THE CELLULAR STRUCTURE OF THE CLASSIFYING SPACES OF FINITE GROUPS

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Abstract. In this paper we obtain a description of the $B\mathbb{Z}/p$-cellularization (in the sense of Dror-Farjoun) of the classifying spaces of all finite groups, for all primes p.

1. Introduction

Let A be a pointed space. A space X is called A-cellular if it can be constructed as an (iterated) pointed homotopy colimit of copies of A. The concept of A-cellularity was developed by Dror-Farjoun ([Far95]) and Chachólski ([Cha96]) with a two-fold goal: to classify the spaces in cellularity classes, whose properties should depend on those of the generator A, and to develop an A-homotopy theory, where the suspensions of A would play the role that the spheres play in classical homotopy. In this context these authors constructed an A-cellularization endofunctor $\mathbf{CW}_A$ of the category of pointed spaces that mimics the well-known cellular approximation. The functor $\mathbf{CW}_A$ turns out to be augmented and idempotent and is characterized by the facts that $\mathrm{map}_*(A, \mathbf{CW}_A X)$ is weakly homotopy equivalent to $\mathrm{map}_*(A, X)$ and every map $A \to X$ factors through $\mathbf{CW}_A X$ in a unique way, up to homotopy.

These ideas have made a great impact both in Homotopy Theory ([CDI06], [CCS07], [RS08]) and outside of it, because they can be generalized in a natural way to any framework where there is a notion of limit (see examples in [DGI01], [BKI08] or [Kie08]). A line of research of great relevance in the last years, and very related to our work, is cellular approximation in the category of groups ([RS01], [Flo07], [DGS07], [DGS08]).

The present paper culminates a program aimed at understanding the mod p homotopy theory of the classifying spaces of finite groups G using tools from $B\mathbb{Z}/p$-homotopy theory, where p is any prime. The main underlying idea in our study is the relationship between the $B\mathbb{Z}/p$-cellular structure of BG and certain strongly closed

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subgroups of G. Recall that given a finite group G, a subgroup R of some Sylow p-subgroup S of G is said to be *strongly closed* in S with respect to G if whenever $x \in R$ and $g \in G$ are such that $gxg^{-1} \in S$, then $gxg^{-1} \in R$. (All relevant definitions, with some discussion, are collected in Subsection 1.1 at the end of this Introduction; some definitions also appear in the Introduction itself to maintain the flow of its overview nature.)

This paper gives a complete description of the $B\mathbb{Z}/p$-cellularization of classifying spaces of all finite groups. The philosophy behind our work is the following: whenever X is a space with a notion of p-fusion — and hence strong closure — knowledge of the strongly closed subobjects of X is deeply related (and in some cases, almost equivalent) to the A-cellular structure of X, for a certain p-torsion space A. This strategy opens up new perspectives for analyzing, from the point of view of (Co)localization Theory, the p-primary structure of a wide class of homotopy meaningful spaces, such as classifying spaces of compact Lie groups, p-local finite groups, p-compact groups or, more generally, p-local compact groups ([BLO07]).

In the specific case of G finite it became evident from [FS07] that the classification of the possible homotopy types of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ would require some kind of description of the strongly closed subgroups that G can possess. For $p = 2$ this task was completed by the second author in [Foo97], while the case of an odd prime was solved in the separate paper [FF08], which is the group-theoretic underpinning of this work. (The combined classification is rendered in a slightly simplified form as Theorem 5.1 of Section 5.) The p odd classification was a crucial ingredient we lacked in order to finish the characterization of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for all finite groups G (the other was the role of the subgroup $\mathcal{O}_A(G)$, see below), solving a problem that was posed by Dror-Farjoun [Far95, 3.C] in the case $G = \mathbb{Z}/p^r$, and partially solved in [Flo07] and [FS07] (see Section 2 below for an analysis of the previous cases). The latter paper showed the relationship between the upper homotopy of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ and a specific strongly closed subgroup $\mathfrak{A}_1(S)$ of G. More precisely, for S a Sylow p-subgroup of G, $\mathfrak{A}_1(S)$ is the unique minimal strongly closed subgroup of S that contains all elements of order p in S. The importance of the subgroup $\mathfrak{A}_1(S)$ comes from the fact that it determines a great part of the structure of $\mathrm{Hom}(\mathbb{Z}/p, G)$, and then of $\mathrm{map}_*(B\mathbb{Z}/p, BG)$.

We are now in position to explain our main results. Given a finite group G and any p-subgroup A of G then, as shown in Subsection 1.1, there is a unique subgroup of G, denoted by $\mathcal{O}_A(G)$, that is maximal with respect to the two properties:

(i) $\mathcal{O}_A(G)$ is a normal subgroup of G, and

(ii) A contains a Sylow p-subgroup of $\mathcal{O}_A(G)$, i.e., $A \cap \mathcal{O}_A(G) \in Syl_p(\mathcal{O}_A(G))$.

We are especially interested in the case when G is generated by its elements of order p and $A = \mathfrak{A}_1(S)$ for some Sylow p-subgroup S of G. In this situation $A = S$ if and only if $A \leq \mathcal{O}_A(G)$, i.e., if and only if A maps to the trivial group under the natural projection of G onto $G/\mathcal{O}_A(G)$. (Section 5 and references [FS07] and [FF08] give illuminating examples of the subgroups just defined.)

Let $BG^\wedge_p$ denote Bousfield-Kan p-completion of BG. For the definition and properties of this functor, we refer the reader to [BK72] for a thorough account, and to [Flo07, Section 2] for a brief survey. Our first main result—which is Theorem 4.3 in the paper—is the following:

Theorem A. *Let G be a finite group generated by its elements of order p, let $A = \mathfrak{A}_1(S)$ be the minimal strongly closed subgroup of S containing all elements of order p in S, and assume $A \neq S$. Let overbars denote passage from G to the quotient group $G/\mathcal{O}_A(G)$. Then there exists a fibration sequence*

$$\mathbf{CW}_{B\mathbb{Z}/p}(BG^\wedge_p) \longrightarrow BG^\wedge_p \longrightarrow B(N_{\overline{G}}(\overline{A})/\,\overline{A})^\wedge_p.$$

Although it is relatively elementary to prove that for any strongly closed subgroup A the subgroup $N_G(A)$ controls fusion of subgroups containing A, it does not generally control fusion inside A, or for subsets that intersect A but do not contain it. The much more delicate problem is to realize that factoring out the "correct" subgroup $\mathcal{O}_A(G)$ combined with an explicit knowledge of the structure of the quotient $G/\mathcal{O}_A(G)$ is crucial, in the sense that there is a part of the fusion inside A that can be "swept under the rug" for the topological considerations. The main result of the classification in [FF08] is a determination of the isomorphism type of $\overline{G}$, and hence of $N_{\overline{G}}(\overline{A})$, under the hypotheses of Theorem A (and more generally as well); roughly speaking, when $\overline{A} \neq \overline{1}$ the quotient group $\overline{G}$ is a direct product of simple groups from certain explicitly listed families, and in each family the subgroups $\overline{A}$ and $N_{\overline{G}}(\overline{A})$ are explicitly determined as well (a more precise rendition of the classification is given as Theorem 5.1). The full force of this classification is then invoked to yield the fusion analysis needed to complete the proofs of Theorem A and Theorem B: namely, that $N_{\overline{G}}(\overline{A})$ always controls strong fusion in $\overline{A}$.

In the special case when $G/\mathcal{O}_A(G)$ is just a single simple group isomorphic to one of the "obstruction groups" listed in the conclusions of Theorem 5.1, in all but one family the subgroup $N_{\overline{G}}(\overline{S})$ controls strong fusion in $\overline{S}$ (and this was the situation for all "obstructions" when $p = 2$). The Sylow-normalizer is always a subgroup of $N_{\overline{G}}(\overline{A})$, however it can be strictly smaller. In that one exceptional family alone, $N_{\overline{G}}(\overline{S})$ is indeed smaller than $N_{\overline{G}}(\overline{A})$, and the smaller normalizer does *not* control strong fusion in $\overline{A}$. This illustrates that the ultimate classification for p odd involves more subtle, unavoidable configurations than those that arose for $p = 2$. These are explicated in more detail in Section 5 and in [FF08].

