed the necessity to carry out new and powerful techniques. Another difference is the flexibility of matrices, what provides a richness and complexity much more bigger than in the scalar case, where there are only a small number of classical families, compared to the plenty of examples barely found in the matrix case.

1.1.4 Matrix discrete Pearson equation

Examples of orthogonal matrix polynomials satisfying second order difference equation like (1.5) have been essentially found by solving an appropriate set of commuting and difference equations. This set includes a matrix analogous to the scalar Pearson equation

$$F_1(x-1)W(x-1) = W(x)F_{-1}^*(x), \tag{1.8}$$

and the commuting equation

$$F_0W = WF_0^*. \tag{1.9}$$

Equations (1.8) and (1.9) are also called *symmetry equations*. Under certain boundary conditions they imply that the orthonormal polynomials with respect to W are eigenfunctions of the second order difference operator

$$L(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1}(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_1(\cdot)F_1(x) \tag{1.10}$$

with Hermitian (left) eigenvalues Λ_n; that is, $L(P_n) = \Lambda_n P_n$, $n \geq 0$.

1.1.5 A functional approach

Sometimes is more convenient to use a functional approach instead of that of weight matrices. Since we use this approach in the first paper composing this thesis, it is convenient to finish this section with a brief summary of functional notation.

Lets denote by

- $\mathbb{C}^{N\times N}$: the set of matrices of size $N \times N$ whose elements are complex numbers.
- $\mathbb{P}^N$: the linear space of matrix polynomials in one real variable with coefficients in $\mathbb{C}^{N\times N}$.
- $\mathbb{P}_n^N$: the linear subspace of $\mathbb{P}^N$ formed by those matrix polynomials with degree at most n.
- $\mathbb{P}$: the linear space of polynomials with complex coefficients.

A moment functional u is a left linear functional

$$u : \mathbb{P}^N \to \mathbb{C}^{N\times N} : \ u(AP + BQ) = Au(P) + Bu(Q),$$

with $A, B \in \mathbb{C}^{N\times N}$ and $P, Q \in \mathbb{P}^N$. For a moment functional u and a matrix polynomial $P \in \mathbb{P}^N$, the duality bracket is defined by

$$\langle P, u\rangle = u(P).$$

We associate to each moment functional u a bilinear form in the linear space of matrix polynomials defined by

$$\langle \cdot,\cdot\rangle_u : \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{C}^{N\times N} : \langle P, Q\rangle_u = \sum_{i,j} A_i \langle x^{i+j} I, u\rangle B_j^*, \tag{1.11}$$

where $P = \sum_n A_n x^n$ and $Q = \sum_n B_n x^n$. We then say that a sequence of matrix polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is orthogonal with respect to the moment functional u if $\langle P_n, P_k\rangle_u = \Lambda_n \delta_{k,n}$, where Λ_n is a nonsingular matrix for $n \geq 0$.

The basic operations such as multiplication by polynomials or the first order difference operators will be defined by duality. In particular, for a polynomial $P = \sum_n A_n x^n$ and a moment functional u, we define the moment functionals Pu and uP as

$$Pu(Q) = u(QP), \quad uP(Q) = \sum_n u(Qx^n) A_n.$$

Notice that for a polynomial $p \in \mathbb{P}$, we have $pIu = upI$. With this definitions, one straightforwardly has $\langle P, Q\rangle_u = \langle P, uQ^*\rangle$. These definitions agree with the usual product of a polynomial P times a weight matrix W.

For a moment functional u, the new moment functionals Δu and ∇u are defined, respectivelly, by

$$\Delta u(P) = -u(\nabla P), \qquad \nabla u(P) = -u(\Delta P).$$

We say that a moment functional u is quasi-definite if $\det(\Delta_n) \neq 0$ for every $n \geq 0$, where Δ_n is the Hankel-block matrix

$$\Delta_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

and μ_k is the moment of order k with respect to u: $\mu_k = \langle x^k I, u\rangle$. Notice that the moments μ_k, $k \geq 0$, of u define uniquely both the moment funcional u and the bilinear form $\langle \cdot,\cdot\rangle_u$ (1.11). It turns out that a moment functional u has a sequence of orthogonal polynomials if and only if u is quasi-definite (see [6]).

Given a moment functional u with moments μ_k, $k \geq 0$, we define its (complex) adjoint u^* as the moment functional with moments μ_k^*, $k \geq 0$. If the moments μ_k, $k \geq 0$, of a moment functional u are Hermitian matrices, i.e. $u = u^*$, we say that u is Hermitian. If, in addition, $\det(\Delta_n) > 0$ for every $n \geq 0$, we say that u is positive definite.

Moment functionals can always be represented by integration with respect to a matrix of measures: using [18], one can see that given a moment functional u there exists a matrix of measures W supported in the real line (whose entries are Borel measures in $\mathbb{R}$) such that $\langle P, u\rangle = \int_{\mathbb{R}} PdW$. The most interesting case is when the matrix moment functional is defined by a weight matrix W. It follows from [39] that a moment functional can be represented by a weight matrix if and only if it is positive definite.

1.2 Goals: characterization theorems and new examples

As we mentioned above, striking developments in the area of orthogonal matrix polynomials have appeared: the discovery of many new phenomena that are absent in the well known scalar theory. One of these phenomena is that the elements of a family of orthogonal matrix polynomials can be eigenfunctions of several linearly independent second order differential or difference operators like (1.6) and (1.7), respectively, (see [6,8,24,52,56] for differential operators, and [28] for difference operators; and references therein), while each of the classical families in the scalar case can be eigenfunctions of only one linearly independent second order differential or difference operator, respectively.

Another new phenomena is that the same matrix operator can be symmetric for several essentially different weight matrices. We say that a matrix operator D is symmetric with respect to a weight matrix W if it satisfies $\langle D(P), Q\rangle = \langle P, D(Q)\rangle$, for all matrix polynomials P and Q, where $\langle P, Q\rangle = \int PdWQ^*$. It provides a new problem of mathematic relevance: the study of the convex cone of weight matrices associated to certain matrix operator (see [38] and [46]).

In 2003 and 2004 the first examples of matrix families satisfying second order differential equations like (1.4) were introduced in the literature [29,55]. In the same way, in 2012 the first matrix families satisfying second order difference equations like (1.5) were published in [28]. In comparison with the scalar theory, the matrix theory seems to be much more wide in this aspect if one take into account the bunch of families of orthogonal matrix polynomials which have been found being eigenfunctions of second order differential or difference operators. This is the reason why the matrix theory takes on more interest, opening a new field of possibilities when we are looking for new examples. One of the more outstanding goals in this field could be the elaboration of a general classification theorem, in the same way as Bochner classification theorem for the classical families [4] or Lancaster classification theorem for the classical discrete families [66]. However, currently, we are far from these classifications.

One of the main purposes of this thesis is to show that for orthogonal matrix polynomials the scalar characterizations for classical discrete families (Theorem 1 in Preliminaries) are no longer equivalent. More precisely, we prove that the equivalence between the orthogonality of the differences of orthogonal polynomials and the discrete Pearson equation for the associated weight matrix still remains true for orthogonal matrix polynomials. Under suitable Hermitian assumptions on the functionals G_2u and G_1u, they also imply that the associated orthogonal polynomials are eigenfunctions of certain second order difference operator, but the converse is, in general, not true. The differential version of this result was obtained in 2007 by Cantero, Moral and Velázquez ([6]).

The other part of this thesis is devoted to the question of the existence of Rodrigues' formulas for these families of orthogonal matrix polynomials, that is, assuming that W satisfies the commuting and difference equations (1.8) and (1.9), is there any efficient and canonical way to produce a sequence of orthogonal matrix polynomials with respect to W? Say in an analogous way as to (1.3) produces the orthogonal polynomials with respect to a classical discrete weight ω.

Even if F_{-1} and W commute in the strong way that $F_{-1}(x)W(y) = W(y)F_{-1}(x)$, x, y in the support of W, orthogonal matrix polynomials which are eigenfunctions of a difference operator like (1.10) do not seem to satisfy, in general, a scalar-type Rodrigues' formula of the form

$$P_n(x) = C_n \Delta^n \left(W(x) \prod_{m=0}^{n-1} F_{-1}(x-m) \right) W^{-1}(x), \quad n \geq 0, \tag{1.12}$$

where C_n, $n \geq 0$, are nonsingular matrices.
(In this thesis we consider discrete weight matrices of the form $W = \sum_{x=0}^{\infty} W(x)\delta_x$, but implicitly assume that the function $W(x)$ is an entire function in the whole complex plane so that the right hand side of (1.12) makes sense for $x \in \mathbb{C}$).

Instead of (1.12), these orthogonal matrix polynomials seem to satisfy some modified Rodrigues' formula. The first instance of that modified Rodrigues' formula appeared in [28]: the expression

$$P_n(x) = \Delta^n \left(\frac{a^x}{\Gamma(x-n+1)} \begin{pmatrix} 1+ab^2n+b^2x^2 & bx \\ bx & 1 \end{pmatrix} \right) W^{-1}(x)$$

defines a sequence of orthogonal matrix polynomials with respect to the discrete weight matrix

$$W = \sum_{x=0}^{\infty} \frac{a^x}{\Gamma(x+1)} \begin{pmatrix} 1+b^2x^2 & bx \\ bx & 1 \end{pmatrix} \delta_x. \tag{1.13}$$

Since now, we take $\frac{1}{\Gamma(y+1)} = 0$ if y is a negative integer, so that the function $1/\Gamma(x)$ is entire.

Similar Rodrigues' formulas of the form

$$P_n(x) = \Delta^n(\xi_n(x))W^{-1}(x), \tag{1.14}$$

have been found for other families of orthogonal polynomials of size 2×2 (see [28]). In all these examples, the functions ξ_n are simple enough as to make the Rodrigues' formula (1.14) useful for the explicit calculation of the sequence of orthogonal polynomials P_n with respect to W.

Under mild conditions on the functions ξ_n in (1.14), one can easily prove that P_n is orthogonal with respect to any polynomial of degree less than n (by performing a sum by parts). The difficulty to find Rodrigues' formulas like (1.14) is that, in general, it is rather involved to guarantee that P_n defined by (1.14) is a polynomial of degree n with nonsingular leading coefficient. In the extant examples, this requirement on P_n has been checked by a direct computation. As a consequence of this, only examples of small size have been worked out (actually size 2×2) before our work in this thesis.

In this thesis we develop a method to find discrete Rodrigues' formulas like (1.14) for orthogonal matrix polynomials which are also eigenfunctions of a second order difference operator. Using it, we produce the first discrete Rodrigues' formulas in arbitrary size for two illustrative examples.

1.3 Summary of the results, discussion and conclusions

1.3.1 Characterization theorems

The following Theorem is one of the main and original results of this thesis. It proves the equivalence between property 2 and property 3 in Theorem 1 (section Preliminaries) for matrix moment functionals.

Theorem 2. *Let u be a quasi-definite moment functional. Consider a sequence of orthogonal matrix polynomials $(P_n)_n$ with respect to u. Then, the following conditions are equivalent:*

(1) The functional u satisfies the discrete Pearson equation

$$\Delta(uG_2^*) = uG_1^*,$$

where $G_2 = G_{2,2}x^2 + G_{2,1}x + G_{2,0}$ and $G_1 = G_{1,1}x + G_{1,0}$ are matrix polynomials of degree not bigger than 2 and 1, respectively, satisfying that $(n-1)G_{22} + G_{11}$ is not singular for $n \geq 1$.

(2) The sequence $(\nabla P_{n+1})_n$ is a family of orthogonal matrix polynomials with respect to a quasi-definite moment functional $\widetilde{u}$.

Furthermore, $\widetilde{u} = uG_2^*$.

The following original result ensures that, under additional assumptions, the discrete Pearson equation $\Delta(uG_2^*) = uG_1^*$ implies that the orthogonal polynomials $(P_n)_n$ with respect to u are eigenfunctions of a second order difference operator.

Theorem 3. *Let u be a quasi-definite moment functional. Consider a sequence of orthogonal matrix polynomials $(P_n)_n$ with respect to u. Assume that the moment functional u satisfies the discrete Pearson equation $\Delta(uG_2^*) = uG_1^*$, and the Hermitian conditions $G_2u = uG_2^*$ and $G_1u = uG_1^*$, where G_2 and G_1 are polynomials of degree at most 2 and 1, respectively. Consider the second order difference operator*

$$D(\cdot) = \Delta\nabla(\cdot)G_2 + \Delta(\cdot)G_1. \tag{1.15}$$

Then $D(P_n) = \Lambda_n P_n$, for certain matrices Λ_n.

For the differential version of these theorems see [6].

At the end of this section we illustrate Theorem 2 and Theorem 3 with two examples in arbitrary size.

We now display an example in size 3×3 showing that the hermitian condition $uG_2^* = G_2u$ is needed in Theorem 3.

Consider the weight matrix

$$W = \sum_{x=0}^{\infty} \frac{a^x}{x!}(I + A)^x (I + A^*)^x \delta_x, \tag{1.16}$$

where A is an arbitrary upper triangular nilpotent matrix of size 3×3

$$A = \begin{pmatrix} 0 & v_1 & v_3 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where $v_1, v_2 \neq 0$.

Using Maple, it is straightforward to show that there always exist polynomials G_2 and G_1 of degrees not bigger that 2 and 1, respectively, such that $\Delta(WG_2^*) = WG_1^*$ (only lineal equations are needed). In fact, the dimension of the linear subspace

$$\{G_2 \in \mathbb{P}_2^N : \text{there exists } G_1 \in \mathbb{P}_1^N \text{ with } \Delta(WG_2^*) = WG_1^*\}$$

is in general equal to 9.

However, it is not difficult to see that only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy

$$a|v_1|^2|v_2|^2 - 2|v_2|^2 + 2|v_1|^2 = 0 \tag{1.17}$$

(restrictions (1.25) for $N = 3$), one of these polynomials satisfies also the Hermitian condition $G_2W = WG_2^*$. This means that the polynomials $(\nabla P_{n+1})_{n\geq 0}$ are always orthogonal with respect to a moment functional but this moment functional can be represented by a weight matrix (in the sense defined in the Preliminaries) only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy (1.17).

Take, for instance, particular non null values of the parameters $v_1 = v_2 = v_3 = 2$ and $a = 1$. Using Maple it is not difficult to generate the first few orthogonal polynomials with respect to W:

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} x - \frac{329}{69} & \frac{314}{69} & -\frac{2}{69} \\ -\frac{58}{69} & x - \frac{17}{69} & -\frac{10}{69} \\ -\frac{2}{69} & -\frac{22}{69} & x - \frac{17}{69} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} x^2 - \frac{5051}{569}x + \frac{5017}{569} & \frac{7788}{569}x - \frac{3308}{569} & -\frac{11300}{569}x + \frac{136}{569} \\ -\frac{500}{569}x + \frac{1068}{569} & x^2 - \frac{1411}{569}x - \frac{137}{569} & \frac{404}{569}x + \frac{4}{569} \\ -\frac{4}{569}x + \frac{136}{569} & -\frac{380}{569}x + \frac{124}{569} & x^2 - \frac{411}{569}x + \frac{41}{569} \end{pmatrix}$$

One can then check that the polynomials P_n, $n \geq 0$, can not be common eigenfunctions of a second order difference operator of the form (1.26) (only lineal equations are needed). This shows that the hypothesis $G_2 u = u G_2^*$ can not be removed in Theorem 3.

With the following example we show that the converse of Theorem 3 is, in general, not true. This example is a particular case of an $N \times N$ weight matrix introduced in [2].

Consider the discrete 2×2 weight matrix defined by

$$W = \sum_{x=0}^{k} \frac{\Gamma(k+1)}{x!\Gamma(k+1-x)} \begin{pmatrix} 1 & \frac{v_1}{k+1} x(x-k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{v_1}{k+1} x(x-k) \\ 0 & 1 \end{pmatrix}^* \delta_x$$

where k is a positive integer.