To describe the second main result let $\Omega_1(G)$ denote the subgroup of G generated by the elements of order p in G. By Propositions 4.1 and 4.3 of [Flo07], the inclusion $\Omega_1(G) \hookrightarrow G$ induces a homotopy equivalence $\mathbf{CW}_{B\mathbb{Z}/p}B\Omega_1(G) \simeq \mathbf{CW}_{B\mathbb{Z}/p}BG$, so these propositions imply that it is enough to consider the case in which G is generated by elements of order p. Thus the following result—which is Theorem 4.4 herein—combines the information obtained in [Flo07] and [FS07] with the present article to give all possible homotopy structures for $\mathbf{CW}_{B\mathbb{Z}/p}BG$.

Theorem B. *Let G be a finite group generated by its elements of order p, let S be a Sylow p-subgroup of G, and let $A = \mathfrak{A}_1(S)$ be the minimal strongly closed subgroup of S containing all elements of order p in S. Let overbars denote passage from G to the quotient group $G/\mathcal{O}_A(G)$. Then the $B\mathbb{Z}/p$-cellularization of BG has one the following shapes:*

1. *If $G = S$ is a p-group then BG is $B\mathbb{Z}/p$-cellular.*
2. *If G is not a p-group and $A = S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q\neq p} BG^{\wedge}_q$.*
3. *If G is not a p-group and $A \neq S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the map $BG \to B(N_{\overline{G}}(\overline{A})/\overline{A})^{\wedge}_p \times \prod_{q\neq p} BG^{\wedge}_q$.*

The classification of groups containing a strongly closed p-subgroup in [FF08] gives a very precise description of the fiber of the augmentation $\mathbf{CW}_{B\mathbb{Z}/p}BG \to BG$ in terms of normalizers of strongly closed p-subgroups in the simple components of $G/\mathcal{O}_A(G)$. In this sense the results here further improve those of [FS07], where this degree of sharpness was only obtained in the description of some concrete examples.

The overall organization of the paper is as follows. Section 2 begins by recapitulating previous results around the $B\mathbb{Z}/p$-cellularization of BG. Section 3 compiles some facts about the structure of mapping spaces from classifying spaces, which are needed afterwards. Section 4 contains the main results and their proofs; and Section 5 illustrates the efficacy of our methods by describing precisely some families of examples. The latter are very illuminating in the sense that they give an alluring glimpse of what "should be" the $B\mathbb{Z}/p$-cellularization of more general objects.

1.1. **Definitions.** Relevant definitions are collected in this subsection, and in the next subsection we list notation used in this paper. Readers may also repair to Section 5, independent of the intervening sections, to plumb some examples that illustrate the concepts. Throughout the paper G is a finite group, p is a prime and S is a Sylow p-subgroup of G.

A subgroup R of S is called *strongly closed* in G if whenever $x \in R$ and $g \in G$ are such that $gxg^{-1} \in S$, then $gxg^{-1} \in R$ (or equivalently, whenever $\langle\, gxg^{-1}, R \,\rangle$ is a p-group, then $gxg^{-1} \in R$, so this property is independent of S). The *p-socle* of any group H is the subgroup of H generated by its elements of order p — denoted by $\Omega_1(H)$ (here p is fixed even if H is not a p-group). The subgroup $\mathfrak{A}_1(S)$ denotes *the unique smallest subgroup of S that contains $\Omega_1(S)$ and is strongly closed in G.* Examples of strongly closed subgroups are given in Section 5. In particular, if $R = S \cap N$ is a Sylow p-subgroup of a normal subgroup N of G, then R is strongly closed. This observation indicates that strongly closed subgroups often, but not always, signal the presence of normal subgroups. It also leads naturally to the following "functor."

Observe that for R any subgroup of S, if N_1 and N_2 are normal subgroups of G with $R \cap N_i \in Syl_p(N_i)$ for both $i = 1, 2$, then $R \cap N_1N_2$ is a Sylow p-subgroup of N_1N_2. Taking the product of all such N_i, let $\mathcal{O}_R(G)$ denote the *unique largest normal subgroup N of G for which $R \cap N \in Syl_p(N)$.* Thus $\mathcal{O}_R(G)$ is characterized by the two properties given earlier in the Introduction. Moreover,

$$R \text{ is a Sylow } p\text{-subgroup of } \langle\, R^G \,\rangle \text{ if and only if } R \leq \mathcal{O}_R(G).$$

Note that $R\mathcal{O}_R(G)/\mathcal{O}_R(G)$ does not contain the Sylow p-subgroup of any nontrivial normal subgroup of $G/\mathcal{O}_R(G)$; in other words, $\mathcal{O}_{\overline{R}}(\overline{G}) = 1$, where overbars denote passage to $G/\mathcal{O}_R(G)$. In the special case when R is strongly closed in G, observe that strong closure passes to all quotient groups, so when analyzing groups where

$R \nleq \mathcal{O}_R(G)$ we often factor out $\mathcal{O}_R(G)$ (and still have a strongly closed subgroup $\overline{R}$ of $\overline{G}$).

For any $R \leq G$, we say a subgroup H of G containing R *controls fusion in* R if any pair of elements of R that are conjugate in G are also conjugate in H. We likewise say H *controls strong fusion* in R if for every subgroup P of R and $g \in G$ such that $gPg^{-1} \leq R$, there exist $h \in H$ and $c \in C_G(P)$, with $g = hc$.

1.2. **Specific Notation.** Unless explicitly stated otherwise, throughout the paper we adopt the additional notation that the group $A = \mathfrak{A}_1(S)$. (For consistency with cited literature, occasionally the letter A is used as a topological space; but this should not cause confusion because of the context of such deviations.) Overbars denote passage to the quotient $G \to G/\mathcal{O}_A(G)$. The normalizer of $\overline{A}$ in $\overline{G}$ is denoted by $\overline{N}$, and the group $\overline{N}/\overline{A}$ will be called Γ.

For any space X let $X^\wedge_p$ denote the Bousfield-Kan p-completion of X. The cofiber of the map $\bigvee B\mathbb{Z}/p \to BG$ (where the wedge is defined over all the homotopy classes of maps from $B\mathbb{Z}/p$ to BG) is called C. Analogously, the cofiber of the corresponding map $\bigvee B\mathbb{Z}/p \to BG^\wedge_p$ (the wedge defined over classes of maps $B\mathbb{Z}/p \to BG^\wedge_p$) is denoted by D. Let $P = \mathbf{P}_{\Sigma B\mathbb{Z}/p}C$, the $\Sigma B\mathbb{Z}/p$-nullification of C (see the remarks following Theorem 2.1 for the definition); and it was checked in the proof of [FS07, Proposition 3.2] that $\mathbf{P}_{\Sigma B\mathbb{Z}/p}D$ has the homotopy type of $P^\wedge_p$, so we will refer to this object with this name.

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2. Previous results

Before undertaking the complete description of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ we describe what is known so far about this problem. As we said in the Introduction, the starting point was the computation done by Dror-Farjoun in [Far95, 3.C], where he establishes that the $B\mathbb{Z}/p$-cellularization of the classifying space of a finite cyclic p-group has the homotopy type of $B\mathbb{Z}/p$.

Subsequently Rodríguez-Scherer investigated in [RS01] the $M(\mathbb{Z}/p,1)$-cellularization, where $M(\mathbb{Z}/p,1)$ denotes the corresponding Moore space for $\mathbb{Z}/p$. When the target is BG, this can be considered a precursor to our study because $M(\mathbb{Z}/p,1)$ can be described as the 2-skeleton of $B\mathbb{Z}/p$. In their description the authors use the concept of cellularization in the category of groups (developed afterwards in [FGS07]). Their work in this subject allows one to prove, in particular, that the $B\mathbb{Z}/p$-cellularization of the classifying space of a p-group is the same as that of its p-socle; as the latter is $B\mathbb{Z}/p$-cellular in this case ([Flo07, Proposition 4.14]), one obtains that $\mathbf{CW}_{B\mathbb{Z}/p}BG \simeq B\Omega_1(G)$ if $G = S$ is a finite p-group.

The aforementioned is proved using a characterization of the cellularization discovered by Chachólski, that is perhaps the most useful tool available to attack these kind of problems. Because of its importance and ubiquity in our context we reproduce it here:

Theorem 2.1. [Cha96, 20.3] *Let A and X be pointed spaces, and let C be the homotopy cofiber of the map $\bigvee_{[A,X]_*} A \to X$, defined as evaluation over all the homotopy classes of maps $A \to X$. Then $\mathbf{CW}_A X$ has the homotopy type of the homotopy fiber of the composite $X \to C \to \mathbf{P}_{\Sigma A} C$.*

Here $\mathbf{P}$ denotes the nullification functor, first defined by A.K. Bousfield in [Bou94]. Recall that given spaces A and X, X is called A-*null* if the natural inclusion $X \hookrightarrow \operatorname{map}(A, X)$ is a weak equivalence. In this way one defines a functor $\mathbf{P}_A : \mathbf{Spaces} \to \mathbf{Spaces}$, coaugmented and idempotent, such that $\mathbf{P}_A X$ is A-null for every X, and such that for every A-null space Y the coaugmentation induces a weak equivalence $\operatorname{map}(\mathbf{P}_A X, Y) \simeq \operatorname{map}(X, Y)$. This functor can also be defined in the pointed category, and its main properties can be found in [Far95] and [Cha96].

In our case the role of A and X in Theorem 2.1 will be played by $B\mathbb{Z}/p$ and BG, respectively. If C is the corresponding cofiber, from now on we shall denote the $\Sigma B\mathbb{Z}/p$-nullification of C by P. As a consequence of Theorem 2.1, describing $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is equivalent to describing P, which is in general a more accessible problem. Next we list some properties of the space P which will be useful in the remaining of our note.

Proposition 2.2. *Let P be as described in the previous paragraph, with G a finite group generated elements by order p. Then the following hold:*

(1) *The spaces P and $P^\wedge_p$ are simply-connected.*
(2) *$P^\wedge_p$ is $H\mathbb{Z}/p$-local.*
(3) *The loops of P and $P^\wedge_p$ are $B\mathbb{Z}/p$-null spaces.*

(4) *P admits a splitting $(\prod_q BG^\wedge_q) \times P^\wedge_p$, where the product is taken over all primes $q \neq p$; moreover $P^\wedge_p$ coincides with the base of the fibration sequence of Theorem 2.1 when the total space is $BG^\wedge_p$.*

Proof. (1) C is by definition a homotopy push-out $* \leftarrow \bigvee B\mathbb{Z}/p \rightarrow BG$, and the generation hypothesis on G implies that the right map is surjective at the level of fundamental groups. Hence by Seifert-Van Kampen C is simply-connected, and then P is so by [Dwy96, Proposition 6.9]. In particular, by [BK72, VII,3.2], $P^\wedge_p$ is also 1-connected.

(2) According to [BK72, VI.5.1] and [Bou75, Proposition 4.3], the homotopy groups of $P^\wedge_p$ are $H\mathbb{Z}/p$-local in the sense of Bousfield, and hence $P^\wedge_p$ is $H\mathbb{Z}/p$-local (as a space) by [Bou75, Theorem 5.5].

(3) By adjunction, $\mathrm{map}_*(B\mathbb{Z}/p, \Omega P)$ is weakly equivalent to $\mathrm{map}_*(\Sigma B\mathbb{Z}/p, P)$, which is weakly trivial, and then ΩP is $B\mathbb{Z}/p$-null. The fibre lemma [BK72, II.5.1] applied to the loop fibration of P implies that $\Omega(P^\wedge_p) \simeq (\Omega P)^\wedge_p$, and the conclusion follows from [Mil84, Lemma 9.9].

(4) See Proposition 2.1 and the proof of Proposition 3.2 in [FF08].

□

Note that the additional hypothesis on the generation of G causes no restriction from the point of view of cellularization, as for every finite G there is a homotopy equivalence $\mathbf{CW}_{B\mathbb{Z}/p}B\Omega_1(G) \simeq \mathbf{CW}_{B\mathbb{Z}/p}BG$ induced by the natural inclusion ([Flo07, Proposition 4.1]). So, one should have in mind that the knowledge of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for G generated by order p elements implies automatically the knowledge of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for every G.

According to [FS07, 3.2], the last statement of our previous Lemma implies that we are actually studying the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$. More precisely, if G is generated by order p elements, the equivalence $\mathbf{CW}_{B\mathbb{Z}/p}(BG^\wedge_p) \simeq (\mathbf{CW}_{B\mathbb{Z}/p}BG)^\wedge_p$ is proved in [FS07, Proposition 3.2]. This is a very peculiar property for spaces which, in general, cannot be decomposed via an arithmetic square.

In the philosophy of [BLO03], the homotopy theory of $BG^\wedge_p$ is codified in the p-fusion data of G. From this point of view it can be observed that the structure of $P^\wedge_p$ strongly depends on the minimal strongly closed p-subgroup $\mathfrak{A}_1(S)$ of S that contains the p-socle of S (called $Cl\ S$ in [FS07]). In particular, it is a consequence of the Puppe sequence and the definition of nullification (see [FS07, Proposition 4.2] for details) that if $\mathfrak{A}_1(S) = S$ then $P_p^\wedge$ is trivial. This leads one to consider the case in which $\mathfrak{A}_1(S)$ is strictly contained in S.

In [FS07] the latter case is studied under the additional assumption that $N_G(S)$ controls (strong) G-fusion in S. Since $\mathfrak{A}_1(S)$ is normal in $N_G(S)$, [FS07] shows that $P_p^\wedge$ is homotopy equivalent to the p-completion of $B(N_G(S)/\mathfrak{A}_1(S))$; this also shows, roughly speaking, that the structure of the mapping space $\mathrm{map}_*(B\mathbb{Z}/p, BG_p^\wedge)$ depends heavily on $\mathfrak{A}_1(S)$. This result is used, in particular, to compute the $B\mathbb{Z}/2$-cellularization of classifying spaces of simple groups (relying on Theorem 5.6 there).

The explicit description of $P_p^\wedge$ in the remaining cases, which lead in turn to a complete knowledge of the structure of the $B\mathbb{Z}/p$-cellularization of BG for every finite group G, is given in Section 4. Before, however, we need to deal with some mapping spaces which play a crucial role in the description.

3. Maps from classifying spaces and Zabrodsky Lemma

It was already evident in our previous work [FS07] that a complete knowledge of the homotopical structure of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ demands, for some spaces X, a certain control of the behavior of the functor $\mathrm{map}_*(-, X)$ when applied to fiber sequences of classifying spaces of finite groups. This is done in part by the classical Zabrodsky Lemma, which we recall here for the reader's convenience, from a recent version of W. Dwyer. We also present in this section a specific version of this statement, which will be useful in the proof of our key result (Theorem 4.3).

Let $F \to E \to B$ be a fibre sequence, with B connected. Denote by $\mathrm{map}(F, X)_{[F]}$ the space of maps $F \to X$ which are null-homotopic, by $\mathrm{map}(E, X)_{[F]}$ the space of maps $E \to X$ which become null-homotopic when restricted to F, and by $[E, X]_{[F]}$ the group of components of $\mathrm{map}(E, X)_{[F]}$, which is identified with the homotopy classes of maps $E \to X$ which restrict trivially to F; observe in particular that all these concepts have their pointed counterparts. Then we have the following:

Theorem 3.1 ([Dwy96], Proposition 3.5). *If X is a space and the inclusion $X \to \mathrm{map}(F, X)_{[F]}$ is an equivalence, then the restriction $\mathrm{map}(B, X) \to \mathrm{map}(E, X)_{[F]}$ is an equivalence too.*

Before introducing our version of this result, which is focused on the case in which the fibre sequence is derived from an extension of groups, we recall some well known properties of the mapping spaces and the $\Sigma B\mathbb{Z}/p$-null spaces. This is undertaken in the next two lemmas.