The orthogonal polynomials $(P_n)_n$ (a finite family in this case) associated to W are eigenfunctions of a second order difference operator, however this weight matrix W does not satisfy any discrete Pearson equation. Using Maple is easy to check that there do not exist any polynomials G_2 and G_1 of degrees not bigger than 2 and 1, respectively, such that $\Delta(W G_2^*) = W G_1^*$ (only lineal equations are needed). In particular, the family $(\nabla P_{n+1})_{n \geq 0}$ can not be orthogonal with respect to any moment functional. This shows that in the matrix orthogonality, the discrete Pearson equation for the weight matrix W is independent of whether the orthogonal polynomials with respect to W are eigenfunctions of a second order difference operator of the form (1.15).

1.3.2 Rodrigues' formula

As we wrote before, one of the purpose of this thesis is to develop a method to explore the existence of Rodrigues' formulas of the form (1.14) for orthogonal matrix polynomials of arbitrary size (for Rodrigues' formulas for orthogonal matrix polynomials which are eigenfunctions of second order differential operators see [26]). The key of this method is to exploit the symmetry equations (1.8) and (1.9) for the weight matrix W, and to use the following lemma as the main tool:

Lemma 4. *Let F_1, F_0 and F_{-1} be matrix polynomials satisfying that*

$$\deg \left(\sum_{l=-1}^{1} l^k F_l \right) \leq k, \quad k = 0, 1, 2. \tag{1.18}$$

Let W and $\mathfrak{R}_n$ be $N \times N$ matrix functions defined in a discrete set $\Omega = \{0, 1, 2, \ldots, \tau\}$, where τ can be a positive integer or infinity. Assume that W is nonsingular for $x \in \Omega$ and satisfies the equations

$$F_0 W = W F_0^*, \quad F_1(x-1) W(x-1) = W(x) F_{-1}^*(x). \tag{1.19}$$

Define the functions $\mathtt{P}_n$, $n \geq 1$, *by*

$$\mathtt{P}_n = \Delta^n(\mathfrak{R}_n)W^{-1}. \tag{1.20}$$

If for a matrix Λ_n, *the function* $\mathfrak{R}_n$ *satisfies*

$$\begin{aligned}\mathfrak{s}_{-1}\left(\mathfrak{R}_n F_1^*\right) + \mathfrak{s}_1 \left[\mathfrak{R}_n \left(\binom{n+1}{2} \Delta^2 F_1^* - n\Delta F_{-1}^* + F_{-1}^*\right)\right] \\ + \mathfrak{R}_n \left(-n\Delta^2 F_1^* + n\Delta F_1^* + F_0^*\right) = \Lambda_n \mathfrak{R}_n\end{aligned} \tag{1.21}$$

for $x \in \Omega$, *then the function* $\mathtt{P}_n$ *satisfies*

$$\mathfrak{s}_{-1}(\mathtt{P}_n)F_{-1} + \mathfrak{s}_0(\mathtt{P}_n)F_0 + \mathfrak{s}_1(\mathtt{P}_n)F_1 = \Lambda_n \mathtt{P}_n, \quad x \in \Omega. \tag{1.22}$$

Once we have a weight matrix W satisfying the equations (1.8) and (1.9), and an appropriate choice of eigenvalues Λ_n, $n \geq 1$, the method consists in using a solution of the equation (1.21) to produce a sequence of orthogonal matrix polynomials with respect to W by means of the Rodrigues' formula (1.20). According to the previous lemma, the function $\mathtt{P}_n$ given by (1.20) satisfies the difference equation (1.22).

It turns out that, in general, these difference equations are not enough to guarantee that $\mathtt{P}_n$, $n \geq 1$, is a polynomial of degree n with nonsingular leading coefficient, because the eigenvalues Λ_n, $n \geq 1$, have, in general, non disjoint spectrum. However, the method seems to work very efficiently as long as the weight matrix W satisfies the equations (1.8) and (1.9) for a couple of linearly independent sets of coefficients $F_{-1,1}$, $F_{0,1}$, $F_{1,1}$ and $F_{-1,2}$, $F_{0,2}$, $F_{1,2}$. In this case, one can do a suitable choice of the eigenvalues $\Lambda_{n,1}$ and $\Lambda_{n,2}$ and real constants a_1, a_2 such that the spectrum of the linear combination $a_1\Lambda_{n,1} + a_2\Lambda_{n,2}$ is disjoint for $n \geq 1$, from where one can deduce that $\mathtt{P}_n$ is a polynomial of degree n with nonsingular leading coefficient. Moreover, the functions $\mathfrak{R}_n$, $n \geq 1$, provided by this method have a so surprisingly simple expression that it suggests the existence of a certain hidden pattern.

1.3.3 Examples

Besides of the obtained results, this thesis would not be concluded without mentioning the examples that illustrate the previous theorems.

Example 1

The interest of the first example lies on that this family satisfies all the four properties of classical families mentioned in the introduction translated to the matrix case: the polynomials $(P_n)_n$ are eigenfunctions of a second order difference operator like (1.10) (in fact of at least two linearly second order difference operators), their differences are again orthogonal polynomials, they can be defined by a matrix Rodrigues' formula, and the corresponding weight matrix satisfy the matrix analogous to the discrete Pearson equation. Besides, they satisfy one more property associated to the classical families, that is, the successive differences $(\nabla^k P_{n+k})_n$ are still orthogonal polynomials for $k \geq 0$.

This example is related to the Charlier weight $\omega(x) = \frac{a^x}{\Gamma(x+1)}$.

This first example is the family of weight matrices

$$W(x) = \sum_{x=0}^{\infty} \frac{a^x}{\Gamma(x+1)}(I+A)^x(I+A^*)^x\delta_x, \tag{1.23}$$

where A is the $N \times N$ nilpotent matrix

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \tag{1.24}$$

$a > 0$, and v_i, $i = 1, \cdots, N-1$, are complex numbers satisfying that

$$(N-i-1)a|v_i|^2|v_{N-1}|^2 - i(N-i)|v_{N-1}|^2 + (N-1)|v_i|^2 = 0. \tag{1.25}$$

The matrix $\mathcal{E}_{i,j}$ stands for the matrix with entry (i,j) equal to 1 and 0 otherwise. For $i = N-1$, the condition (1.25) is an identity. For $N = 2$, this example depends on just one parameter v_1 and gives the weight matrix (1.13). For $N = 2$, the condition (1.25) always fulfils.

Let us note that since A is nilpotent of order N then $(I+A)^x(I+A^*)^x$ is a matrix polynomial of degree $2N-2$.

This family was introduced in [2] where it is proved that, without any restriction for the parameters v_i, the orthogonal polynomials with respect to W are also eigenfunctions of a second order difference operator of the form

$$D_1(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1,1}(x) + \mathfrak{s}_0(\cdot)F_{0,1}(x) + \mathfrak{s}_1(\cdot)F_{1,1}(x), \tag{1.26}$$

where

$$F_{-1,1}(x) = (I+A)^{-1}x, \quad F_{0,1}(x) = -J - (I+A)^{-1}x, \quad F_{1,1} = a(I+A),$$

and J denotes the diagonal matrix $J = \sum_{i=1}^{N}(N-i)\mathcal{E}_{i,i}$.

Denote by $\mathcal{A}$ the set formed by all nilpotent matrices of the form (1.24) whose entries v_i, $i = 1, \cdots, N-1$, satisfy the constrains (1.25).

We then construct matrix polynomials G_2 and G_1 of degrees 2 and 1, respectively, such that

$$\Delta(WG_2^*) = WG_1^*, \quad G_2W = WG_2^*.$$

What is equivalent to say that W satisfies the matrix discrete Pearson equation (1.8) and the commuting equation (1.9). As a consequence of Theorem 2, we get that the orthogonal polynomials $(P_n)_n$ with respect to W have differences $(\nabla P_{n+1})_{n\geq 0}$ which are also orthogonal.

The orthogonality weight matrix for $(\nabla P_{n+1})_{n\geq 0}$ is $\widetilde{W} = G_2W$ which it turns to be a weight matrix again (supported in the set of positive integer $\{1, 2, \cdots\}$). Moreover, we prove that $\widetilde{W}$ is essentially of the form (1.23): indeed, there are a nonsingular matrix M and a nilpotent matrix $\widetilde{A} \in \mathcal{A}$ (both independet of x) such that

$$\widetilde{W}(x) = MW_{\widetilde{A}}(x-1)M^*, x \geq 1.$$

In particular, this implies that for all $k \geq 1$, the differences of order k, $(\nabla^k P_{n+k})_{n\geq 0}$ are again orthogonal polynomials.

Theorem 3 also implies that the orthogonal polynomials $(P_n)_n$ are eigenfunctions of the second order difference operator

$$\begin{aligned} D_2(\cdot) &= \Delta\nabla(\cdot)G_2 + \Delta(\cdot)G_1 \\ &= \mathfrak{s}_{-1}(\cdot)F_{-1,2}(x) + \mathfrak{s}_0(\cdot)F_{0,2}(x) + \mathfrak{s}_1(\cdot)F_{1,2}(x) \end{aligned}$$

where

$$\begin{aligned} F_{-1,2}(x) &= \left[(I+A)^{-1} - I\right]x^2 + \left(\frac{N-1}{a|v_{N-1}|^2}I + J\right)x, \\ F_{1,2}(x) &= \left[(I+A)^{-1} - I\right]x^2 + \big((I+A)^{-1} - aA + 2J - NI\big)x \\ &\quad + a(I+A)\left(\frac{N-1}{a|v_{N-1}|^2}I + J\right)(I+A^*), \\ F_{0,2}(x) &= -F_{-1,2}(x) - F_{1,2}(x). \end{aligned}$$

This difference operator is however linearly independent to (1.26).

The Rodrigues' formula provided by our method for this example is the following:

Theorem 5. *Assume that the moduli of the entries $|v_i|$, $i = 1, \ldots, N-1$, of the matrix A (1.24) satisfy (1.25). Then, a sequence of orthogonal polynomials with respect to the weight matrix W (1.23) can be defined by using* *the Rodrigues' formula*

(1.27)
$$P_n(x) = \Delta^n \left(\frac{a^x}{\Gamma(x-n+1)} (I+A)^x \mathfrak{L}_{n,1} (I+A^*)^x \right) W_1^{-1}(x), \quad n \geq 1,$$

where $\mathfrak{L}_{n,1}$ *is the diagonal matrix independent of* x *and with entries*

(1.28)
$$\mathfrak{L}_{n,1} = \sum_{i=1}^{N} \prod_{k=i}^{N-1} \left(1 + \frac{an|v_k|^2}{k(N-k)} \right) \mathcal{E}_{ii}$$

(for $i > j$ *we take* $\prod_{l=i}^{j} = 1$*).*

The assumption (1.25) on the parameters seems to be necessary. We have symbolic computational evidence which shows that if (1.25) does not hold then, for any choice of the matrix $\mathfrak{L}_{n,1}$ (diagonal or not), the polynomial P_n in (1.27) has degree bigger than n or singular leading coefficient.

Example 2

The second example is as interesting as the first one. On the one hand, it also satisfies the four classical properties mentioned above, and on the other hand, it is, as far as we know, the first time this example appears in the literature.

This example is related to the Meixner weight $\omega(x) = \frac{a^x \Gamma(x+c)}{\Gamma(x+1)}$.

This example is the family of weight matrices

(1.29)
$$W = \sum_{x \geq 0} \frac{a^x}{\Gamma(x+1)} (I+A)^x \Gamma\left((x+c)I + J\right) (I+A^*)^x \delta_x,$$

where A is defined by (1.24), $0 < a < 1$, $c > 0$, J is the diagonal matrix

(1.30)
$$J = \sum_{i=1}^{N} (N-i) \mathcal{E}_{i,i}$$

and the complex numbers v_i, $i = 1, \cdots, N-1$, satisfy that

(1.31)
$$\frac{(N-i-1)a|v_i|^2|v_{N-1}|^2}{1-a} - i(N-i)|v_{N-1}|^2 + (N-1)|v_i|^2 = 0.$$

The properties of the matrices A (nilpotent of order N) and J (diagonal) imply that $(I+A)^x \Gamma\left((x+c)I + J\right)(I+A^*)^x / \Gamma(x+c)$ is a matrix polynomial of degree $2N-2$.

This weight matrix satisfies the symmetry equations (1.8) and (1.9) for a couple of sets of linearly independent coefficients $F_{-1,j}$, $F_{0,j}$, $F_{1,j}$, $j = 1, 2$.

$$D_i(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1,i}(x) + \mathfrak{s}_0(\cdot)F_{0,i}(x) + \mathfrak{s}_1(\cdot)F_{1,i}(x), \quad i = 1, 2$$

where

$$F_{-1,1} = (I+A)^{-1}x,\ F_{1,1} = aIx+a(I+A)(cI+J),\ F_{0,1} = -J-\left(aI + (I + A)^{-1}\right)x,$$

and

$$\begin{aligned}
F_{-1,2} =&[I - (I + A)^{-1}]x^2 + \left(\frac{(N-1)(a-1)}{a|v_{N-1}|^2}I - J\right)x,\\
F_{1,2} =&[I - (I + A)^{-1}]x^2 + a(I + A)\left(\frac{(N-1)(a-1)}{a|v_{N-1}|^2}I - J\right)(cI + J + A^*)\\
&+\left[\left((a-1)\left(\frac{N-1}{|v_{N-1}|^2} - N\right) + a\right)I + (a-2)J + aA(cI + J) - (I + A)^{-1}\right]x,\\
F_{0,2} =& - F_{-1,2} - F_{1,2}.
\end{aligned}$$

It is proved that for the set of coefficients $F_{i,1}$, $i = -1, 0, 1$, the symmetry equations (1.8) and (1.9) are fulfilled. As a consequence of Theorem 2, it is obtained that the differences $(\nabla P_{n+1})_n$ are again orthogonal polynomials.

The Rodrigues' formula provided by our method for this example is the following:

Theorem 6. *Assume that the moduli of the entries $|v_i|$, $i = 1, \dots, N-1$, of the matrix A (1.24) satisfy (1.31). Then, a sequence of orthogonal polynomials with respect to the weight matrix W (1.29) can be defined by using the Rodrigues' formula ($n \geq 1$)*

$$P_n(x) = \Delta^n\left(\frac{a^x}{\Gamma(x-n+1)}(I + A)^x\Gamma\left((x + c)I + J\right)\mathfrak{L}_{n,2}(I + A^*)^x\right)W^{-1}(x),$$

where $\mathfrak{L}_{n,2}$ is the diagonal matrix independent of x and with entries

$$\mathfrak{L}_{n,2} = \sum_{i=1}^{N}\prod_{k=i}^{N-1}\left(1 + \frac{an|v_k|^2}{k(N-k)(1-a)}\right)\mathcal{E}_{ii}. \tag{1.32}$$

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Orthogonal matrix polynomials whose differences are also orthogonal☆

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Abstract

We characterize orthogonal matrix polynomials $(P_n)_n$ whose differences $(\nabla P_{n+1})_n$ are also orthogonal by means of a discrete Pearson equation for the weight matrix W with respect to which the polynomials $(P_n)_n$ are orthogonal. We also construct some illustrative examples. In particular, we show that contrary to what happens in the scalar case, in the matrix orthogonality the discrete Pearson equation for the weight matrix W is, in general, independent of whether the orthogonal polynomials with respect to W are eigenfunctions of a second order difference operator with polynomial coefficients.
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1. Introduction

Classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouk and Hahn) are characterized by a number of equivalent properties. The following ones are three of them:

1. They are eigenfunctions of a second order difference operator of the form

$$\sigma\Delta\nabla + \tau\Delta,$$

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where σ and τ are polynomials of degrees not bigger than 2 and 1, respectively, and Δ and ∇ denote the usual first order difference operators $\Delta(p) = p(x+1) - p(x)$ and $\nabla(p) = p(x) - p(x-1)$.
2. The sequence of its differences is again orthogonal with respect to a measure.
3. Each classical discrete weight measure w satisfies a discrete Pearson difference equation $\Delta(g_2 w) = g_1 w$, where g_1 and g_2 are polynomials of degrees at most 1 and 2, respectively.

The purpose of this paper is to show that for orthogonal matrix polynomials these three properties are no longer equivalent. More precisely, we prove that the equivalence between (2) and (3) above still remains true for orthogonal matrix polynomials. Under suitable Hermitian assumptions on wg_2 and wg_1, they also imply the property (1), but the converse is not true.

A matrix moment functional is a (left) linear functional $u : \mathbb{P}^N \to \mathbb{C}^{N\times N}$, where $\mathbb{P}^N$ denotes the linear space of $N \times N$ matrix polynomials. In this paper, we consider orthogonality with respect to a matrix moment functional, that is, with respect to the bilinear form $\langle \cdot, \cdot \rangle_u : \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{C}^{N\times N}$ defined by $\langle P, Q\rangle_u = \sum_{i,j} A_i u(x^{i+j} I) B_j^*$, where $P = \sum_n A_n x^n$ and $Q = \sum_n B_n x^n$. We then say that a sequence of matrix polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is orthogonal with respect to u if $\langle P_n, P_k\rangle_u = \Lambda_n \delta_{k,n}$, where Λ_n is a nonsingular matrix for $n \geq 0$. We say that the moment functional u is quasi-definite if it has a sequence of orthogonal polynomials. The most interesting case appears when the moment functional is defined by a weight matrix W. A weight matrix W is a $N \times N$ matrix of measures supported in the real line such that $W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$, having finite moments and satisfying that $\int P(x) dW(x) P^*(x)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular. Each weight matrix W defines a matrix moment functional, which we denote also by W: $W(P) = \int P dW$. The corresponding bilinear form is then defined by $\langle P, Q\rangle_W = \int P dW Q^*$. The theory of matrix valued orthogonal polynomials starts with two papers by M. G. Kreĭn in 1949, see [9,10].

The paper is organized as follows. Section 2 will be devoted to basic definitions and facts. In particular, we define the operational calculus with moment functionals. As usual, the basic operations such as multiplication by polynomials or the first order difference operators will be defined by duality. In particular, for a polynomial $P = \sum_n A_n x^n$ and a moment functional u, we define the moment functionals Pu and uP as

$$Pu(Q) = u(QP), \tag{1.1}$$

$$uP(Q) = \sum_n u(Qx^n) A_n. \tag{1.2}$$

Notice that for a polynomial $p \in \mathbb{P}$, we have $pIu = upI$. With these definitions, one straightforwardly has $\langle P, Q\rangle_u = \langle P, uQ^*\rangle$. These definitions agree with the usual product of a polynomial P times a weight matrix W. For a moment functional u, the new moment functionals Δu and ∇u are defined by

$$\Delta u(P) = -u(\nabla P), \qquad \nabla u(P) = -u(\Delta P), \tag{1.3}$$

respectively.

In Section 3, we prove the equivalence between the properties (2) and (3) above for matrix moment functionals. More precisely, we prove the following theorem.

Theorem 1. *Let u be a quasi-definite moment functional. Consider a sequence of orthogonal matrix polynomials $(P_n)_n$ with respect to u. Then, the following conditions are equivalent:*

(1) *The functional u satisfies the discrete Pearson equation $\Delta(uG_2^*) = uG_1^*$, where $G_2 = G_{2,2}x^2 + G_{2,1}x + G_{2,0}$ and $G_1 = G_{1,1}x + G_{1,0}$ are matrix polynomials of degrees not bigger than 2 and 1, respectively, satisfying that $(n-1)G_{22} + G_{11}$ is not singular for $n \geq 1$.*
(2) *The sequence $(\nabla P_{n+1})_n$ is a family of orthogonal matrix polynomials with respect to a quasi-definite moment functional $\widetilde{u}$.*

Furthermore, $\widetilde{u} = uG_2^*$.

In Section 4, we prove that, under additional assumptions, the discrete Pearson equation $\Delta(uG_2^*) = uG_1^*$ implies that the orthogonal polynomials $(P_n)_n$ with respect to u are eigenfunctions of a second order difference operator.

Theorem 2. *Let u be a quasi-definite moment functional. Consider a sequence of orthogonal matrix polynomials $(P_n)_n$ with respect to u. Assume that the moment functional u satisfies the discrete Pearson equation $\Delta(uG_2^*) = uG_1^*$, and the Hermitian conditions $G_2u = uG_2^*$ and $G_1u = uG_1^*$, where G_2 and G_1 are polynomials of degrees at most 2 and 1, respectively. Consider the second order difference operator*

$$D = \Delta\nabla(\cdot)G_2 + \Delta(\cdot)G_1.$$

Then $D(P_n) = \Lambda_n P_n$, for certain matrices Λ_n.

For the differential version of these theorems see [2].

In Section 5 we display some illustrative examples. For a positive real number $a > 0$, and non null complex numbers v_i, $i = 1, \ldots, N-1$, we first consider the family of $N \times N$ weight matrices

$$W = \sum_{x=0}^{\infty} \frac{a^x}{x!}(I + A)^x (I + A^*)^x \delta_x, \tag{1.4}$$

where A is the nilpotent matrix (of order N)

$$A = \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.5}$$

These weight matrices were introduced in [1] where it is proved that the orthogonal polynomials with respect to W are also eigenfunctions of a second order difference operator of the form

$$\Delta\nabla(\cdot)F_2 + \Delta(\cdot)F_1 + \cdot F_0. \tag{1.6}$$

In this paper, we assume that the parameters v_i, $i = 1, \ldots, N-1$, satisfy the constraints

$$(N-i-1)a|v_i|^2|v_{N-1}|^2 + (N-1)|v_i|^2 - i(N-i)|v_{N-1}|^2 = 0. \tag{1.7}$$

Denote by $\mathcal{A}$ the set formed by all nilpotent matrices of the form (1.5) whose entries v_i, $i = 1, \ldots, N-1$, satisfy the constraints (1.7).

We then construct matrix polynomials G_2 and G_1 of degrees 2 and 1, respectively, such that $\Delta(WG_2^*) = WG_1^*$, and $G_2W = WG_2^*$. As a consequence of Theorem 1, we get that the orthogonal polynomials $(P_n)_n$ with respect to W have differences $(\nabla P_{n+1})_{n\geq 0}$ which are also orthogonal. The orthogonality weight matrix for $(\nabla P_{n+1})_{n\geq 0}$ is $\widetilde{W} = G_2 W$ which it turns to be a weight matrix again (supported in the set of positive integer $\{1, 2, \ldots\}$). Moreover, we prove that $\widetilde{W}$ is essentially of the form (1.4): indeed, there are a nonsingular matrix M and a nilpotent matrix $\widetilde{A} \in \mathcal{A}$ such that $\widetilde{W}(x) = M W_{\widetilde{A}}(x-1) M^*$ for all $x \geq 1$. In particular, this implies that for all $k \geq 1$, the differences of order k, $(\nabla^k P_{n+k})_{n\geq 0}$ are again orthogonal polynomials.

Theorem 2 also implies that the orthogonal polynomials $(P_n)_n$ are eigenfunctions of the second order difference operator

$$D = \Delta\nabla(\cdot)G_2 + \Delta(\cdot)G_1.$$

This difference operator is however linearly independent to the one found in [1]. This shows once again one of the striking developments in this area: the discovery of many new phenomena that are absent in the well known scalar theory. One of these phenomena is that the elements of a family of orthogonal matrix polynomials can be eigenfunctions of several linearly independent second order difference operators while the elements of each of the classical discrete families in the scalar case are eigenfunctions of only one linearly independent second order difference operator, respectively (see [5] for more details; for the differential version of this result see [4,6] and the references therein).

We study in more detail the case 3×3, where we consider the weight matrix W defined by (1.4) where A is now an arbitrary 3×3 upper triangular nilpotent matrix

$$A = \begin{pmatrix} 0 & v_1 & v_3 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We show that there are always polynomials G_2 and G_1 of degrees not bigger than 2 and 1, respectively, such that $\Delta(W G_2^*) = W G_1^*$. However, only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy (1.7), one of these polynomials $G_2 \neq 0$ satisfies also the Hermitian condition $G_2 W = W G_2^*$. This means that the polynomials $(\nabla P_{n+1})_{n\geq 0}$ are always orthogonal with respect to a moment functional but this moment functional can be represented by a weight matrix (in the sense defined above) only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy (1.7). We also show that, in general, the orthogonal polynomials $(P_n)_n$ are not eigenfunctions of a second order difference operator (unless $v_3 = 0$). This shows that the hypothesis $G_2 W = W G_2^*$ cannot be removed in Theorem 2.

We finally display a weight matrix W whose orthogonal polynomials $(P_n)_n$ are eigenfunctions of a second order difference operator of the form (1.6). However this weight matrix W does not satisfy any discrete Pearson equation. This shows that in the matrix orthogonality, the discrete Pearson equation for the weight matrix W is independent of whether the orthogonal polynomials with respect to W are eigenfunctions of a second order difference operator of the form (1.6).

2. Preliminaries

We will denote by $\mathbb{C}^{N\times N}$ the set of matrices of size $N \times N$ whose elements are complex numbers, and by $\mathbb{P}^N$ the linear space of matrix polynomials in one real variable with coefficients in $\mathbb{C}^{N\times N}$. The linear subspace of $\mathbb{P}^N$ formed by those matrix polynomials with degree at most n will be denoted by $\mathbb{P}_n^N$. $\mathbb{P}$ will stand for the linear space of polynomials with complex coefficients.

A moment functional u is a left linear functional $u : \mathbb{P}^N \to \mathbb{C}^{N\times N}$: $u(AP + BQ) = Au(P) + Bu(Q)$, with $A, B \in \mathbb{C}^{N\times N}$ and $P, Q \in \mathbb{P}^N$. For a moment functional u and a matrix polynomial $P \in \mathbb{P}^N$, the duality bracket is defined by $\langle P, u\rangle = u(P)$. We will use the duality bracket $\langle P, u\rangle$ instead of the functional notation $u(P)$. Notice that we write the moment functional in the right hand side of the bracket contrarily to the standard notation. This is because of the following reason. As we will explain below a moment functional can always be represented by integrating with respect to a matrix of measures W: $\langle P, u\rangle = \int P(t)dW(t)$. We write the moment functional in the right hand side of the bracket to stress that in this representation the matrix of measures is multiplying on the right.

We associate to each moment functional u a bilinear form $\langle\cdot,\cdot\rangle_u : \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{C}^{N\times N}$ in the linear space of matrix polynomials defined by

$$\langle P, Q\rangle_u = \sum_{i,j} A_i \langle x^{i+j} I, u\rangle B_j^*, \tag{2.1}$$

where $P = \sum_n A_n x^n$ and $Q = \sum_n B_n x^n$. We then say that a sequence of matrix polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is orthogonal with respect to the moment functional u if $\langle P_n, P_k\rangle_u = \Lambda_n \delta_{k,n}$, where Λ_n is a nonsingular matrix for $n \geq 0$. An easy consequence of the orthogonality is that orthogonal polynomials $(P_n)_n$ with respect to a moment functional satisfy a three term recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0, \tag{2.2}$$

where $P_{-1} = 0$, and $\alpha_n, \gamma_n, n \geq 1$, are non-singular matrices.

We say that a moment functional u is quasi-definite if $\det(\Delta_n) \neq 0$ for every $n \geq 0$, where Δ_n is the Hankel-block matrix

$$\Delta_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

and μ_k is the moment of order k with respect to u: $\mu_k = \langle x^k I, u\rangle$. Notice that the moments μ_k, $k \geq 0$, of u define uniquely both the moment functional u and the bilinear form $\langle\cdot,\cdot\rangle_u$ (2.1). It turns out that a moment functional u has a sequence of orthogonal polynomials if and only if u is quasi-definite (see [2]).

Given a moment functional u with moments μ_k, $k \geq 0$, we define its (complex) adjoint u^* as the moment functional with moments μ_k^*, $k \geq 0$. If the moments μ_k, $k \geq 0$, of a moment functional u are Hermitian matrices, i.e. $u = u^*$, we say that u is Hermitian. If, in addition, $\det(\Delta_n) > 0$ for every $n \geq 0$, we say that u is positive definite.

Moment functionals can always be represented by integration with respect to a matrix of measures: using [7], one can see that given a moment functional u there exists a matrix of measures W supported in the real line (whose entries are Borel measures in $\mathbb{R}$) such that $\langle P, u\rangle = \int_{\mathbb{R}} PdW$. The most interesting case is when the matrix moment functional is defined by a weight matrix W. A weight matrix W is an $N \times N$ matrix of measures supported in the real line such that $W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$, having finite moments and satisfying that $\int P(x)dW(x)P^*(x)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular. It follows from [8] that a moment functional can be represented by a weight matrix if and only if it is positive definite.

Along this paper, we will use without an explicit mention the usual properties listed below of the difference operators Δ and ∇

1. $\nabla\Delta = \Delta\nabla = \Delta - \nabla$.
2. $\Delta f(x) = \nabla f(x+1)$.
3. $\Delta[f(x)g(x)] = f(x)\Delta g(x) + [\Delta f(x)]\, g(x+1)$,
4. $\nabla[f(x)g(x)] = f(x-1)\nabla g(x) + [\nabla f(x)]\, g(x)$.
5. $\sum_a^{b-1} f(x)\Delta g(x) = f(x)g(x)\Big|_a^b - \sum_a^{b-1} [\Delta f(x)]\, g(x+1)$,
6. $\sum_{a+1}^{b} f(x)\nabla g(x) = f(x)g(x)\Big|_a^b - \sum_{a+1}^{b} [\nabla f(x)]\, g(x-1)$.

The difference operators Δ and ∇ acting on moment functional are defined by

$$\langle P, \Delta u\rangle = -\langle \nabla(P), u\rangle, \qquad \langle P, \nabla u\rangle = -\langle \Delta(P), u\rangle. \tag{2.3}$$

A simple computation gives that

$$\begin{array}{ll} \Delta(Pu) = P(x+1)\Delta u + (\Delta P)u, & \nabla(Pu) = P(x-1)\nabla u + (\nabla P)u, \\ \Delta(uP) = (\Delta u)P(x+1) + u\Delta P, & \nabla(uP) = (\nabla u)P(x-1) + u\nabla P. \end{array}$$

Given a discrete weight matrix

$$W = \sum_{x=a}^{b} W(x)\delta_x,$$

supported in $\{a, a+1, \ldots, b-1, b\}$ (a can be $-\infty$ and b can be $+\infty$), the matrices of measures ΔW and ∇W are defined in the usual way by

$$\Delta W = \sum_{x=a-1}^{b} (W(x+1) - W(x))\delta_x, \tag{2.4}$$

$$\nabla W = \sum_{x=a}^{b+1} (W(x) - W(x-1))\delta_x, \tag{2.5}$$

respectively, where by definition $W(b+1) = W(a-1) = 0$ (if $a = -\infty$ or $b = \infty$, we take $a - 1 = -\infty$ or $b + 1 = +\infty$, respectively). It is worth noting that the support of ΔW is $\{a-1, a, a+1, \ldots, b-1, b\}$ and that it is different to the support of W, except when $a = -\infty$. In the same way, the support of ∇W is $\{a, a+1, \ldots, b-1, b, b+1\}$ and it is different to the support of W, except when $b = +\infty$. A simple computation shows that if we consider the weight matrix W as a moment functional these definitions of ΔW and ∇W agree with (2.3).

We finish this section introducing the concept of quasi-orthogonal polynomials with respect to a moment functional u. A family of polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is quasi-orthogonal of order r with respect to a moment functional u if $\langle P_n, x^k I\rangle_u = 0$ for $k = 0, \ldots, n-r-1$.

Theorem 3.4 of [3] shows that if $(P_n)_n$ are orthogonal with respect to the moment functional u and quasi-orthogonal of order r with respect to the moment functional $\widetilde{u}$ then there exists a matrix polynomial Q of degree r such that $\widetilde{u} = uQ$.

3. Orthogonal polynomials whose differences are also orthogonal

In this section we prove Theorem 1 in the Introduction.