Lemma 3.2. *Let A and X be spaces with X simply-connected. Then there is a bijection $[A, X]_* \to [A, X]$ between the set of components of the respective mapping spaces* $\text{map}_*(A, X)$ *and* $\text{map}(A, X)$.

Proof. It is enough to consider the long exact homotopy sequence of the classical fibre sequence

$$\text{map}_*(A, X) \to \text{map}(A, X) \to X,$$

and to take account of the fact that $\pi_0\text{map}_*(A, X) = [A, X]_*$ and $\pi_0\text{map}(A, X) = [A, X]$. □

Lemma 3.3. *Let p be a prime. If G is a compact Lie group and X is a simply-connected p-complete, $\Sigma B\mathbb{Z}/p$-null space, then X is ΣBG-null.*

Proof. Recall that by definition of nullification, a space X is ΣA-null for a certain space A if and only if ΩX is A-null. Then ΩX is p-complete and $B\mathbb{Z}/p$-null, so according to [Dwy96, Theorem 1.2] it is BG-null. Hence X is ΣBG-null. □

We may now state our specific version of Zabrodsky's Lemma:

Proposition 3.4. *Let $G_1 \to G_2 \xrightarrow{h} G_3$ be an extension of finite groups, $BG_1 \to BG_2 \to BG_3$ the corresponding fibre sequence of classifying spaces. If X is a simply-connected, p-complete $\Sigma B\mathbb{Z}/p$-null space for a prime p, f induces a bijection $[BG_3, X] \simeq [BG_2, X]_{[BG_1]}$, which remains true in the pointed category.*

Proof. We follow the proof of [CCS07, Lemma 2.3]. By the previous Lemma 3.3, X is ΣBG_1-null, so by definition $\text{map}_*(\Sigma BG_1, X)$ is contractible, and in particular the component of the constant in $\text{map}_*(BG_1, X)$ is so. Then, the restriction of the fibre sequence of the proof of 3.2 (with $A = BG_1$) to the components of constant maps gives that the evaluation $\text{map}(BG_1, X)_{[BG_1]} \to X$ is an equivalence. Now Theorem 3.1 implies that the map $\text{map}(BG_3, X) \to \text{map}(BG_2, X)_{[BG_1]}$ induced by $BG_2 \xrightarrow{Bh} BG_3$ is also an equivalence, and therefore $[BG_3, X] \simeq [BG_2, X]_{[BG_1]}$. As X is simply-connected, Lemma 3.2 ensures that the bijection remains true for classes of pointed maps, so we are done. □

This proposition can be read, in particular, as an extension property:

Corollary 3.5. *In the conditions of the previous proposition, if* $f : BG_2 \to X$ *is a map in* $\mathrm{map}(BG_2, X)_{[BG_1]}$*, then* f *factors through a map* $g : BG_3 \to X$ *that is unique up to unpointed homotopy. Moreover, if we assume* f *to be pointed, then the factorization is unique up to pointed homotopy.*

Remark 3.6. *Our usual candidates for* X *will be* p*-completions of classifying spaces of* p*-perfect finite groups, which are simply-connected* [BK72, Proposition II.5]*,* p*-complete* [BK72, Proposition II.5] *and* $\Sigma B\mathbb{Z}/p$*-null* [Mil84, Lemma 9.9]*, the latter because* BG *is always so. We will make regular use of this fact.*

We include a group-theoretic lemma that will be needed in the proof of Proposition 3.8 following, and also in the description of the structure of $A = \mathfrak{A}_1(S)$ in Section 4.

Lemma 3.7. *Let* T *be a nontrivial* p*-group, let* $\{x_1, \dots, x_k\}$ *be any minimal set of generators of* T*, and let* H *be the normal subgroup generated by all* T*-conjugates of* $x_1, \dots, x_{k-1}$*. Then* H *is a proper subgroup of* T *with* $T = H\langle x_k \rangle$*, and* T/H *is isomorphic to a (cyclic) quotient group of* $\langle x_k \rangle$*. In particular, if all* x_i *have order* p*, then* $H = \Omega_1(H)$ *and* $T/H \cong \mathbb{Z}/p$*.*

Proof. Let $x_1, \dots, x_k$ be a minimal set of generators of T, let $H_0 = \langle x_1, \dots, x_{k-1} \rangle$, and let H be the normal closure of H_0 in T. By minimality of k we have $H_0 < T$. Then H_0 lies in some maximal subgroup, M, of T, and by basic p-group theory $M \trianglelefteq T$; thus we have $H \le M < T$ too. Since $T = \langle H_0, x_k \rangle$ we get $T = H\langle x_k \rangle$; and so T/H is isomorphic to a quotient of $\langle x_k \rangle$. This proves the first assertion.

If all x_i have order p, then since H is generated by conjugates of these, $H = \Omega_1(H)$, and the second assertion of the lemma now follows from the first. □

We finish the section with an adaptation of Proposition 2.4 in [CCS07], which will be necessary when dealing with maps whose source is a strongly closed subgroup.

Proposition 3.8. *Let* $P \xrightarrow{i} Q \xrightarrow{p} G$ *be an extension of finite groups,* X *a simply-connected* $\Sigma B\mathbb{Z}/p$*-null* p*-complete space, and* $f : BQ \to X$ *a map that restricts trivially to* BP*. Assume that* G *is a* p*-group generated by a collection* C *of elements such that, for every* $x \in C$*, there exists* $y \in Q$ *such that* $p(y) = x$ *and the restriction of* f *to* $B\langle y \rangle$ *is null-homotopic. Then* f *is null-homotopic.*

Proof. According to the previous corollary, there exists an extension $\hat{f} : BG \to X$ of f such that f is null-homotopic if and only if $\hat{f}$ is. We will check the latter by induction on the order of the group G. If $G = \langle x \rangle$ with x an element of order p, then the composite $B\langle y \rangle \to B\langle x \rangle \to X$ is null-homotopic, and $B\langle x \rangle \to X$ is so by applying Corollary 3.5 to the fibration $BK \to B\langle y \rangle \to B\langle x \rangle$, with K the kernel of $\langle y \rangle \to \langle x \rangle$. So we are done in this case.

For the general case, apply Lemma 3.7 to $T = G$ and the set $C = \{x_1, \ldots, x_k\}$, which we may assume is a minimal set of generators. We obtain a proper normal subgroup H of G and an extension $H \to G \to \mathbb{Z}/p^r$ with $r \geq 1$, where the quotient is generated by the image of $x = x_k$. Let Q' be the pull-back of the extension of the statement along the inclusion $H \to G$, let $Q' \to Q$ be the corresponding induced homomorphism, and let $h : BQ' \to BQ$ be the induced map between classifying spaces. Now H and the extension $P \to Q' \to H$ satisfy the assumptions in the statement, and moreover $|H| < |G|$ so by induction the composition $f \circ h$ is trivial, and the restriction of $\hat{f}$ to BH is so. Now consider the diagram:

$$\begin{array}{ccccc}
B(\langle x \rangle \cap X) & \longrightarrow & BH & & \\
\downarrow & & \downarrow & \overset{x}{\searrow} & \\
B(\langle x \rangle) & \longrightarrow & BG & \xrightarrow{\hat{f}} & X \\
\downarrow & & \downarrow & \underset{f'}{\nearrow} & \\
B\mathbb{Z}/p^r & \xrightarrow{Id} & B\mathbb{Z}/p^r & &
\end{array}$$

By Corollary 3.5 applied to the second vertical fibre sequence, we only need to see that f' is trivial, and again by the same result applied now to the first, we obtain that f' is trivial because $\hat{f}$ is trivial too. So $\hat{f}$ is null-homotopic and we are done. □

4. Describing the $B\mathbb{Z}/p$-cellularization of BG

Unless otherwise stated, throughout this section G will always be a finite group generated by its elements of order p. As said above, this is a technical reduction which does not affect the generality of our main result. We continue to use notation specified at the end of Section 1.