Proof. (1) ⇒ (2). A simple computation using the definitions and basic properties introduced in the previous section gives

$$\begin{aligned}\langle \nabla P_n, x^k I\rangle_{uG_2^*} &= \langle \nabla P_n, uG_2^* x^k\rangle = -\langle P_n, \Delta(uG_2^* x^k)\rangle \\ &= -\langle P_n, \Delta(uG_2^*)(x+1)^k + uG_2^*\Delta(x^k)\rangle \\ &= -\langle P_n, (x+1)^k G_1\rangle_u - \langle P_n, \Delta(x^k)G_2\rangle_u.\end{aligned}$$

If $0 \le k \le n-2$, since $(x+1)^k G_1$ and $\Delta(x^k)G_2$ are polynomials of degree at most $n-1$, we get

$$\langle \nabla P_n, x^k I\rangle_{uG_2^*} = 0. \tag{3.1}$$

For $k = n-1$, we have

$$\begin{aligned}\langle \nabla P_n, x^{n-1} I\rangle_{uG_2^*} &= -\langle P_n, (x+1)^{n-1} G_1\rangle_u - \langle P_n, \Delta(x^{n-1})G_2\rangle_u \\ &= -\langle P_n, x^n I\rangle_u G_{11}^* - (n-1)\langle P_n, x^n I\rangle_u G_{22}^* \\ &= -\langle P_n, x^n I\rangle_u [G_{11} + (n-1)G_{22}]^*. \end{aligned}\tag{3.2}$$

Since both matrices $\langle P_n, x^n I\rangle_u$ and $G_{11}^* + (n-1)G_{22}^*$ are nonsingular matrices, we get that also $\langle \nabla P_n, x^{n-1} I\rangle_{uG_2^*}$ is a nonsingular matrix.

This proves the orthogonality of the family $(\nabla P_{n+1})_n$ with respect to uG_2^*.

(2) ⇒ (1). We start writing P_n as a linear combination of ∇P_{n-1}, ∇P_n and ∇P_{n+1}.

Since $(P_n)_n$ is a family of orthogonal polynomials, they satisfy a three term recurrence relation

$$xP_n = \alpha_n P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}, \quad n \ge 0,$$

where we define $P_{-1} = 0$.

Applying now the operator ∇, we get

$$(\nabla x)P_n + (x-1)\nabla P_n = \alpha_n \nabla P_{n+1} + \beta_n \nabla P_n + \gamma_n \nabla P_{n-1},$$

and thus

$$P_n = \alpha_n \nabla P_{n+1} - x\nabla P_n + (\beta_n + 1)\nabla P_n + \gamma_n \nabla P_{n-1}. \tag{3.3}$$

On the other hand, since $(\nabla P_{n+1})_n$ is also a family of orthogonal polynomials, they satisfy a three term recurrence relation

$$x\nabla P_n = \widetilde{\alpha}_n \nabla P_{n+1} + \widetilde{\beta}_n \nabla P_n + \widetilde{\gamma}_n \nabla P_{n-1}, \quad n \ge 1.$$

Replacing in (3.3) we obtain

$$\begin{aligned}P_n &= \alpha_n \nabla P_{n+1} - \widetilde{\alpha}_n \nabla P_{n+1} - \widetilde{\beta}_n \nabla P_n - \widetilde{\gamma}_n \nabla P_{n-1} + (\beta_n + 1)\nabla P_n + \gamma_n \nabla P_{n-1} \\ &= a_n \nabla P_{n+1} + b_n \nabla P_n + c_n \nabla P_{n-1}, \end{aligned}\tag{3.4}$$

where $a_n = \alpha_n - \widetilde{\alpha}_n$, $b_n = \beta_n + 1 - \widetilde{\beta}_n$ and $c_n = \gamma_n - \widetilde{\gamma}_n$.

We now prove that $(P_n)_n$ is quasi-orthogonal of order 2 with respect to $\widetilde{u}$, i.e., $\langle P_n, x^k I\rangle_{\widetilde{u}} = 0$ for $k = 0, \ldots, n-3$.

Since $(\nabla P_{n+1})_n$ is orthogonal with respect to $\widetilde{u}$, it follows that $\langle \nabla P_n, x^k I\rangle_{\widetilde{u}} = 0$ for $k \le n-2$, thus

$$\langle P_n, x^k I\rangle_{\widetilde{u}} = \left\langle a_n \nabla P_{n+1} + b_n \nabla P_n + c_n \nabla P_{n-1}, x^k I\right\rangle_{\widetilde{u}} = 0 \tag{3.5}$$

for k from 0 to $n-3$. Theorem 3.4 of [3] implies that there exists a matrix polynomial G_2^* of degree 2 such that $\widetilde{u} = uG_2^*$.

We next show that $(P_n)_n$ are quasi orthogonal of order 1 with respect to $\Delta(uG_2^*)$. Indeed

$$\begin{aligned}\langle P_n, x^k I\rangle_{\Delta(uG_2^*)} &= \langle x^k P_n, I\rangle_{\Delta(uG_2^*)} = \langle x^k P_n, \Delta(uG_2^*)\rangle = -\langle \nabla(x^k P_n), uG_2^*\rangle \\ &= -\langle (x-1)^k \nabla P_n, uG_2^*\rangle - \langle \nabla(x^k) P_n, uG_2^*\rangle \\ &= -\langle \nabla P_n, uG_2^*(x-1)^k\rangle - \langle P_n, uG_2^* \nabla(x^k)\rangle \\ &= -\langle \nabla P_n, (x-1)^k I\rangle_{uG_2^*} - \langle P_n, \nabla(x^k) I\rangle_{uG_2^*}.\end{aligned}$$

The orthogonality of ∇P_{n+1} with respect to uG_2^* and (3.5) imply that $\langle P_n, x^k I\rangle_{\Delta(uG_2^*)} = 0$ for $0 \le k \le n-2$. This proves that $(P_n)_n$ is a quasi-orthogonal family of order 1 with respect to $\Delta(uG_2^*)$, and according to Theorem 3.4 of [3] there exists a matrix polynomial G_1^* of degree 1 such that $\Delta(uG_2^*) = uG_1^*$.

We finally prove that the matrices $G_{11} + (n-1)G_{22}$ are nonsingular for $n \ge 1$. Indeed, proceeding as in the proof of (1) $\Rightarrow$ (2), we get (see (3.2))

$$\langle \nabla P_n, x^{n-1} I\rangle_{uG_2^*} = -\langle P_n, x^n I\rangle_u [G_{11} + (n-1)G_{22}]^*.$$

Since the matrix $\langle \nabla P_n, x^{n-1} I\rangle_{uG_2^*}$ is nonsingular for $n \ge 1$, we conclude that also $G_{11} + (n-1)G_{22}$ is nonsingular. $\square$

It is easy to see that, under the additional assumption $uG_2^* = G_2 u$, the condition

$$\langle \nabla P_n, x^k I\rangle_{uG_2^*} = 0, \quad k \le n-2$$

(see (3.1)) is equivalent to

$$\nabla P_n G_2 = f_n P_{n+1} + g_n P_n + h_n P_{n-1},$$

for certain matrices f_n, g_n and h_n. In the scalar orthogonality, this last identity is usually called the first structure relation. The identity

$$P_n = a_n \nabla P_{n+1} + b_n \nabla P_n + c_n \nabla P_{n-1}$$

(see (3.4)) is called the second structure relation. Hence, in the previous proof we have also shown the equivalence between the conditions of Theorem 1 and the first and second structure relations (under the additional assumption $uG_2^* = G_2 u$).

4. Discrete Pearson equations and second order difference operators

In this section we prove Theorem 2 in the Introduction.

To do that, we introduce the concept of symmetry. For a linear operator $D : \mathbb{P}^N \to \mathbb{P}^N$, we say that D is symmetric with respect to the moment functional u if $\langle D(P), Q\rangle_u = \langle P, D(Q)\rangle_u$, for all polynomials $P, Q \in \mathbb{P}^N$.

Lemma 4.1. *Let D be a symmetric operator with respect to the quasi-definite moment functional u. Assume that in addition we have* $\deg(D(P)) \le \deg(P)$ *for every polynomial $P \in \mathbb{P}^N$. Then, the orthogonal polynomials with respect to u are also eigenfunctions of D.*

Proof. Indeed, one can write $D(P_n) = \sum_{k=0}^{n} A_k P_k$. Then for $k = 0, \ldots, n-1$, we have

$$A_k \langle P_k, P_k \rangle_u = \langle D(P_n), P_k \rangle_u = \langle P_n, D(P_k) \rangle_u = 0,$$

(since $D(P_k)$ is a polynomial of degree less than n) and thus $A_k = 0$. Hence $D(P_n) = A_n P_n$. □

We are now ready to prove Theorem 2.

Proof. Using the previous lemma, it is enough to prove that $\deg(D(P)) \le \deg(P)$ for all polynomials $P \in \mathbb{P}$, and that the operator D is symmetric with respect to the moment functional u. The first condition follows easily taking into account that $\deg(G_2) \le 2$ and $\deg(G_1) \le 1$. The symmetry of D with respect to u is equivalent to prove that $\langle D(x^i I), x^j I \rangle_u = \langle x^i I, D(x^j I) \rangle_u$, $i, j \ge 0$.

Taking into account that $G_2 u = u G_2^*$ and $G_1 u = u G_1^*$, the Pearson equation can be rewritten as $\Delta(G_2 u) = G_1 u$. It is easy to see that this equation is equivalent to the equation $\nabla(G_1 u + G_2 u) = G_1 u$. Using it, the proof now follows from the following computations (where we use the definitions and basic properties of first order difference operators which can be found in Section 2).

$$\begin{aligned}
\langle D(x^i I), x^j I \rangle_u &= \langle D(x^i I), u x^j I \rangle = \langle (\Delta \nabla x^i) G_2 + (\Delta x^i) G_1, u x^j \rangle \\
&= \langle \Delta \nabla x^i I, G_2 u x^j \rangle + \langle \Delta x^i I, G_1 u x^j \rangle \\
&= \langle x^i I, \nabla \Delta (G_2 u x^j) \rangle - \langle x^i I, \nabla (G_1 u x^j) \rangle \\
&= \langle x^i I, \nabla (\Delta (G_2 u)(x+1)^j + G_2 u \Delta x^j - G_1 u x^j) \rangle \\
&= \langle x^i I, \nabla ((G_2 u + G_1 u) \Delta x^j) \rangle \\
&= \langle x^i I, (\nabla (G_2 u + G_1 u)) \Delta (x-1)^j + (G_2 u + G_1 u) \nabla \Delta x^j \rangle \\
&= \langle x^i I, G_1 u \Delta (x-1)^j + G_1 u (\Delta x^j - \Delta (x-1)^j) + G_2 u \nabla \Delta x^j \rangle \\
&= \langle x^i I, G_2 u \nabla \Delta x^j + G_1 u \Delta x^j \rangle \\
&= \langle x^i I, u G_2^* \nabla \Delta x^j + u G_1^* \Delta x^j \rangle \\
&= \langle x^i I, (\nabla \Delta x^j) G_2 + (\Delta x^j) G_1 \rangle_u = \langle x^i I, D(x^j I) \rangle_u. \quad \square
\end{aligned}$$

5. Examples

In this section we consider three illustrative examples. Examples 1 and 2 illustrate Theorems 1 and 2 (in particular, that the assumptions $G_2 W = W G_2^*$ and $G_1 W = W G_1^*$ are necessary in Theorem 2). Example 3 shows that in the matrix orthogonality, the discrete Pearson equation for the weight matrix W does not imply that the orthogonal polynomials with respect to W are eigenfunctions of a second order difference operator of the form (1.6).

Example 1. For a positive real number $a > 0$, and non null complex numbers v_i, $i = 1, \ldots, N-1$, we consider the family of $N \times N$ weight matrices

$$W = \sum_{x=0}^{\infty} \frac{a^x}{x!} (I + A)^x (I + A^*)^x \delta_x, \tag{5.1}$$

where A is the nilpotent matrix defined by (1.5).

Consider also the diagonal matrix

$$J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the weight matrix $\widetilde{W}$ defined by

$$\widetilde{W} = \sum_{x=1}^{\infty} \frac{a^x}{(x-1)!}(I+A)^x \left(\frac{N-1}{a|v_{N-1}|^2} I + J \right) (I+A^*)^x \delta_x. \tag{5.2}$$

The following theorem provides an example with arbitrary size $N \times N$ of a weight matrix whose orthogonal polynomials have differences which are also orthogonal with respect to a weight matrix.

Theorem 3. *Let W and $\widetilde{W}$ be the weight matrices defined by (5.1) and (5.2), where we assume that the parameters v_i, $i = 1, \ldots, N-2$, satisfy the constraints*

$$(N-i-1)a|v_i|^2|v_{N-1}|^2 + (N-1)|v_i|^2 - i(N-i)|v_{N-1}|^2 = 0. \tag{5.3}$$

Let G_2 and G_1 be the matrix polynomials of degrees 2 and 1, respectively, given by

$$G_2(x) = \left[(I+A)^{-1} - I \right] x^2 + \left(\frac{N-1}{a|v_{N-1}|^2} I + J \right) x, \tag{5.4}$$

$$\begin{aligned} G_1(x) \ &= \ \left[J - \left(N + \frac{N-1}{a|v_{N-1}|^2} \right) I - aA + (I+A)^{-1} \right] x \\ &\quad + a(I+A) \left(\frac{N-1}{a|v_{N-1}|^2} I + J \right) (I+A^*). \end{aligned} \tag{5.5}$$

Then, we have

1. $\widetilde{W} = G_2 W$, *and hence* $G_2 W = W G_2^*$.
2. $\Delta(G_2 W) = G_1 W$, *and so* $G_1 W = \widetilde{W} G_1^*$.

Moreover, if we write $(P_n)_n$ for a sequence of orthogonal polynomials with respect to W, then they are eigenfunctions of the second order difference operator

$$\Delta\nabla(\cdot)G_2 + \Delta(\cdot)G_1, \tag{5.6}$$

and the sequence $(\nabla P_{n+1})_{n \geq 0}$ is orthogonal with respect to $\widetilde{W}$.

Proof. To simplify the writing, we set

$$\widetilde{J} = \frac{N-1}{a|v_{N-1}|^2} I + J, \tag{5.7}$$

so that

$$\widetilde{W} = \sum_{x=1}^{\infty} \frac{a^x}{(x-1)!}(I+A)^x \widetilde{J} (I+A^*)^x \delta_x.$$

We assume the following claim which will be proved later.

Claim. For a nonnegative integer x, we have

$$G_2(x)(I+A)^x = (I+A)^x \widetilde{J}x, \tag{5.8}$$

$$G_1(x)(I+A)^x = (I+A)^x \left[a(I+A)\widetilde{J}(I+A^*) - \widetilde{J}x\right]. \tag{5.9}$$

We now proceed in three steps.

Step 1. $\widetilde{W} = G_2 W$ and $G_2 W = W G_2^*$.

Indeed, since $G_2(0) = 0$, we have $\widetilde{W}(0) = 0 = G_2(0)W(0)$. For $x \geq 1$, using (5.8), we get

$$\begin{aligned}\widetilde{W}(x) &= \frac{a^x}{(x-1)!}(I+A)^x \widetilde{J}(I+A^*)^x = \frac{a^x}{x!}(I+A)^x \widetilde{J}x(I+A^*)^x \\ &= \frac{a^x}{x!} G_2(x)(I+A)^x(I+A^*)^x = G_2(x)W(x).\end{aligned}$$

Since $\widetilde{W}$ is Hermitian, we also get the Hermitian condition $G_2 W = W G_2^*$.

Step 2. $\Delta(G_2 W) = G_1 W$ and $G_1 W = W G_1^*$.

Since $G_2(0) = 0$, we have (see (2.4))

$$\Delta(G_2 W) = \sum_{x=0}^{\infty} (G_2(x+1)W(x+1) - G_2(x)W(x))\delta_x.$$

Using now (5.9) and the first step, we have for $x \geq 0$,

$$\begin{aligned}\Delta(G_2 W)(x) &= \Delta(\widetilde{W})(x) = \widetilde{W}(x+1) - \widetilde{W}(x) \\ &= \frac{a^x}{x!}(I+A)^x \left[a(I+A)\widetilde{J}(I+A^*) - \widetilde{J}x\right](I+A^*)^x \\ &= \frac{a^x}{x!} G_1(x)(I+A)^x(I+A^*)^x = G_1(x)W(x).\end{aligned}$$

Then $G_1 W = \Delta(G_2 W) = \Delta(W G_2^*) = W G_1^*$.