In [FS07] it was already anticipated that a complete description of the $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups would depend on a structure theorem for groups that contain a non-trivial strongly closed p-subgroup that is not a Sylow p-subgroup; at that time such a classification was only known for $p = 2$ ([Foo97]). But even in this case there were examples of groups such that $\mathfrak{A}_1(S) \neq S$ and $N_G(S)$ does not control fusion in S — in other words, groups that were beyond the scope of [FS07].

The key step missing in that paper was the role of the subgroup $\mathcal{O}_A(G)$, whose importance was already evident, in an independent group-theoretic context, in [Foo97]. First, $\mathcal{O}_A(G)$ is by definition a subgroup of G, so in the case $A = \mathfrak{A}_1(S)$ it is likely that it controls a "part" of the structure of $\mathbf{CW}_{B\mathbb{Z}/p}BG$. Moreover, $\mathcal{O}_A(G)$ is normal in G, so one might expect a strong relationship between the cofiber of Chachólski fibration in Theorem 2.1 for $BG^\wedge_p$ and for $B(G/\mathcal{O}_A(G))^\wedge_p$. The significance of $\mathcal{O}_A(G)$ is also suggested by the fact that passing to the quotient $G/\mathcal{O}_A(G)$ gives a considerable simplification in the p-fusion structure of this quotient group in the sense that — as we shall see in Section 5 — there are normalizers (of subgroups/elements) that do not control fusion in G with images that do so in the quotient. These ideas, combined with the explication of the role of $\mathcal{O}_A(G)$ in the classification result [FF08, Theorem 1.2], allow us to prove the main theorem, Theorem 4.3, which covers all extant cases for $\mathbf{CW}_{B\mathbb{Z}/p}BG$, subsuming all by a uniform treatment.

At this point it is worth recalling from [FS07] the inductive construction of A. For S a Sylow p-subgroup of G define $Cl_0(S) = \Omega_1(S)$, the subgroup of S generated by its elements of order p. Let $Cl_{i+1}(S)$ be built from $Cl_i(S)$ as the group generated by all elements $gxg^{-1} \in S$ with $g \in G$ and $x \in Cl_i(S)$. Then A is the (finite) union $\cup_i Cl_i(S)$, and hence is the unique minimal strongly closed subgroup of S containing all elements of order p in S. We have the following:

Lemma 4.1. *Let p be a prime, X a simply-connected p-complete, $\Sigma B\mathbb{Z}/p$-null space, and let $\gamma : BA \to X$ be a map that restricts trivially to every map $B\mathbb{Z}/p \to BA$. Then γ is also null-homotopic.*

Proof. Suppose first that A is generated by elements of order p. We use induction on the order of A. If $|A| = p$, the result is clear. If not, suppose the result is true for every group A_1 such that $\Omega_1(A_1) = A_1$ and $|A_1| < |A|$. As A is a p-group generated by elements of order p, Lemma 3.7 gives an extension $H \to A \to A/H$ such that $H = \Omega_1(H)$, $H \trianglelefteq A$ and $A/H = \mathbb{Z}/p$. Then, applying the induction hypothesis and Corollary 3.5 to the fibre sequence $BH \to BA \to B(A/H)$, we obtain that, up to homotopy, there is an extension

$$\begin{array}{ccc} BA & \longrightarrow & X \\ \downarrow & \nearrow & \\ B(A/H) & & \end{array}$$

and the horizontal map is trivial if and only if the diagonal one is so. But the latter holds by the induction hypothesis applied now to $A/H = \mathbb{Z}/p$, so we are done with this case.

Consider now the general case, with A not necessarily generated by elements of order p. We must see that γ is trivial when restricting to $Cl_i(S)$ for every i. The statement is clear for $i = 0$, because this subgroup is generated by elements of order p. Assume by induction that the restriction to $BCl_i(S)$ for a certain fixed i is null-homotopic, and consider the fibre sequence $BCl_i(S) \to BCl_{i+1}(S) \to B(Cl_{i+1}(S)/Cl_i(S))$, which is defined because $Cl_i(S) \trianglelefteq Cl_{i+1}(S)$. By induction the composite $BCl_i(S) \to BCl_{i+1}(S) \to A \xrightarrow{\gamma} X$ is inessential. Now Corollary 3.5 applied to the previous fibre sequence gives that the map $BCl_{i+1}(S) \to X$ factors through a map $s : B(Cl_{i+1}(S)/Cl_i(S)) \to X$, and it is null-homotopic if and only if s is. But generators of the quotient group $Cl_{i+1}(S)/Cl_i(S)$ are classes of conjugates of elements of $Cl_i(S)$ on which γ restricts trivially, so s is homotopic to the constant when restricted to these generators. Then the previous fibre sequence satisfies the conditions of Lemma 3.8, and therefore s is null-homotopic, as needed to complete the induction. □

In the proof of the Theorem 4.3, we will also need a group-theoretic property of the quotient group $\Gamma = \overline{N}/\overline{A}$.

Lemma 4.2. *The group Γ is p-perfect, i.e., has no normal subgroup of index p.*

Proof. By contradiction, assume that Γ is not p-perfect, and let $\psi : \overline{N} \to \mathbb{Z}/p$ be a surjection that sends $\overline{A}$ to the identity. As $\overline{N}$ controls fusion in $\overline{G}$ and the inclusion $\overline{N} < \overline{G}$ is a mod p homology isomorphism, ψ factors through a homomorphism on $\overline{G}$. Since G is generated by its elements of order p and A contains all elements of order p in some Sylow p-subgroup of G, no proper normal subgroup of G contains A, and then no proper normal subgroup of $\overline{G}$ contains $\overline{A}$. This is a contradiction, and we are done. □

Alternatively, Theorem B of [Gol75] says that for any strongly closed A we have $(G' \cap S)A = (N_G(A)' \cap S)A$. Thus if G has no normal subgroup of index p containing A, neither does $N_G(A)/A$. Corollary 1.5 in [FF08] therefore also shows Γ is p-perfect since each L_i is simple (see also Corollary 5.2).

Now we are in a position to prove the principal result of this section. Recall that G is generated by elements of order p, and P denotes the $\Sigma B\mathbb{Z}/p$-nullification of the homotopy cofibre of the map $\bigvee B\mathbb{Z}/p \to BG$, where the wedge is taken over all the homotopy classes of maps $B\mathbb{Z}/p \to BG^\wedge_p$. We call D this homotopy cofibre. Again overbars denote passage to the quotient $G \to G/\mathcal{O}_A(G)$, A is the unique smallest subgroup of the p-Sylow $S < G$ that contains $\Omega_1(S)$ and is strongly closed in G, and $\overline{N}$ is the normalizer of $\overline{A}$ in $\overline{G}$.

Theorem 4.3. *In the previous notation, $P^\wedge_p$ is homotopy equivalent to the p-completion of the classifying space of $\overline{N}/\overline{A}$.*

Proof. We denote by $h : BG^\wedge_p \to D$ the natural map, and by $\eta : D \to P^\wedge_p$ the canonical coaugmentation. Moreover, if $A_1 < A_2$, we will call i_{A_1,A_2} the group inclusion $A_1 \hookrightarrow A_2$. Unless it is stated otherwise, we assume throughout this proof that we work in the pointed category. Observe that when the target space X is p-complete, $\Sigma B\mathbb{Z}/p$-null and simply-connected, we can use the "pointed version" of the Zabrodsky Lemma given by Proposition 3.4 and Corollary 3.5.

We claim there are maps $B\Gamma^\wedge_p \to P^\wedge_p$ and $P^\wedge_p \to B\Gamma^\wedge_p$ that are homotopy inverses to one another. First we define $P^\wedge_p \xrightarrow{g} B\Gamma^\wedge_p$. Recall that, as $\overline{N}$ controls $\overline{G}$-fusion in $\overline{S}$, the inclusion $\overline{N} \hookrightarrow \overline{G}$ induces a homotopy equivalence $B\overline{N}^\wedge_p \overset{(Bi_{\overline{N},\overline{G}})^\wedge_p}{\simeq} B\overline{G}^\wedge_p$ (see for example [MP98, Proposition 2.1]) . Now consider the diagram

(4.1)
$$\begin{tikzcd}
\bigvee B\mathbb{Z}/p \arrow[r, "v"] & BG^\wedge_p \arrow[r, "h"] \arrow[d, "B\pi^\wedge_p"'] & D \arrow[r, "\eta"] \arrow[dddl, dashed, "g'"] & P^\wedge_p \arrow[dddll, dashed, "g"] \\
& B\overline{G}^\wedge_p & & \\
& B\overline{N}^\wedge_p \arrow[u, "(Bi_{\overline{N},\overline{G}})^\wedge_p", "\simeq"'] \arrow[d, "Bp^\wedge_p"'] & & \\
& B\Gamma^\wedge_p & &
\end{tikzcd}$$

and call α the composite of all vertical maps.

According to the definition of $\overline{G}$ and A, the composite $B\mathbb{Z}/p \to BG_p^\wedge \xrightarrow{\alpha} B\Gamma_p^\wedge$ is inessential for every map $B\mathbb{Z}/p \to BG_p^\wedge$. This implies that the composite $\alpha \circ v$ is so, and hence there exists the lifting g'. Now Remark 3.6 and the universal properties of p-completion and $\Sigma B\mathbb{Z}/p$-nullification imply that g' also lifts to g, and that is the map we were looking for.

To construct $f : B\Gamma_p^\wedge \to P_p^\wedge$ consider now the composite $BA \xrightarrow{(Bi_{A,G})_p^\wedge} BG_p^\wedge \xrightarrow{\eta\circ h} P_p^\wedge$. As the induced homomorphism of fundamental groups is trivial by construction when restricted to every element of order p of A, Lemma 4.1 implies that the composite is null-homotopic. In particular, it is also inessential when precomposing with the map $B(\mathcal{O}_A(G)\cap A) \xrightarrow{Bi_{\mathcal{O}_A(G)\cap A,A}} BA$. As $P_p^\wedge$ is p-complete (by [FS07, 3.2]), $\mathcal{O}_A(G)\cap A$ is p-Sylow in $\mathcal{O}_A(G)$ by definition of $\mathcal{O}_A(G)$, and the hypothesis of [Dwy96, Theorem 1.4] hold by Proposition 2.2, we obtain that the composite $B\mathcal{O}_A(G) \xrightarrow{(Bi_{\mathcal{O}_A(G),G})_p^\wedge} BG_p^\wedge \longrightarrow P_p^\wedge$ is again homotopically trivial. Then by Proposition 3.4 applied to the vertical fibre sequence below, there exists a lifting f'

$$\begin{array}{ccc} B\mathcal{O}_A(G) & & \\ \Big\downarrow{\scriptstyle Bi_{\mathcal{O}_A(G),G}} & & \\ BG & \xrightarrow{\eta\circ h\circ(-)_p^\wedge} & P_p^\wedge \\ \Big\downarrow{\scriptstyle B\pi} & \nearrow{\scriptstyle f'} & \\ B\overline{G} & & \end{array}$$

making the triangle homotopy commutative; here $(-)_p^\wedge$ denotes the p-completion $BG \to BG_p^\wedge$. Now we have another map $B\overline{N} \to P_p^\wedge$ given by the composite

$$B\overline{N} \longrightarrow B\overline{G} \xrightarrow{f'} P_p^\wedge,$$

where we have used the fact that $P_p^\wedge$ is p-complete.

The next diagram is clearly commutative by construction:

$$\begin{array}{ccccccc} BA & & \xrightarrow{\quad Bi_{A,G} \quad} & & BG & & \\ \Big\downarrow{\scriptstyle B\pi_{|A}} & & & & \Big\downarrow{\scriptstyle B\pi} & \searrow & \\ B\overline{A} & \xrightarrow{Bi_{\overline{A},\overline{N}}} & B\overline{N} & \xrightarrow{Bi_{\overline{N},\overline{G}}} & B\overline{G} & \longrightarrow & P_p^\wedge. \end{array}$$

Note that the composite $BA \xrightarrow{Bi_{A,G}} BG \xrightarrow{\eta\circ h\circ(-)^\wedge_p} P^\wedge_p$ is homotopically trivial, and hence the composite $BA \longrightarrow B\overline{A} \xrightarrow{Bi_{\overline{A},\overline{N}}} B\overline{G} \xrightarrow{f'} P^\wedge_p$ is too. By Proposition 3.4 the composite $B\overline{A} \xrightarrow{Bi_{\overline{A},\overline{N}}} B\overline{N} \xrightarrow{Bi_{\overline{N},\overline{G}}} B\overline{G} \xrightarrow{f'} P^\wedge_p$ is also homotopically trivial, and thus again by Corollary 3.5 applied to the extension $\overline{A} \to \overline{N} \to \Gamma$, the map factors through $B\Gamma$. As $P^\wedge_p$ is p-complete, we obtain a map f that fits in the following commutative diagram:

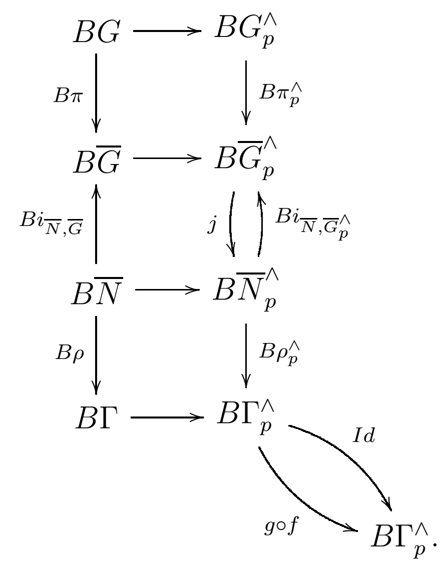
$$\begin{array}{ccc} B\overline{N}^\wedge_p & \xrightarrow{(f')^\wedge_p} & P^\wedge_p \\ \downarrow & \nearrow_{f} & \\ B\Gamma^\wedge_p. & & \end{array}$$

This is the map f that we wanted. As $B\overline{N}^\wedge_p$ is homotopy equivalent to $B\overline{G}^\wedge_p$, we have in particular another commutative diagram, where α is defined in the obvious way by composing a homotopy inverse of the previous equivalence with the projection $B\pi$:

$$\begin{array}{ccc} BG^\wedge_p & \xrightarrow{\eta\circ h} & P^\wedge_p \\ {\scriptstyle\alpha}\downarrow & \nearrow_{f} & \\ B\Gamma^\wedge_p. & & \end{array} \tag{4.2}$$

It remains to prove that $f\circ g \simeq \mathrm{Id}_{P^\wedge_p}$ and $g\circ f \simeq \mathrm{Id}_{B\Gamma^\wedge_p}$. In the first case, the universal property of the nullification functor implies that it is enough to prove that $f \circ g'$ is homotopic to the coaugmentation η. Now we only need to prove that $f \circ g' \simeq \eta$; and moreover, as $P^\wedge_p$ is also $\Sigma B\mathbb{Z}/p$-null, we can use the Puppe sequence of the cofibre sequence $BG^\wedge_p \xrightarrow{h} D \to \vee\Sigma B\mathbb{Z}/p$ to establish that $f \circ g' \simeq \eta$ if and only if $f \circ g' \circ h \simeq \eta \circ h$. According to diagram 4.1, $g' \circ h$ is homotopic to α, and by diagram 4.2, $f \circ \alpha \simeq \eta \circ h$, so we are done.