Step 3. The polynomials $(P_n)_n$ are eigenfunctions of the second order difference operator (5.6) and the polynomials $(\nabla P_{n+1})_{n\geq 0}$ are orthogonal with respect to the weight matrix $\widetilde{W}$.

The first statement follows straightforwardly from Theorem 2.

The definition of G_2 and G_1 (see ((5.4) and (5.5)) show that

$$(n-1)G_{22} + G_{11} = n(I+A)^{-1} + J - \left(n - 1 + N + \frac{N-1}{a|v_{N-1}|^2}\right) I - aA.$$

Since this matrix is upper triangular, we get

$$\det((n-1)G_{22} + G_{11}) = \prod_{i=1}^{N}\left(1 - i - \frac{N-1}{a|v_{N-1}|^2}\right) \neq 0$$

(since $a > 0$). The orthogonality of $(\nabla P_{n+1})_{n\geq 0}$ with respect to the weight matrix $\widetilde{W}$ now follows from Theorem 1 and steps 1 and 2.

We now prove the claim (in two steps).

Step A. For a nonnegative integer x, we have

$$G_2(x)(I+A)^x = (I+A)^x \widetilde{J}x.$$

Taking into account the definition of G_2 (5.4), this is equivalent to

$$(I+A)^x \widetilde{J} - \widetilde{J}(I+A)^x = -Ax(I+A)^{x-1}. \tag{5.10}$$

We now call $F_1(x) = (I + A)^x \widetilde{J} - \widetilde{J}(I + A)^x$ and $F_2(x) = -Ax(I + A)^{x-1}$. Since A is a nilpotent matrix of order N, F_1 and F_2 are both polynomials of degree not bigger than N. So, it will be enough to prove that

$$\Delta^k F_1(0) = \Delta^k F_2(0), \quad k = 0, \dots, N.$$

Since

$$\Delta^k (I + A)^x = (I + A)^x A^k = A^k (I + A)^x, \tag{5.11}$$

$$\Delta^k \big(x(I + A)^x\big) = x(I + A)^x A^k + k(I + A)^{x+1} A^{k-1}, \tag{5.12}$$

a simple computation gives

$$\Delta^k F_1(x) = A^k (I + A)^x \widetilde{J} - \widetilde{J}(I + A)^x A^k,$$
$$\Delta^k F_2(x) = -xA^{k+1}(I + A)^{x-1} - kA^k (I + A)^x.$$

And hence for $k = 0, \dots, N$.

$$\Delta^k F_1(0) = A^k \widetilde{J} - \widetilde{J} A^k, \qquad \Delta^k F_2(0) = -kA^k.$$

A simple computation using the definition of $\widetilde{J}$ (5.7) gives

$$A^k \widetilde{J} - \widetilde{J} A^k = -k \sum_{i=1}^{N-1} v_i v_{i+1} \cdots v_{i+k-1} \mathcal{E}_{i,i+k} = -kA^k.$$

And the step A is proved.

Step B. For a nonnegative integer x, we have

$$G_1(x)(I + A)^x = (I + A)^x \left[a(I + A)\widetilde{J}(I + A^*) - \widetilde{J}x\right].$$

The constraints (5.3) on the parameters v_i are used in this step (and hence in the step 2 above). We proceed as before. Call

$$H_1(x) = (I + A)^x \left[a(I + A)\widetilde{J}(I + A^*) - \widetilde{J}x\right],$$
$$H_2(x) = G_1(x)(I + A)^x.$$

Again, H_1 and H_2 are both polynomials of degree not bigger than N. Hence it will be enough to prove that for $k = 0, \dots, N$

$$\Delta^k H_1(0) = \Delta^k H_2(0).$$

Writing $G_1(x) = G_{11}x + G_{10}$, a simple computation gives for $k = 0$

$$H_1(0) = a(I + A)\widetilde{J}(I + A^*) = G_{10} = H_2(0).$$

For $k \geq 1$, from (5.11) and (5.12) we get straightforwardly

$$\Delta^k H_1(x) = (I + A)^x \left[a(I + A)A^k \widetilde{J}(I + A^*) - xA^k \widetilde{J} - kA^{k-1}(I + A)\widetilde{J}\right],$$
$$\Delta^k H_2(x) = [xG_{11}A + kG_{11}(I + A) + G_{10}A]\,(I + A)^x A^{k-1}.$$

Hence,

$$\Delta^k H_1(0) = (I + A)A^{k-1} \left[aA\widetilde{J}(I + A^*) - k\widetilde{J}\right],$$
$$\Delta^k H_2(0) = k\,[G_{11}(I + A) + G_{10}A]\,A^{k-1}.$$

It is easy to see that the equality $\Delta^k H_1(0) = \Delta^k H_2(0)$ can be rewritten in the form

$$A^k G_{10} - G_{10} A^k = k G_{11}(I + A) A^{k-1} + k(I + A) A^{k-1} \widetilde{J}.$$

The only component of this equation appears in the positions $(i, i + k - 1)$, $(i, i + k)$ and $(i, i + k + 1)$ of the matrix, and a careful computation shows that the equation holds by using (1.7). The step B is also proved. $\square$

Write $\mathcal{A}$ for the set formed by all nilpotent matrices $A = A(v_1, \ldots, v_{N-1})$ defined by (1.5) whose parameters $v_1, \ldots, v_{N-1}$ satisfy the constraints (5.3). For a matrix $A \in \mathcal{A}$, we denote by W_A the weight matrix (5.1). We now prove that, for every matrix $A \in \mathcal{A}$ the weight $\widetilde{W}$ (5.2) is, up to a normalization and a shift in the variable, again of the form $W_{\widetilde{A}}$ for a certain matrix $\widetilde{A} \in \mathcal{A}$.

Proposition 5.1. *Given non null complex numbers v_i, $i = 1, \ldots, N - 1$, satisfying the constraints (5.3), define new complex numbers w_j, $i = j, \ldots, N - 1$, by*

$$w_j = \frac{\lambda_{j+1}}{\lambda_j} v_j, \tag{5.13}$$

where $\lambda_j = \sqrt{(N - j)a|v_{N-1}|^2 + N - 1} \in \mathbb{R}$. Write $A = A(v_1, \ldots, v_{N-1})$ and $\widetilde{A} = A(w_1, \ldots, w_{N-1})$. Then

1. *The parameters w_i, $i = 1, \ldots, N - 1$, also satisfy the constraints (5.3), and hence $\widetilde{A} \in \mathcal{A}$.*
2. *Moreover, if we write M for the matrix of numbers*

$$M = \frac{1}{v_{N-1}} (I + A) \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix}, \tag{5.14}$$

then, $\widetilde{W}(x) = M W_{\widetilde{A}}(x - 1) M^*$, $x \geq 1$, where $\widetilde{W}$ is the weight matrix (5.2).

Proof. Part 1 is an easy consequence of the definition of w_j and (5.3).

In order to prove part 2, we have to check that for $x \geq 1$

$$\frac{a^x}{(x-1)!} (I + A)^x \widetilde{J} (I + A^*)^x = \frac{a^{x-1}}{(x-1)!} M (I + \widetilde{A})^{x-1} (I + \widetilde{A}^*)^{x-1} M^*,$$

where $\widetilde{J}$ is defined by (5.7).

The previous identity is a consequence of the following one

$$M(I + \widetilde{A})^{x-1} = \sqrt{a} (I + A)^x \widetilde{J}^{1/2}.$$

We denote then $E_1(x) = M(I + \widetilde{A})^{x-1}$ and $E_2(x) = \sqrt{a}(I + A)^x \widetilde{J}^{1/2}$, and proceeding as in the proof of Theorem 3 (steps A and B), we prove that $\Delta^k E_1(1) = \Delta^k E_2(1)$ for $k = 0, \ldots, N - 1$. Since both E_1 and E_2 are polynomials of degree $N - 1$, we then have $E_1 = E_2$. Using that $\Delta^k (I + A)^x = (I + A)^x A^k$, we get

$$\Delta^k E_1(1) = M \widetilde{A}^k, \qquad \Delta^k E_2(1) = \sqrt{a}(I + A) A^k \widetilde{J}^{1/2}.$$

We have then to prove that

$$M\widetilde{A}^k = \sqrt{a}(I + A)A^k \widetilde{J}^{1/2}.$$

But this identity follows from the definition of M if we take into account that

$$\sqrt{a}\widetilde{J}^{1/2} = \frac{1}{v_{N-1}} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix}^{-1} A \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix}. \quad \square$$

Corollary 5.2. *Let W be the weight matrix defined by (5.1), where we assume that the parameters v_i, $i = 1, \ldots, N-2$, satisfy the constraints (5.3). Then for $k \geq 1$, the polynomials $(\nabla^k P_{n+k})_n$ are again orthogonal with respect to a weight matrix.*

The weight matrices (5.1) were introduced in [1] without the constraints (5.3). It is proved there that the orthogonal polynomials with respect to W are eigenfunctions of the following second order difference operator

$$\widetilde{D} = \Delta\nabla(\cdot)(I + A)^{-1}x + \Delta(\cdot)\left[a(I + A) - (I + A)^{-1}x\right] + (\cdot)\left[a(I + A) - J\right].$$

According to Theorem 1, when the parameters v_i, $i = 1, 2, \ldots, N-1$, satisfy the constraints (1.7), the orthogonal polynomials $(\nabla P_n)_{n\geq 1}$ are also eigenfunctions of the second order difference operator (5.6). This difference operator is however linearly independent to the one found in [1]. This phenomenon never happens in the scalar case where the elements of each of the classical discrete families of orthogonal polynomials are eigenfunctions of only one linearly independent second order difference operator, respectively (see [5] for more details).

Among other things, we next show that for the weight matrix W (5.1) the constraints (5.3) seem to be necessary for the existence of polynomials G_2 and G_1 (of degrees at most 2 and 1, respectively) satisfying the Pearson equation $\Delta(WG_2^*) = WG_1^*$ and the Hermitian condition $G_2W = WG_2^*$.

Example 2. Consider the weight matrix

$$W = \sum_{x=0}^{\infty} \frac{a^x}{x!}(I + A)^x (I + A^*)^x \delta_x, \tag{5.15}$$

where A is now an arbitrary upper triangular nilpotent matrix of size 3×3

$$A = \begin{pmatrix} 0 & v_1 & v_3 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where $v_1, v_2 \neq 0$.

Using Maple, it is straightforward to show that there always exist polynomials G_2 and G_1 of degrees not bigger than 2 and 1, respectively, such that $\Delta(WG_2^*) = WG_1^*$ (only linear equations are needed). In fact, the dimension of the linear subspace

$$\{G_2 \in \mathbb{P}_2^N : \text{there exists } G_1 \in \mathbb{P}_1^N \text{ with } \Delta(WG_2^*) = WG_1^*\}$$

is in general equal to 9.

However, it is not difficult to see that only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy (5.3), one of these polynomials satisfies also the Hermitian condition $G_2 W = WG_2^*$. This means that the polynomials $(\nabla P_{n+1})_{n\geq 0}$ are always orthogonal with respect to a moment functional but this moment functional can be represented by a weight matrix (in the sense defined in the introduction) only when $v_3 = 0$ and the parameters v_1 and v_2 satisfy the constraints (5.3).

Take now particular non null values of the parameters. Using Maple it is not difficult to generate the first few orthogonal polynomials with respect to W. For instance for $v_1 = v_2 = v_3 = 2$ and $a = 1$, we have

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} x - \frac{329}{69} & \frac{314}{69} & -\frac{2}{69} \\ -\frac{58}{69} & x - \frac{17}{69} & -\frac{10}{69} \\ -\frac{2}{69} & -\frac{22}{69} & x - \frac{17}{69} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} x^2 - \frac{5051}{569}x + \frac{5017}{569} & \frac{7788}{569}x - \frac{3308}{569} & -\frac{11300}{569}x + \frac{136}{569} \\ -\frac{500}{569}x + \frac{1068}{569} & x^2 - \frac{1411}{569}x - \frac{137}{569} & \frac{404}{569}x + \frac{4}{569} \\ -\frac{4}{569}x + \frac{136}{569} & -\frac{380}{569}x + \frac{124}{569} & x^2 - \frac{411}{569}x + \frac{41}{569} \end{pmatrix}.$$

One can then check that the polynomials P_n, $n \geq 0$, cannot be common eigenfunctions of a second order difference operator of the form (1.6) (only linear equations are needed). This shows that the hypothesis $G_2 W = WG_2^*$ cannot be removed in Theorem 2.

Example 3. We finally display a weight matrix W whose orthogonal polynomials $(P_n)_n$ are eigenfunctions of a second order difference operator, however this weight matrix W does not satisfy any discrete Pearson equation. Indeed, consider the discrete 2×2 weight matrix defined by

$$W = \sum_{x=0}^{k} \frac{\Gamma(k+1)}{x!\Gamma(k+1-x)} \begin{pmatrix} 1 & \frac{v_1}{k+1}x(x-k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{v_1}{k+1}x(x-k) \\ 0 & 1 \end{pmatrix}^* \delta_x$$

where k is a positive integer. This weight matrix is a particular case of an $N \times N$ weight matrix introduced in [1]. The orthogonal polynomials $(P_n)_{n=0}^{k-1}$ with respect to W (they are now finitely many) are also eigenfunctions of a second order difference operator. Using Maple is easy to check that there do not exist any polynomials G_2 and G_1 of degrees not bigger than 2 and 1, respectively, such that $\Delta(WG_2^*) = WG_1^*$ (only linear equations are needed). In particular, the family $(\nabla P_{n+1})_{n\geq 0}$ cannot be orthogonal with respect to any moment functional. This shows that in the matrix orthogonality, the discrete Pearson equation for the weight matrix W is independent of whether the orthogonal polynomials with respect to W are eigenfunctions of a second order difference operator of the form (1.6).

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Rodrigues' formulas for orthogonal matrix polynomials satisfying second-order difference equations

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We develop a method to find discrete Rodrigues' formulas for orthogonal matrix polynomials which are also eigenfunctions of a second-order difference operator. Using it, we produce Rodrigues' formulas for two illustrative examples of arbitrary size.

Keywords: matrix orthogonal polynomials; difference operators and equations; Charlier polynomials; Meixner polynomials

1991 Mathematics Subject Classifications: 33E30; 42C05; 47B39

1. Introduction and results

It is well known that the orthogonal polynomials of Charlier, Meixner, Krawtchouk and Hahn have a number of very interesting extra properties. Among those properties are the following two:

(1) each one of these classical discrete families is eigenvalues of a second-order difference operator of the form

$$f_{-1}(x)\mathfrak{s}_{-1} + f_0(x)\mathfrak{s}_0 + f_1(x)\mathfrak{s}_1,$$

where $\mathfrak{s}_l$ denotes the shift operator $\mathfrak{s}_l(f) = f(x+l)$ and f_i, $i = -1, 0, 1$, are polynomials of degree not larger than 2 (independent of n) satisfying that $\deg(\sum_{l=-1}^{1} l^k f_l) \leq k,\ k = 0, 1, 2$;

(2) they can be obtained using a discrete Rodrigues' formula:

$$p_n(x) = \frac{\Delta^n(w(x)\prod_{m=0}^{n-1} f_{-1}(x-m))}{w(x)}, \tag{1.1}$$

where Δ is the first-order difference operator $\Delta(f) = f(x+1) - f(x)$ and w is the corresponding classical discrete weight.

Each one of these characterizations is the result of a different effort, and they are usually associated with the names of O. Lancaster and W. Hahn, respectively. Actually, these properties can be seen to follow from the so-called Pearson discrete equation for the weight w: $f_{-1}(x)w(x) = f_1(x-1)w(x-1)$, which also characterizes the four classical discrete weights of

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Charlier, Meixner, Krawtchouk and Hahn. (For a historical account of this and other related subjects, see, for instance, [1,2]).