Let us see that $g \circ f$ is homotopic to the identity of $B\Gamma^{\wedge}_p$. We have the following commutative diagram:



Here the horizontal arrows represent the p-completion map $(-)^{\wedge}_p$, the homomorphism $\rho : \overline{N} \to \Gamma$ is the canonical projection, and j is a homotopy inverse of the equivalence $B\overline{N}^{\wedge}_p \overset{Bi_{\overline{N},\overline{G}}}{\simeq} B\overline{G}^{\wedge}_p$. As $B\Gamma^{\wedge}_p$ is already p-complete, the universal property of p-completion show that is enough to check that $g \circ f \circ (-)^{\wedge}_p$ is homotopic to $(-)^{\wedge}_p$. Now, applying Corollary 3.5 to the fibre sequence $B\overline{A} \to B\overline{N} \xrightarrow{B\rho} B\Gamma$ (with $X = B\Gamma^{\wedge}_p$) we obtain that $g \circ f \circ (-)^{\wedge}_p \simeq (-)^{\wedge}_p$ if and only if $g \circ f \circ (-)^{\wedge}_p \circ B\rho \simeq (-)^{\wedge}_p \circ B\rho$. Again because $B\Gamma^{\wedge}_p$ is p-complete, the latter is equivalent to prove that the map $B\rho^{\wedge}_p$ is homotopic to $g \circ f \circ B\rho^{\wedge}_p$. Again by completeness and Corollary 3.5 applied to the fibration $B\mathcal{O}_A(G) \to BG \xrightarrow{B\pi} B\overline{G}$, it is seen that we should check that $B\rho^{\wedge}_p \circ j \circ B\pi \simeq B\rho^{\wedge}_p \circ j \circ B\pi \circ g \circ f$.

Now it is clear by construction of α that $B\rho^{\wedge}_p \circ j \circ B\pi^{\wedge}_p$ is homotopic to α. Therefore, we need to see that $g \circ f \circ \alpha \simeq \alpha$. By diagram 4.2 the latter is homotopic to $g \circ \eta \circ h$, and has the same homotopy class as α by diagram 4.1. So we are done. □

When we combine the previous statement with the last statement of Proposition 2.2 we obtain a complete description of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for every p and every finite group G.

Theorem 4.4. *Let G be a finite group generated by its elements of order p, let $S \in Syl_p(G)$, and let $A = \mathfrak{A}_1(S)$ be the minimal strongly closed subgroup of S containing*

$\Omega_1(S)$. Let overbars denote passage from G to the quotient group $G/\mathcal{O}_A(G)$. Then the $B\mathbb{Z}/p$-cellularization of BG has one the following shapes:

1. *If $G = S$ is a p-group then BG is $B\mathbb{Z}/p$-cellular.*
2. *If G is not a p-group and $A = S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q\neq p} BG^\wedge_q$.*
3. *If G is not a p-group and $A \neq S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the map $BG \to B(N_{\overline{G}}(\overline{A})/\overline{A})^\wedge_p \times \prod_{q\neq p} BG^\wedge_q$.*

Remark 4.5. *It is worth recalling here that if G is not generated by order p elements, its $B\mathbb{Z}/p$-cellularization can be computed by using the aforementioned homotopy equivalence $\mathbf{CW}_{B\mathbb{Z}/p}B\Omega_1(G) \simeq \mathbf{CW}_{B\mathbb{Z}/p}BG$ induced by inclusion, and then applying the previous theorem to $\Omega_1(G)$.*

Theorems 1.1 and 1.2 and Corollary 1.5 in [FF08] determine $N_G(\overline{A})/\overline{A}$, whose structure is very rigid and depends on a restricted set of well-known simple groups. It is very likely that an analogous classification can be obtained exactly in the same way for $\mathbf{CW}_{B\mathbb{Z}/p^r}BG$, $r > 1$, but we have restricted ourselves to the case $r = 1$ for the sake of simplicity (cf. also [CCS07, Theorem 3.6]).

In the cases where $\mathcal{O}_A(G) = 1$ and $A \trianglelefteq G$ — which are implicit in the computations — the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$ is the homotopy fiber of the natural map $BG^\wedge_p \to B(G/A)^\wedge_p$. It is then tempting to identify $\mathbf{CW}_{B\mathbb{Z}/p}BG$ with BA. But this would mean, in particular, that $\mathrm{map}_*(BZ/p, BG^\wedge_p)$ would be discrete. However, an analysis of the fibration sequence

$$\mathrm{map}_*(B\mathbb{Z}/p, BG^\wedge_p) \to \mathrm{map}(B\mathbb{Z}/p, BG^\wedge_p) \to BG^\wedge_p$$

together with the description of its total space — which is given, for example, in [BK02, Appendix] — shows that $\mathrm{map}_*(B\mathbb{Z}/p, BG^\wedge_p)$ is non-discrete in general, and then usually $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is not an aspherical space.

It is conceivable that our results can also have interesting consequences from the point of view of homotopical representations of groups. In [FS07, Section 6] the results on cellularization gave rise to specific examples of nontrivial maps $BG \to BU(n)^\wedge_p$ that enjoyed two particular properties: they did not come from group homomorphisms $G \to U(n)$, and they were trivial when precomposing with any map $B\mathbb{Z}/p \to BG$. While there are a number of examples in the literature with the first feature (see for example [BW95] or [MT89]), no representations were known at this point for which the second property holds. The classification results of this paper give hope of finding a systematic and complete treatment of all these kinds of representations. We plan to undertake this task in a separate paper.

In the next section we show the applicability of our results by computing the $B\mathbb{Z}/p$-cellularization of various specific families of classifying spaces. We have chosen the simple groups (as they have shown their cornerstone role in the computation of $\mathbf{CW}_{B\mathbb{Z}/p}BG$), certain split extensions that signaled there was something beyond the results of [FS07], and certain nonsplit extensions of $G_2(q)$ that illuminate the roles of the normalizers of A and S in the $B\mathbb{Z}/p$-cellular context.

5. Examples

In Theorem 4.4 a description of the $B\mathbb{Z}/p$-cellularization of BG for every group G and every prime p is given, so in this section we describe some families of concrete examples for which $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is interesting. We begin by providing more explanation and some examples of the group-theoretic concepts.

As observed earlier, if N is any normal subgroup of G, the Sylow p-subgroup $S\cap N$ of N is strongly closed in G. More generally, if A_0 is a strongly closed p-subgroup of a group G_0 and $\pi : G \to G_0$ is any surjective homomorphism with kernel N, then any Sylow p-subgroup, A, of the complete preimage $\pi^{-1}(A_0)$ is strongly closed in G with $N \leq \mathcal{O}_A(G)$. Since such extensions of a given G_0 are essentially arbitrary, it is natural to seek the "minimal" groups possessing strongly closed subgroups, A, namely those with $\mathcal{O}_A(G) = 1$. This is achieved by passing to the quotient $\overline{G} = G/\mathcal{O}_A(G)$ wherein $\overline{A}$ is still strongly closed (but possibly trivial) and does not contain the Sylow p-subgroup of any nontrivial normal subgroup. The main result of [FF08] is a characterization of all these "minimal configurations." For the reader's convenience we now state a slightly weaker but less recondite version of that classification for all primes p.