The theory of orthogonal matrix polynomials starts with two papers by Krein in 1949, [3,4]. Each sequence of orthogonal matrix polynomials $(P_n)_n$ is associated with a weight matrix W and satisfies that P_n, $n \geq 0$, is a matrix polynomial of degree n with nonsingular leading coefficient and $\int P_n \mathrm{d}WP_m^* = \Gamma_n \delta_{n,m}$, where Γ_n, $n \geq 0$, is a positive-definite matrix. When $\Gamma_n = I$, we say that the polynomials $(P_n)_n$ are orthonormal.

However, the first examples of orthogonal matrix polynomials $(P_n)_n$, which are eigenfunctions of a second-order difference operator of the form

$$L(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1}(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_1(\cdot)F_1(x), \tag{1.2}$$

(with left eigenvalues) appeared in 2012 [5–7]. Here F_{-1}, F_0 and F_1 are matrix polynomials satisfying $\deg(\sum_{l=-1}^{1} l^k F_l) \leq k,\ k = 0, 1, 2.$

These examples have been essentially found by solving an appropriate set of commuting and difference equations. This set includes a matrix analogous to the Pearson equation of the scalar case and is the following one

$$F_0 W = W F_0^*, \tag{1.3}$$

$$F_1(x-1)W(x-1) = W(x)F_{-1}^*(x). \tag{1.4}$$

Under certain boundary conditions, these equations imply that the orthonormal polynomials with respect to W are eigenfunctions of the second-order difference operator (1.2) with Hermitian (left) eigenvalues Λ_n; that is, $L(P_n) = \Lambda_n P_n$, $n \geq 0$.

The families of orthogonal matrix polynomials found using these methods are among those that are likely to play in the case of matrix orthogonality and the role of the classical discrete families of Charlier, Meixner, Krawtchouk and Hahn in the case of scalar orthogonality.

This paper is devoted to the question of the existence of Rodrigues' formulas for these families of orthogonal matrix polynomials, that is, assuming that W satisfies the commuting and difference equations (1.3) and (1.4), and is there any efficient and canonical way to produce a sequence of orthogonal matrix polynomials with respect to W? Say in an analogous way as to (1.1) produces the orthogonal polynomials with respect to a classical discrete weight w.

Even if F_{-1} and W commute in the strong form that $F_{-1}(x)W(y) = W(y)F_{-1}(x)$, x, y in the support of W, orthogonal matrix polynomials which are eigenfunctions of a difference operator like (1.2) do not seem to satisfy, in general, a scalar-type Rodrigues' formula of the form

$$P_n(x) = C_n \Delta^n \left(W(x) \prod_{m=0}^{n-1} F_{-1}(x-m) \right) W^{-1}(x), \quad n \geq 0, \tag{1.5}$$

where C_n, $n \geq 0$, are nonsingular matrices.

(In this paper, we consider discrete weight matrices of the form $W = \sum_{x=0}^{\infty} W(x)\delta_x$, but implicitly assume that the function $W(x)$ is an entire function in the whole complex plane so that the right-hand side of (1.5) makes sense for $x \in \mathbb{C}$.)

Instead of (1.5), these orthogonal matrix polynomials seem to satisfy some modified Rodrigues' formula. The first instance of that modified Rodrigues' formula appeared in [5]: the expression

$$P_n(x) = \Delta^n \left(\frac{a^x}{\Gamma(x-n+1)} \begin{pmatrix} 1 + ab^2 n + b^2 x^2 & bx \\ bx & 1 \end{pmatrix} \right) W^{-1}(x)$$

defines a sequence of orthogonal matrix polynomials with respect to the discrete weight matrix

$$W = \sum_{x=0}^{\infty} \frac{a^x}{\Gamma(x+1)} \begin{pmatrix} 1+b^2x^2 & bx \\ bx & 1 \end{pmatrix} \delta_x. \tag{1.6}$$

Along this paper, we take $1/\Gamma(y+1) = 0$ if y is a negative integer, so that the function $1/\Gamma(x)$ is entire.

Similar Rodrigues' formulas of the form

$$P_n(x) = \Delta^n(\xi_n(x))W^{-1}(x) \tag{1.7}$$

have been found for other families of orthogonal polynomials of size 2×2.[5] In all these examples, the functions ξ_n are simple enough as to make Rodrigues' formula (1.7) useful for the explicit calculation of the sequence of orthogonal polynomials P_n with respect to W.

Under mild conditions on the functions ξ_n in (1.7), one can easily prove that P_n is orthogonal with respect to any polynomial of degree less than n (by performing a sum by parts). The difficulty to find Rodrigues' formulas like (1.7) is that, in general, it is rather involved to guarantee that P_n defined by (1.7) is a polynomial of degree n with nonsingular leading coefficient. In the extant examples, this requirement on P_n has been checked by a direct computation. As a consequence of this, only examples of small size have been worked out (actually size 2×2).

The purpose of this paper is to develop a method to explore the existence of Rodrigues' formulas of the form (1.7) for orthogonal matrix polynomials of arbitrary size (for Rodrigues' formulas for orthogonal matrix polynomials, which are eigenfunctions of second-order differential operators, see [8]). The key of this method is to exploit the set of commuting and difference equations (1.3) and (1.4) for the weight matrix W, and use the following lemma as the main tool:

Lemma 1.1 *Let F_1, F_0 and F_{-1} be matrix polynomials satisfying that*

$$\deg\left(\sum_{l=-1}^{1} l^k F_l\right) \leq k, \quad k = 0, 1, 2. \tag{1.8}$$

Let W and $\mathfrak{R}_n$ be $N \times N$ matrix functions defined in a discrete set $\Omega = \{0, 1, 2, \dots, \tau\}$, where τ can be a positive integer or infinity. Assume that W is nonsingular for $x \in \Omega$ and satisfies the equations

$$F_0 W = W F_0^*, \quad F_1(x-1)W(x-1) = W(x)F_{-1}^*(x). \tag{1.9}$$

Define the functions $\mathcal{P}_n$, $n \geq 1$, by

$$\mathcal{P}_n = \Delta^n(\mathfrak{R}_n)W^{-1}. \tag{1.10}$$

If for a matrix Λ_n, the function $\mathfrak{R}_n$ satisfies

$$\begin{aligned} &\mathfrak{s}_{-1}(\mathfrak{R}_n F_1^*) + \mathfrak{s}_1\left[\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2 F_1^* - n\Delta F_{-1}^* + F_{-1}^*\right)\right] \\ &\quad + \mathfrak{R}_n\left(-n\Delta^2 F_1^* + n\Delta F_1^* + F_0^*\right) = \Lambda_n \mathfrak{R}_n \end{aligned} \tag{1.11}$$

for $x \in \Omega$, then the function $\mathcal{P}_n$ satisfies

$$\mathfrak{s}_{-1}(\mathcal{P}_n)F_{-1} + \mathfrak{s}_0(\mathcal{P}_n)F_0 + \mathfrak{s}_1(\mathcal{P}_n)F_1 = \Lambda_n \mathcal{P}_n, \quad x \in \Omega. \tag{1.12}$$

Once we have a weight matrix W satisfying Equations(1.3) and (1.4), and an appropriate choice of eigenvalues Λ_n, $n \geq 1$, our method consists in using a solution of Equation (1.11) to produce a sequence of orthogonal matrix polynomials with respect to W by means of Rodrigues' formula (1.10). According to the previous lemma, the function $\mathtt{P}_n$ given by (1.10) satisfies the difference equation (1.12).

It turns out that, in general, these difference equations are not enough to guarantee that $\mathtt{P}_n$, $n \geq 1$, is a polynomial of degree n with nonsingular leading coefficient, because the eigenvalues Λ_n, $n \geq 1$, have, in general, non disjoint spectrum. However, the method seems to work very efficiently as long as the weight matrix W satisfies Equations (1.3) and (1.4) for a couple of linearly independent sets of coefficients $F_{-1,1}, F_{0,1}, F_{1,1}$ and $F_{-1,2}, F_{0,2}, F_{1,2}$. In this case, one can do a suitable choice of the eigenvalues $\Lambda_{n,1}$ and $\Lambda_{n,2}$ and real constants a_1, a_2 such that the spectrum of the linear combination $a_1\Lambda_{n,1} + a_2\Lambda_{n,2}$ is disjoint for $n \geq 1$, from where one can deduce that $\mathtt{P}_n$ is a polynomial of degree n with nonsingular leading coefficient. Moreover, the functions $\mathfrak{R}_n$, $n \geq 1$, provided by this method have a so surprisingly simple expression that it suggests the existence of a certain hidden pattern.

The existence of orthogonal matrix polynomials being eigenfunctions of several linearly independent second-order difference operators as (1.2) is a new phenomenon of the matrix orthogonality (see [5]; for differential operators see [9–13], and the references therein). In the scalar case, the classical discrete families of Charlier, Meixner, Krawtchouk and Hahn are eigenfunctions of only one second-order difference operator, up to multiplicative constants.

Using our method, we have found Rodrigues' formulas for the following two illustrative examples.

The first example is the weight matrix

$$W_1 = \sum_{x \geq 0} \frac{a^x}{\Gamma(x+1)} (I + A)^x (I + A^*)^x \delta_x, \tag{1.13}$$

where A is the $N \times N$ nilpotent matrix

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \tag{1.14}$$

$a > 0$, and v_i, $i = 1, \ldots, N-1$, are complex numbers satisfying that

$$(N - i - 1)a|v_i|^2|v_{N-1}|^2 - i(N - i)|v_{N-1}|^2 + (N - 1)|v_i|^2 = 0. \tag{1.15}$$

The matrix $\mathcal{E}_{i,j}$ stands for the matrix with entry (i, j) equal to 1 and 0 otherwise. For $i = N - 1$, the condition (1.15) is an identity. For $N = 2$, this example depends on just one parameter v_1 and gives the weight matrix (1.6). For $N = 2$, the condition (1.15) always fulfils.

Let us note that since A is nilpotent of order N, $(I + A)^x (I + A^*)^x$ is a matrix polynomial of degree $2N - 2$.

It was proved in [6] and [7] that this weight matrix satisfies Equations (1.3) and (1.4) for a couple of sets of linearly independent coefficients $F_{-1,j}, F_{0,j}, F_{1,j}$, $j = 1, 2$ (see (4.1) and (4.3) below). Rodrigues' formula provided by our method for this example is the following:

THEOREM 1.2 *Assume that the moduli of the entries $|v_i|$, $i = 1, \ldots, N-1$, of the matrix A (1.14) satisfy (1.15). Then, a sequence of orthogonal polynomials with respect to the weight matrix W_1 (1.13) can be defined by using Rodrigues' formula*

$$P_n(x) = \Delta^n \left(\frac{a^x}{\Gamma(x - n + 1)} (I + A)^x \mathfrak{L}_{n,1} (I + A^*)^x \right) W_1^{-1}(x), \quad n \geq 1, \tag{1.16}$$

where $\mathfrak{L}_{n,1}$ *is the diagonal matrix independent of x and with entries*

$$\mathfrak{L}_{n,1} = \sum_{i=1}^{N} \prod_{k=i}^{N-1} \left(1 + \frac{an|v_k|^2}{k(N-k)}\right) \mathcal{E}_{ii} \tag{1.17}$$

(*for* $i > j$ *we take* $\prod_{l=i}^{j} = 1$).

The assumption (1.15) on the parameters seems to be necessary. We have symbolic computational evidence which shows that if (1.15) does not hold then, for any choice of the matrix $\mathfrak{L}_{n,1}$ (diagonal or not), the polynomial P_n in (1.16) has degree bigger than n or singular leading coefficient.

The second example is the weight matrix

$$W_2 = \sum_{x \geq 0} \frac{a^x}{\Gamma(x+1)} (I+A)^x \Gamma((x+c)I + J)(I+A^*)^x \delta_x, \tag{1.18}$$

where A is defined by (1.14), $0 < a < 1$, $c > 0$, J is the diagonal matrix

$$J = \sum_{i=1}^{N} (N-i)\mathcal{E}_{i,i} \tag{1.19}$$

and the complex numbers v_i, $i = 1, \ldots, N-1$, satisfy that

$$\frac{(N-i-1)a|v_i|^2|v_{N-1}|^2}{1-a} - i(N-i)|v_{N-1}|^2 + (N-1)|v_i|^2 = 0. \tag{1.20}$$

The properties of the matrices A (nilpotent of order N) and J (diagonal) imply that $(I + A)^x\Gamma((x+c)I + J)(I+A^*)^x/\Gamma(x+c)$ is a matrix polynomial of degree $2N-2$.

As far as we know, this is the first time this weight matrix appears in the literature. We show that this weight matrix satisfies Equations (1.3) and (1.4) for a couple of sets of linearly independent coefficients $F_{-1,j}, F_{0,j}, F_{1,j}, j = 1, 2$ (see (4.2) and (4.3) below). Rodrigues' formula provided by our method for this example is the following:

Theorem 1.3 *Assume that the moduli of the entries* $|v_i|$, $i = 1, \ldots, N-1$, *of the matrix A* (1.14) *satisfy* (1.20). *Then, a sequence of orthogonal polynomials with respect to the weight matrix* W_2 (1.18) *can be defined by using Rodrigues' formula*

$$P_n(x) = \Delta^n \left(\frac{a^x}{\Gamma(x-n+1)} (I+A)^x \Gamma((x+c)I + J) \mathfrak{L}_{n,2} (I+A^*)^x \right) W_2^{-1}(x), \quad n \geq 1,$$

where $\mathfrak{L}_{n,2}$ *is the diagonal matrix independent of x and with entries*

$$\mathfrak{L}_{n,2} = \sum_{i=1}^{N} \prod_{k=i}^{N-1} \left(1 + \frac{an|v_k|^2}{k(N-k)(1-a)}\right) \mathcal{E}_{i,i}. \tag{1.21}$$

2. Preliminaries

A weight matrix W is an $N \times N$ matrix of measures supported in the real line satisfying that (1) $W(A)$ is positive semi-definite for any Borel set $A \in \mathbb{R}$, (2) W has finite moments of every order and (3) $\int P(t)\,\mathrm{d}W(t)P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular. All the examples considered in this paper are discrete weight matrices of the form

$$W = \sum_{x\in\mathbb{N}} W(x)\delta_x. \tag{2.1}$$

For a discrete weight matrix $W = \sum_{x\in\mathbb{N}} W(x)\delta_x$ supported in a countable set X of real numbers, the Hermitian sesquilinear form defined by $\langle P, Q\rangle = \int P\,\mathrm{d}WQ^*$ takes the form

$$\langle P, Q\rangle = \sum_x P(x)W(x)Q^*(x).$$

If W does not satisfy condition (3) above, we will say that W is degenerate. That happens, for instance, if W is supported in finitely many points (as is the case of the discrete classical families of Krawtchouk and Hahn). That condition (3) is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W, P_n of degree n with nonsingular leading coefficient. For a discrete weight matrix as (2.1) Condition (3) is fulfilled, in particular, when $W(x)$ is positive definite for infinitely many $x \in \mathbb{N}$.

We then say that a sequence of matrix polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is orthogonal with respect to W if $\langle P_n, P_k\rangle = \Lambda_n\delta_{k,n}$, where Λ_n is a nonsingular matrix for $n \geq 0$. Since each orthogonal polynomial P_n has degree n with nonsingular leading coefficient, any matrix polynomial of degree less than or equal to n can be expressed as a linear combination of P_k, $0 \leq k \leq n$, with matrix coefficients (multiplying on the left or on the right). That property, together with the orthogonality, defines the sequence of orthogonal polynomials uniquely from W up to multiplication on the left by a sequence of nonsingular matrices (multiplication by unitary matrices for the orthonormal polynomials).

Along this paper, we will use without an explicit mention the usual properties listed below of the first-order difference operator Δ defined by $\Delta(p) = p(x+1) - p(x)$,

$$\Delta[f(x)g(x)] = f(x)\Delta g(x) + (\Delta f(x))g(x+1), \tag{2.2}$$

$$\sum_a^{b-1} f(x)\Delta g(x) = f(x)g(x)|_a^b - \sum_a^{b-1}(\Delta f(x))g(x+1), \tag{2.3}$$

$$\Delta^n f(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x+n-k). \tag{2.4}$$

Given a discrete weight matrix

$$W = \sum_{x=a}^b W(x)\delta_x,$$

supported in $\{a, a+1, \dots, b-1, b\}$ (a can be $-\infty$ and b can be $+\infty$), the matrices of measures ΔW and ∇W are defined in the usual way by

$$\Delta W = \sum_{x=a-1}^{b} (W(x+1) - W(x))\delta_x, \quad \nabla W = \sum_{x=a}^{b+1} (W(x) - W(x-1))\delta_x,$$

respectively, where by definition $W(b+1) = W(a-1) = 0$ (if $a = -\infty$ or $b = \infty$, we take $a-1 = -\infty$ or $b+1 = +\infty$, respectively). It is worth to note that the support of ΔW is $\{a-1, a, a+1, \dots, b-1, b\}$ and that it is different to the support of W, except when $a = -\infty$. In the same way, the support of ∇W is $\{a, a+1, \dots, b-1, b, b+1\}$ and it is different to the support of W, except when $b = +\infty$.

3. Proof of Lemma 1.1

The proof is a matter of computation. We start denoting $E_n = \Delta^n \mathfrak{R}_n$ to simplify the notation, so $\mathtt{P}_n = E_n W^{-1}$.

From (1.9) we deduce

$$\begin{aligned} W^{-1}(x)F_0(x) &= F_0^*(x)W^{-1}(x), \\ W^{-1}(x+1)F_1(x) &= F_{-1}^*(x+1)W^{-1}(x), \\ W^{-1}(x-1)F_{-1}(x) &= F_1^*(x-1)W^{-1}(x). \end{aligned} \tag{3.1}$$

Then, the equation

$$\mathfrak{s}_{-1}(\mathtt{P}_n)F_{-1} + \mathfrak{s}_0(\mathtt{P}_n)F_0 + \mathfrak{s}_1(\mathtt{P}_n)F_1 = \Lambda_n \mathtt{P}_n \tag{3.2}$$

is equivalent to

$$\begin{aligned} &E_n(x-1)W^{-1}(x-1)F_{-1}(x) + E_n(x)W^{-1}(x)F_0(x) \\ &\quad + E_n(x+1)W^{-1}(x+1)F_1(x) = \Lambda_n E_n(x)W^{-1}(x). \end{aligned}$$

Applying (3.1) and multiplying by $W(x)$ on the right, we have

$$E_n(x-1)F_1^*(x-1) + E_n(x)F_0^*(x) + E_n(x+1)F_{-1}^*(x+1) = \Lambda_n E_n(x). \tag{3.3}$$

Denote now

$$H_2 = F_1, \quad H_1 = F_{-1} - F_1, \quad H_0 = F_1 + F_0 + F_{-1}. \tag{3.4}$$

The assumptions (1.8) imply that

$$\deg(H_i) \le i, \quad i = 0, 1, 2. \tag{3.5}$$

Using (3.4) we obtain that Equation (3.3) is equivalent to

$$\Delta\nabla(E_n H_2^*) + \Delta(E_n H_1^*) + E_n H_0^* = \Lambda_n E_n.$$

Using (3.5) and the well-known identity

$$\Delta^n(FG) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k F(x+n-k)\Delta^{n-k}G(x),$$

we obtain

$$\begin{aligned} E_n H_0^* &= (\Delta^n \mathfrak{R}_n)H_0^* = \Delta^n(\mathfrak{R}_n H_0^*), \\ E_n H_1^* &= (\Delta^n \mathfrak{R}_n)H_1^* = \Delta^n(\mathfrak{R}_n(H_1^* - n\Delta H_1^*)) + \Delta^{n-1}(-n\mathfrak{R}_n \Delta H_1^*), \\ E_n H_2^* &= (\Delta^n \mathfrak{R}_n)H_2^* = \Delta^n\left(\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2 H_2^* - n\Delta H_2^* + H_2^*\right)\right) \\ &\quad + \Delta^{n-1}(\mathfrak{R}_n(n^2\Delta^2 H_2^* - n\Delta H_2^*)) + \Delta^{n-2}\left(\mathfrak{R}_n\binom{n}{2}\Delta^2 H_2^*\right). \end{aligned}$$

So, Equation (3.2) is equivalent to

$$\nabla\Delta^{n+1}\left(\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2H_2^*-n\Delta H_2^*+H_2^*\right)\right)+\nabla\Delta^n(\mathfrak{R}_n(n^2\Delta^2H_2^*-n\Delta H_2^*))$$
$$+\nabla\Delta^{n-1}\left(\mathfrak{R}_n\binom{n}{2}\Delta^2H_2^*\right)+\Delta^{n+1}(\mathfrak{R}_n(H_1^*-n\Delta H_1^*))$$
$$+\Delta^n(\mathfrak{R}_n(H_0^*-n\Delta H_1^*))=\Lambda_n\Delta^n\mathfrak{R}_n.$$

this can be rewritten as

$$\Delta^n\Bigg[\Delta\nabla(\mathfrak{R}_nH_2^*)+\Delta\left(\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2H_2^*-n\Delta H_2^*-n\Delta H_1^*+H_1^*\right)\right)$$
$$+\mathfrak{R}_n\left(\binom{n}{2}\Delta^2H_2^*-n\Delta H_1^*+H_0^*\right)-\Lambda_n\mathfrak{R}_n\Bigg]=0.$$

Replacing the coefficients H_0, H_1 and H_2 by F_{-1}, F_0 and F_1 according to (3.4), we can express this equation in terms of the shift operators

$$\Delta^n\Bigg[\mathfrak{s}_{-1}(\mathfrak{R}_nF_1^*)+\mathfrak{s}_1\left(\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2F_1^*-n\Delta F_{-1}^*+F_{-1}^*\right)\right)$$
$$+\mathfrak{R}_n(-n\Delta^2F_1^*+n\Delta F_1^*+F_0^*)-\Lambda_n\mathfrak{R}_n\Bigg]=0. \tag{3.6}$$

Equation (3.6) obviously holds if

$$\mathfrak{s}_{-1}(\mathfrak{R}_nF_1^*)+\mathfrak{s}_1\left(\mathfrak{R}_n\left(\binom{n+1}{2}\Delta^2F_1^*-n\Delta F_{-1}^*+F_{-1}^*\right)\right)$$
$$+\mathfrak{R}_n(-n\Delta^2F_1^*+n\Delta F_1^*+F_0^*)-\Lambda_n\mathfrak{R}_n=0.$$

4. Proof of Theorem 1.2

The weight matrix (1.13) was introduced in [6], where it was proved that the orthogonal matrix polynomials with respect to W_1 are eigenfunctions of a second-order difference operator of the form (1.2), where

$$F_{-1,1}(x)=(I+A)^{-1}x,\quad F_{0,1}(x)=-J-(I+A)^{-1}x,\quad F_{1,1}(x)=a(I+A) \tag{4.1}$$

and J is the diagonal matrix defined by (1.19). Something more interesting can be proved when the complex numbers v_i, $i=1,\dots,N-1$, are non-null and satisfy the constraints

$$(N-i-1)a|v_i|^2|v_{N-1}|^2+(N-1)|v_i|^2-i(N-i)|v_{N-1}|^2=0. \tag{4.2}$$

Indeed in [7], we have proved that the orthogonal matrix polynomials with respect to W_1 are eigenfunctions of other second-order difference operator (linearly independent to the previous one) of the form (1.2) with

$$\begin{aligned} F_{-1,2}(x) &= [(I+A)^{-1} - I]x^2 + \left(\frac{N-1}{a|v_{N-1}|^2} I + J\right) x, \\ F_{1,2}(x) &= [(I+A)^{-1} - I]x^2 + ((I+A)^{-1} - aA + 2J - NI)x \\ &\quad + a(I+A)\left(\frac{N-1}{a|v_{N-1}|^2} I + J\right)(I+A^*), \\ F_{0,2}(x) &= -F_{-1,2}(x) - F_{1,2}(x). \end{aligned} \tag{4.3}$$

Moreover, the differences of the orthogonal polynomials with respect to W_1 are again orthogonal with respect to a weight matrix.

To produce Rodrigues' formula for this example, we consider

$$\mathfrak{R}_{n,1}(x) = \frac{a^x}{\Gamma(x-n+1)} (I+A)^x \mathfrak{L}_{n,1} (I+A^*)^x, \tag{4.4}$$

where the diagonal matrix $\mathfrak{L}_{n,1}$ is given by (1.17). Remember the expression of the weight matrix W_1:

$$W_1(x) = \sum_{x \geq 0} \frac{a^x}{\Gamma(x+1)} (I+A)^x (I+A^*)^x \delta_x. \tag{4.5}$$

We have to prove that an nth orthogonal polynomial with respect to W_1 is given by the formula

$$P_n = \Delta^n(\mathfrak{R}_{n,1}) W_1^{-1}.$$

We first claim that for a suitable choice of eigenvalues $\Lambda_{n,1}$ and $\Lambda_{n,2}$, the function $\mathfrak{R}_{n,1}$ satisfies the difference equations (1.11) corresponding to the set of difference coefficients (4.1) and (4.3).

Lemma 4.1 *For*

$$\Lambda_{n,1} = a(I+A) - J - n(I+A)^{-1}$$

and

$$\Lambda_{n,2} = n^2((I+A)^{-1} - I) + n\left(J - aA - \left(N - 1 + \frac{N-1}{a|v_{N-1}|^2}\right) I\right)$$

the function $\mathfrak{R}_{n,1}$ *satisfies the two following second-order difference equations* $(i = 1, 2)$

$$\begin{aligned} \mathfrak{s}_{-1}(\mathfrak{R}_{n,1} F_{1,i}^*) + \mathfrak{s}_1 \left[\mathfrak{R}_{n,1}\left(\binom{n+1}{2}\Delta^2 F_{1,i}^* - n\Delta F_{-1,i}^* + F_{-1,i}^*\right)\right] \\ + \mathfrak{R}_{n,1}(-n\Delta^2 F_{1,i}^* + n\Delta F_{1,i}^* + F_{0,i}^*) = \Lambda_{n,i} \mathfrak{R}_{n,1}, \end{aligned}$$

where $F_{j,i}$, $j = -1, 0, 1$ *and* $i = 1, 2$, *are given by* (4.1) *and* (4.3).

We will prove the lemma at the end of this section.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 We proceed in three steps.

Step 1*:* P_n is a polynomial of degree n.

Since A is a nilpotent matrix of order N and $\mathfrak{L}_{n,1}$ is nonsingular, the functions $a^{-x}\Gamma(x-n+1)\mathfrak{R}_{n,1}(x)$ and $a^x W_1^{-1}(x)/\Gamma(x+1)$ are polynomials of degree $2N-2$, so P_n is a polynomial (see (4.4) and (4.5)). Write m for the degree of P_n and C_m for its leading coefficient, so $C_m \neq 0$. Comparing leading coefficients in (1.12), we obtain

$$C_m \Lambda_{m,i} = \Lambda_{n,i} C_m, \quad i = 1, 2. \tag{4.6}$$

Write $\lambda_{n,k,i}$ for the eigenvalues of $\Lambda_{n,k}$, $i = 1, \ldots, N$, $k = 1, 2$. From (4.1) and (4.3), we have that

$$\lambda_{n,1,i} = a - N + i - n, \quad \lambda_{n,2,i} = n\left(1 - i - \frac{N-1}{a|v_{N-1}|^2}\right).$$

We assume that $m \neq n$ and proceed by *reductio ad absurdum*. We first prove that if $\lambda_{n,1,j} = \lambda_{m,1,i}$ for some i, j, $1 \leq i, j \leq N$, then $\lambda_{n,2,j} \neq \lambda_{m,2,i}$.

If we take $\lambda_{n,1,j} = \lambda_{m,1,i}$, then we have $m = n + i - j$. Since $n \neq m$, then $i \neq j$. Thus,

$$\lambda_{m,2,i} - \lambda_{n,2,j} = (j - i)\left(i + n - 1 + \frac{N-1}{a|v_{N-1}|^2}\right) \neq 0.$$

We now take two numbers $a_1, a_2 \in \mathbb{R}$ such that $a_1, a_2 \neq 0$ and

$$\frac{a_1}{a_2} \neq \frac{\lambda_{n,2,j} - \lambda_{m,2,i}}{\lambda_{m,1,i} - \lambda_{n,1,j}}, \quad \text{if } \lambda_{m,1,i} \neq \lambda_{n,1,j},\ 1 \leq i, j \leq N.$$

Since $\Lambda_{n,i}$, $i = 1, 2$, are upper triangular, it is easy to see that the matrices $\Theta_k = a_1\Lambda_{k,1} + a_2\Lambda_{k,2}$, $k = n, m$, do not share any eigenvalue.

From (4.6), we obtain that

$$C_m \Theta_m = \Theta_n C_m.$$

Since Θ_m and Θ_n do not share any eigenvalue, we obtain that $C_m = 0$ (see [14, p.220]), which it contradicts that $C_m \neq 0$, so we have $m = n$.

Step 2: P_n is orthogonal to x^k, $k = 0, \ldots, n-1$, with respect to W_1.

Using (2.4), we can write for $j = 0, \ldots, n$:

$$\Delta^{n-j}\mathfrak{R}_{n,1}(x + j - 1) = \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} \mathfrak{R}_{n,1}(x + n - l - 1).$$

Hence, for $k = 0, \ldots, n-1$, using that $1/\Gamma(y+1) = 0$ when y is a negative integer, we obtain from (4.4)

$$\begin{aligned} \Delta^{j-1}(x^k)\Delta^{n-j}\mathfrak{R}_{n,1}(x + j - 1)|_{x=0} &= \Delta^{j-1}(x^k)|_{x=0} \\ &\times \sum_{l=0}^{n-j}(-1)^l\binom{n-j}{l}\frac{a^{n-l-1}}{\Gamma(-l)}(I + A)^{n-l-1}\mathfrak{L}_{n,1}(I + A^*)^{n-l-1} = 0. \end{aligned} \tag{4.7}$$

In the same way, we see that

$$\Delta^{j-1}(x^k)\Delta^{n-j}\mathfrak{R}_{n,1}(x + j - 1)|_{x=+\infty} = 0. \tag{4.8}$$

Since $P_n = \Delta^n(\mathfrak{R}_{n,1})W_1^{-1}$, summing by parts (see (2.3)) and using (4.7) and (4.8), we obtain for $k = 0, \ldots, n-1$:

$$\begin{aligned} \sum_{x=0}^{\infty} x^k P_n(x) W_1(x) &= \sum_{x=0}^{\infty} x^k \Delta^n \mathfrak{R}_{n,1}(x) \\ &= x^k \Delta^{n-1}\mathfrak{R}_{n,1}(x)|_0^{\infty} - \sum_{x=0}^{\infty} \Delta(x^k)\Delta^{n-1}\mathfrak{R}_{n,1}(x+1) \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{x=0}^{\infty} \Delta(x^k)\Delta^{n-1}\mathfrak{R}_{n,1}(x+1) \\
&= \cdots \\
&= (-1)^k \Delta^k(x^k) \sum_{x=0}^{\infty} \Delta^{n-k}\mathfrak{R}_{n,1}(x+k) \\
&= (-1)^k \Delta^k(x^k)\Delta^{n-k-1}\mathfrak{R}_{n,1}(x+k)|_0^{\infty} = 0.
\end{aligned}
$$

Step 3*:* The leading coefficient of P_n is nonsingular.

We write $\hat{P}_n$ for the nth monic orthogonal polynomial with respect to W_1. From Steps 1 and 2, we have $P_n = C_n\hat{P}_n$ with C_n the leading coefficient of P_n. Hence,

$$
\sum_{x\geq 0} P_n(x)W_1(x)x^n = C_n \sum_{x\geq 0} \hat{P}_n(x)W_1(x)x^n = C_n \sum_{x\geq 0} \hat{P}_n(x)W_1(x)\hat{P}_n^*(x) = C_n\langle \hat{P}_n, \hat{P}_n\rangle.
$$

Since $\langle \hat{P}_n, \hat{P}_n\rangle$ is positive definite, we deduce that C_n will be nonsingular if and only if $\sum_{x\geq 0} P_n(x)W_1(x)x^n$ is nonsingular. By using (2.3), we obtain

$$
\begin{aligned}
\sum_{x\geq 0} P_n(x)W_1(x)x^n &= \sum_{x\geq 0} x^n\Delta^n\mathfrak{R}_{n,1}(x) = (-1)^n \sum_{x\geq 0}\Delta^n(x^n)\mathfrak{R}_{n,1}(x+n) \\
&= (-1)^n n! \sum_{x\geq 0}\mathfrak{R}_{n,1}(x+n) \\
&= (-1)^n n! \left(\sum_{x\geq 0} \frac{a^{x+n}}{\Gamma(x+1)} (I+A)^{x+n}\mathfrak{L}_{n,1}(I+A^*)^{x+n} \right).
\end{aligned}
$$

And for $u \in \mathbb{C}^N\backslash\{0\}$

$$
\begin{aligned}
u\left(\sum_{x\geq 0} P_n(x)W_1(x)x^n\right)u^* &= u\left((-1)^n n!\left(\sum_{x\geq 0}\frac{a^{x+n}}{\Gamma(x+1)}(I+A)^{x+n}\mathfrak{L}_{n,1}(I+A^*)^{x+n}dx\right)\right)u^* \\
&= (-1)^n n!\left(\sum_{x\geq 0}\frac{a^{x+n}}{\Gamma(x+1)}u(I+A)^{x+n}\mathfrak{L}_{n,1}(I+A^*)^{x+n}u^*\right).
\end{aligned}
$$

Since $(I+A)^{x+n}\mathfrak{L}_{n,1}(I+A^*)^{x+n}$ is positive definite, we have $u(I+A)^{x+n}\mathfrak{L}_{n,1}(I+A^*)^{x+n}u^* > 0$ for $x = 0, 1, \ldots$ and hence $u(\sum_{x\geq 0} P_n(x)W_1(x)x^n)u^* > 0$. So $\sum_{x\geq 0} P_n(x)W_1(x)x^n$ is nonsingular. ■

We conclude this section with the proof of Lemma 4.1.

Let us start with the case $i = 1$. We will use Lemma 1.1 and proceed in two steps. To simplify the notation, we write $\mathfrak{R}_{n,1} = \mathfrak{R}$, $\mathfrak{L}_{n,1} = \mathfrak{L}_n$, $F_{j,1} = F_j$, $j = -1, 0, 1$ and $\Lambda_{n,1} = \Lambda_n$, where $\mathfrak{R}_{n,1}$, $\mathfrak{L}_{n,1}$, $F_{j,1}$, $j = -1, 0, 1$, $\Lambda_{n,1}$ are defined in Lemma 4.1 and W_1 is defined in (4.5). In this case, we neither use the constraints (1.15) nor the expression (1.17) for the matrix $\mathfrak{L}_{n,1}$. In fact, we only will use that this matrix $\mathfrak{L}_{n,1}$ is independent of x and diagonal.

Step 1. *The hypothesis (1.8) and (1.9) in Lemma 1.1 holds.* This is proved in [6] (see the proof of Theorem 1.2 in p.47–48).

Step 2. *The function* $\mathfrak{R}$ *satisfies the second-order difference equation*

$$\mathfrak{s}_{-1}(\mathfrak{R}F_1^*) + \mathfrak{s}_1\left[\mathfrak{R}\left(\binom{n+1}{2}\Delta^2 F_1^* - n\Delta F_{-1}^* + F_{-1}^*\right)\right] + \mathfrak{R}(F_0^* - n\Delta^2 F_1^* + n\Delta F_1^*) = \Lambda_n \mathfrak{R}.$$

Using (4.1) and that $\mathfrak{R}$ is Hermitian, this second-order difference equation reduces to

$$\mathfrak{s}_{-1}(F_1\mathfrak{R}) + \mathfrak{s}_1[(-n\Delta F_{-1} + F_{-1})\mathfrak{R}] + F_0\mathfrak{R} = \mathfrak{R}\Lambda_n^*. \tag{4.9}$$

A simple computation using the definition of the matrices A and J (see (1.14) and (1.19)) gives the identity

$$J(I+A)^x - (I+A)^x J = xA(I+A)^{x-1}. \tag{4.10}$$

Using it, together with (4.1), we can write

$$\begin{aligned}
\mathfrak{s}_{-1}(F_1\mathfrak{R}) &= \frac{a^x}{\Gamma(x-n)}(I+A)^x \mathfrak{L}_n (I+A^*)^{x-1},\\
\mathfrak{s}_1[(F_{-1} - n\Delta F_{-1})\mathfrak{R}] &= \frac{a^{x+1}}{\Gamma(x-n+1)}(I+A)^x \mathfrak{L}_n (I+A^*)^{x+1},\\
F_0\mathfrak{R} &= -\frac{a^x}{\Gamma(x-n+1)}(I+A)^x (J+xI)\mathfrak{L}_n (I+A^*)^x,\\
\mathfrak{R}\Lambda_n^* &= \frac{a^x}{\Gamma(x-n+1)}(I+A)^x \mathfrak{L}_n (a(I+A^*) - J - (xA^* + nI)(I+A^*)^{-1})(I+A^*)^x.
\end{aligned}$$

Multiplying these identities on the left by $(\Gamma(x-n+1)/a^x)(I+A)^{-x}$ and on the right by $(I + A^*)^{-x}$, we obtain that Equation (4.9) is equivalent to

$$\mathfrak{L}_n((x-n)(I+A^*)^{-1} + a(I+A^*)) - (J+xI)\mathfrak{L}_n = \mathfrak{L}_n(a(I+A^*) - J - (xA^* + nI)(I+A^*)^{-1}).$$

A simple computation shows that this is true as long as we assume that the matrix $\mathfrak{L}_n$ is diagonal.

We now prove the case $i = 2$. In this case, we need to use the restrictions (4.2) for the parameters v_k, $k = 1, \dots, N-1$, and the definition (1.17) for the matrix $\mathfrak{L}_n$. To simplify the notation, we rename $F_{j,2} = F_j$, $j = -1, 0, 1$ and $\Lambda_{n,2} = \Lambda_n$. Denote $\tilde{J}_1 = ((N-1)/a|v_{N-1}|^2)I + J$ and $F_l(x) = F_l^2 x^2 + F_l^1 x + F_l^0$, $l = -1, 0, 1$.

We start with six technical identities we will need later.

$$(xF_{-1}^2 + F_{-1}^1)(I+A)^x = (I+A)^x F_{-1}^1, \tag{4.11}$$

$$(x(F_1^1 - F_{-1}^1) + F_1^0)(I+A)^x = (I+A)^x(F_1^0 - xF_{-1}^1), \tag{4.12}$$

$$(F_1^1 - F_{-1}^1)(I+A)^x = (I+A)^x(F_1^1 - F_{-1}^1) + x(I+A)^{x-1}A, \tag{4.13}$$

$$F_1^2(I+A)^x = (I+A)^x F_1^2, \tag{4.14}$$

$$(\mathfrak{L}_n A - A\mathfrak{L}_n)\tilde{J}_1 = nA\mathfrak{L}_n, \tag{4.15}$$

$$\tilde{J}_1[(I+A^*)\mathfrak{L}_n - \mathfrak{L}_n(I+A^*)] = n\mathfrak{L}_n A^*. \tag{4.16}$$

The identities (4.11)–(4.13) are direct consequences of the identities (5.8)–(5.10) in [7]. The identity (4.14) is obvious since F_1^2 is a function of A. The identity (4.15) is easy to deduce from the definition of $\mathfrak{L}_n$ (1.17) and the constrains (4.2). Finally, the identity (4.16) follows straightforwardly from (4.15).

We now proceed in the same way as in the previous case.

Step 1. *The hypothesis (1.8) and (1.9) in Lemma 1.1 holds.* It straightforwardly follows from Theorem 3 in [7] (with the notation of [7], $G_2 = F_{-1}$, $G_1 = F_1 - F_{-1}$).

Step 2. *The function $\mathfrak{R}$ satisfies the second-order difference equation*

$$\mathfrak{s}_{-1}(\mathfrak{R}F_1^*) + \mathfrak{s}_1\left[\mathfrak{R}\left(\binom{n+1}{2}\Delta^2F_1^* - n\Delta F_{-1}^* + F_{-1}^*\right)\right] + \mathfrak{R}\left(F_0^* - n\Delta^2F_1^* + n\Delta F_1^*\right) = \Lambda_n\mathfrak{R}. \tag{4.17}$$

Since $\mathfrak{R}$ is Hemitian, this second-order difference equation reduces to

$$\mathfrak{s}_{-1}(F_1\mathfrak{R}) + \mathfrak{s}_1\left[\left(\binom{n+1}{2}\Delta^2F_1 - n\Delta F_{-1} + F_{-1}\right)\mathfrak{R}\right] + (F_0 - n\Delta^2F_1 + n\Delta F_1)\mathfrak{R} = \mathfrak{R}\Lambda_n^*. \tag{4.18}$$

Using (4.11) and (4.12), we obtain

$$\mathfrak{s}_{-1}(F_1\mathfrak{R}) = \frac{a^{x-1}}{\Gamma(x-n)}(I+A)^{x-1}F_1^0\mathfrak{L}_n(I+A^*)^{x-1}. \tag{4.19}$$

Using (4.11) and (4.14) and taking into account that $F_{-1}^2 = F_1^2$, we obtain

$$\begin{aligned}\mathfrak{s}_1&\left[\left(\binom{n+1}{2}\Delta^2F_1 - n\Delta F_{-1} + F_{-1}\right)\mathfrak{R}\right]\\ &= \frac{a^{x+1}}{\Gamma(x-n+1)}(I+A)^{x+1}(F_{-1}^1 - nF_1^2)\mathfrak{L}_n(I+A^*)^{x+1}.\end{aligned} \tag{4.20}$$

Using (4.11)–(4.14), we have

$$\begin{aligned}(F_0 - n\Delta^2F_1 + n\Delta F_1)\mathfrak{R} = {}& \frac{a^x}{\Gamma(x-n+1)}(I+A)^x(n(x-1)F_1^2 + nF_1^1 - F_1^0\\ &- xF_{-1}^1 + nx(I+A)^{-1}A)\mathfrak{L}_n(I+A^*)^x.\end{aligned} \tag{4.21}$$

Finally, using (4.13) and taking into account that $\Lambda_n = (n^2-n)F_1^2 + n(F_1^1 - F_{-1}^1)$, we obtain

$$\mathfrak{R}\Lambda_n^* = \frac{a^x}{\Gamma(x-n+1)}(I+A)^x\mathfrak{L}_n[(n^2-n)F_1^2 + n(F_1^1 - F_{-1}^1) + nx(I+A)^{-1}A]^*(I+A^*)^x. \tag{4.22}$$

Multiplying now in (4.18) by $(\Gamma(x-n+1)/a^x)(I+A)^{-x}$ on the left and by $(I+A^*)^{-x}$ on the right, and using the identities (4.19)–(4.22), we obtain that Equation (4.17) is equivalent to

$$\begin{aligned}&(x-n)\tilde{J}(I+A^*)\mathfrak{L}_n(I+A^*)^{-1} + a(nA + (I+A)\tilde{J})\mathfrak{L}_n(I+A^*)\\ &\quad + [n(I-aA) - x\tilde{J} - a(I+A)\tilde{J}(I+A^*)]\mathfrak{L}_n\\ &\quad\quad = n\mathfrak{L}_n[(x-n)A(I+A)^{-1} + I - aA - \tilde{J}]^*.\end{aligned} \tag{4.23}$$

Now, we just need to use (4.16) to obtain that Equation(4.23) holds. So Equation (4.17) holds.

4.1. *Proof of Theorem 1.3*

The second example is the family of $N \times N$ weight matrices

$$W_2 = \sum_{x\ge 0}\frac{a^x}{\Gamma(x+1)}(I+A)^x\Gamma((x+c)I+J)(I+A^*)^x\delta_x, \tag{4.24}$$

where A is the nilpotent matrix defined by (1.14), and the parameters v_i, $i = 1,\dots,N-1$, satisfy the constrains (1.20).

As far as we know, this is the first time this weight matrix appears in the literature.

We have found a couple of linearly independent sets of coefficients $F_{-1,1}, F_{0,1}, F_{1,1}$ and $F_{-1,2}$, $F_{0,2}$, $F_{1,2}$ such that the orthogonal polynomials with respect to W_2 are eigenfunctions of the associated second-order difference operator.

Lemma 4.2 *The orthogonal matrix polynomials with respect to W_2 (4.24) are eigenfunctions of the second-order difference operator*

$$D(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1} + \mathfrak{s}_0(\cdot)F_0 + \mathfrak{s}_1(\cdot)F_1$$

for these two sets of coefficients

$$F_{-1,1} = (I+A)^{-1}x, \quad F_{1,1} = aIx + a(I+A)(cI+J), \quad F_{0,1} = -J - (aI + (I+A)^{-1})x$$

and

$$\begin{aligned}
F_{-1,2} &= [I-(I+A)^{-1}]x^2 + \left(\frac{(N-1)(a-1)}{a|v_{N-1}|^2}I - J\right)x,\\
F_{1,2} &= [I-(I+A)^{-1}]x^2 + a(I+A)\left(\frac{(N-1)(a-1)}{a|v_{N-1}|^2}I - J\right)(cI+J+A^*)\\
&\quad + \left[\left((a-1)\left(\frac{N-1}{|v_{N-1}|^2} - N\right) + a\right)I + (a-2)J + aA(cI+J) - (I+A)^{-1}\right]x,\\
F_{0,2} &= -F_{-1,2} - F_{1,2}.
\end{aligned}$$

Proof We only sketch the proof.

For $i=1$, we can proceed as follows. It is enough to prove that the operator D is symmetric with respect to the weight matrix W_2, and then use Lemma 1.1 in [6] to deduce that the orthogonal polynomials with respect to W_2 are eigenfunctions of D. Using Theorem 2.1 of [6], in order to prove the symmetry of D with respect to W_2, it is enough to prove that

$$\mathfrak{s}_{-1}(F_{1,1}W_2) = W_2F_{-1,1}^*, \quad F_{0,1}W_2 = W_2F_{0,1}^*, \quad W_2(0)F_{-1,1}^*(0) = 0.$$

This can be proved by a careful computation using (4.10).

For $i=2$, we can proceed as follows. We write $G_2 = F_{-1,2}$ and $G_1 = F_{1,2} - F_{-1,2}$. Using Theorem 2 of [7], it is enough to prove that

$$G_2W_2 = W_2G_2^*, \quad \Delta(G_2W_2) = G_1W_2.$$

This can be proved in a similar way as Theorem 3 of [7] but using here the identities

$$\begin{aligned}
G_2(x)(I+A)^x &= (I+A)^x\tilde{J}_2x,\\
G_1(x)(I+A)^x &= (I+A)^x(a(I+A)\tilde{J}_2(cI+J+A^*) + (a-1)x\tilde{J}_2 + axA\tilde{J}_2),
\end{aligned}$$

where $\tilde{J}_2 = ((N-1)(a-1)/a|v_{N-1}|^2)I - J$, instead of the identities (5.8) and (5.9) (used in [7]). ∎

To produce Rodrigues' formula for this example, we consider

$$\mathfrak{R}_{n,2}(x) = \frac{a^x}{\Gamma(x-n+1)}(I+A)^x\Gamma((x+c)I+J)\mathfrak{L}_{n,2}(I+A^*)^x,$$

where the diagonal matrix $\mathfrak{L}_{n,2}$ is given by (1.21).

Theorem 1.3 states that an nth orthogonal polynomial with respect to W_2 is given by the formula

$$P_n = \Delta^n(\mathfrak{R}_{n,2})W_2^{-1}.$$

The proof of Theorem 1.3 is similar as that of Theorem 1.2 and it is omitted. In this case, the choice of the eigenvalues $\Lambda_{n,1}$ and $\Lambda_{n,2}$ (see Lemma 4.1) is the following

$$\Lambda_{n,1} = (a-1)(nI+J) + ac(I+A) + nA(I+A)^{-1} + aAJ,$$
$$\Lambda_{n,2} = n^2A(I+A)^{-1} + n\left(aA(cI+J) + (1-a)\left[(N-1)\left(\frac{1-a}{av_{N-1}^2}+1\right)I - J\right]\right).$$

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