Theorem 5.1. *Let A be any nontrivial strongly closed p-subgroup of G. Assume $\mathcal{O}_A(G) = 1$ and G is generated by conjugates of A. Then G has a normal subgroup L such that*

$$L \cong L_1 \times L_2 \times \cdots \times L_r$$

where each L_i is a simple group, $A_i = A \cap L_i$ is strongly closed but not Sylow in L_i and $A = (A_1 \times \cdots A_r)A_F$ where the subgroup A_F (possibly trivial) normalizes each

L_i and induces field automorphisms on L_i. Each L_i is one of the following types: (i) a Chevalley group of arbitrary BN-rank in characteristic $\neq p$ with abelian but not elementary abelian Sylow p-subgroups, (ii) a Chevalley group of BN-rank 1 in characteristic $= p$, (iii) $G_2(q)$ with $(q, 3) = 1$ and $p = 3$, or (iv) one of seven sporadic groups (with p specified in each case).

Here L is the socle of G (also $L = F^*(G)$) and the exact structure of G is described in [FF08]. Moreover, the full classification describes all possibilities for A_i, its normalizer, and how the conditions in family (i) may easily be determined. The "field automorphism subgroup" A_F of A can only be nontrivial when there are factors from family (i).

In the special case when $A = \mathfrak{A}_1(S)$, the condition that G be generated by its elements of order p is equivalent to G being generated by conjugates of A. The classification shows that when $A = \mathfrak{A}_1(S) \neq S$ and $\mathcal{O}_A(G) = 1$, then $A = \Omega_1(S)$ and A is elementary abelian (although S itself need not be abelian). We shall see examples in Subsection 5.2 where $\mathfrak{A}_1(S)$ is not abelian when $\mathcal{O}_A(G) \neq 1$.

In our context, two important (slightly simplified renditions of) consequences of this theorem are:

Corollary 5.2. *Under the hypotheses of Theorem 5.1*

1. *$N_G(A)$ controls strong fusion in S.*
2. *If $A_F = 1$ then $\Gamma = N_G(A)/A = N_{L_1}(A_1)/A_1 \times \cdots \times N_{L_r}(A_r)/A_r$.*

When $A_F \neq 1$ the general classification also describes Γ precisely—it may be viewed as a subgroup of the direct product in (2); and in all cases Γ is p-perfect.

One example where $L = L_1$ belongs to family (i) but $A_F \neq 1$ is when $G = PSL_{11}(q)\langle f \rangle$ with $p = 5$, $q = 3^5$, and f is a field automorphism of order 5 (this example is explicated in greater detail in [FF08]). Here the simple group $L = PSL_{11}(q)$ has an abelian Sylow 5-subgroup of type (25,25). If $f \in S \in Syl_5(G)$, then

$$A = \mathfrak{A}_1(S) = \Omega_1(S) = \langle f, \Omega_1(S \cap L) \rangle \cong Z_5 \times Z_5 \times Z_5$$

and A is evidently strongly closed in S with respect to G. Furthermore $A^* = \Omega_1(S \cap L)$ is a strongly closed subgroup of L. Both G and L are generated by their elements of order 5. One may also calculate that

$$\Gamma = N_G(A)/A \cong (H \times H)\langle t \rangle \times GL_3(3)$$

where H is a split extension $Z_{80} \cdot Z_4$ which has no subgroup of index 5, and t has order 2, commutes with f and interchanges the two copies of H. Thus Γ has no subgroup of index 5, even when G itself has a normal subgroup of index 5.

In Subsections 5.2 and 5.3 we explore control of fusion in extensions of the simple groups possessing strongly closed subgroups: in particular, we examine when $N_G(S)$, which is a subgroup of $N_G(A)$, also controls fusion in S.

5.1. **Simple groups.** In the previous discussion we sketched the list of all simple groups G possessing a strongly closed p-subgroup A that is not a Sylow p-subgroup. By simplicity, G is generated by its elements of order p, and $\mathcal{O}_A(G) = 1$. In [FS07, 5.6-5.8] the $B\mathbb{Z}/2$-cellularization of the classifying spaces of all the simple groups was computed. (For $p = 2$ only the groups $U_3(2^n)$ and $Sz(2^n)$ can occur; these belong to family (ii).) In this section we invoke the classification for p odd, Theorem 5.1, to see how for every simple group G the cellularization $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is included in the different cases of Theorem 4.4. By this classification, when G is simple and $\mathfrak{A}_1(S) \neq S$ we always have $N_G(\mathfrak{A}_1(S)) = N_G(S)$ (see Corollary 2.8 in [FF08]). We thus obtain the following characterization:

Proposition 5.3. *Let G be a simple group, let p a prime and let S be a Sylow p-subgroup of G. Then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ has one of the following two structures:*

1. *If $\mathfrak{A}_1(S) = S$, then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q \neq p} BG^\wedge_q$.*
2. *If $\mathfrak{A}_1(S) \neq S$, then we have a fibration sequence*

$$\mathbf{CW}_{B\mathbb{Z}/p}BG \to BG \to B(N_G(S)/\mathfrak{A}_1(S))^\wedge_p \times \prod_{q \neq p} BG^\wedge_q.$$

Note that the inclusion $N_G(S) \hookrightarrow N_G(\mathfrak{A}_1(S))$ induces a homotopy equivalence $BN_G(S)^\wedge_p \simeq BN_G(\mathfrak{A}_1(S))^\wedge_p$ when $N_G(S)$ (and then $N_G(\mathfrak{A}_1(S))$) controls p-fusion in S. This happens (p odd) for every simple group such that $\mathfrak{A}_1(S) \neq S$, and in particular a comparison of the fibration sequences of (Theorem 2.1) for $A = B\mathbb{Z}/p$ and $X = BN_G(S)^\wedge_p$ and $X = BN_G(\mathfrak{A}_1(S))^\wedge_p$ (which are homotopy equivalent) respectively, gives that the induced map $B(N_G(S)/\mathfrak{A}_1(S))^\wedge_p \simeq B(N_G(\mathfrak{A}_1(S))/\mathfrak{A}_1(S))^\wedge_p$ is also a homotopy equivalence.

The general structure of the normalizers that appear in the previous characterization is described in Sections 2 and 4.1 of [FF08]. For a specific example, take p odd and let q be any power of an odd prime such that $p^2 \mid q - 1$, so that a Sylow

p-subgroup S of $G = PSL_2(q)$ is cyclic of order $\geq p^2$ (for example, $p = 3$ and $q = 19$). One easily computes that for $A = \mathfrak{A}_1(S) = \Omega_1(S) < S$ we have $N_G(A) = N_G(S)$ is dihedral of order $q-1$ and so $N_G(A)/A$ is dihedral of order $(q-1)/p$.

5.2. **Split extensions.** We turn now to the case of non-simple groups. Here we give some explicit examples of $B\mathbb{Z}/p$-cellularization of split extensions that are beyond the scope of [FS07], and show the usefulness of Theorems 4.3 and 4.4.

In [FS07] the $B\mathbb{Z}/p$-cellularization of BG is described when G is generated by elements of order p, $\mathfrak{A}_1(S)$ is a proper subgroup of S, and the normalizer of S controls strong fusion in S. No example was given there of a group for which the first two conditions hold but not the third. George Glauberman suggested an example of a group of the latter type: a wreath product $(\mathbb{Z}/2) \wr Sz(2^n)$. In this section we generalize this example, showing that many split extensions for which these conditions hold can be constructed. The computation of the cellularization of their classifying space is then easy from Corollaries 1.4 and 1.5 in [FF08].

Let R be a simple group, and assume that there exists a Sylow p-subgroup T of R for which $\mathfrak{A}_1(T) \neq T$ (for instance any simple group in family (i) of Theorem 5.1). Let E be any elementary abelian p-group on which R acts faithfully. Then $S = ET$ is a Sylow p-subgroup of the semi-direct product $G = E \rtimes R$, and if we denote $A = E\mathfrak{A}_1(T)$, it is clear that $E = \mathcal{O}_A(G)$. As the extension is split, the canonical projection $G \to R$ sends $\mathfrak{A}_1(S)$ to $\mathfrak{A}_1(T)$. Note that because $\mathfrak{A}_1(T)$ acts faithfully on E, $\mathfrak{A}_1(S)$ is non-abelian. In such a group G *neither* $N_G(A)$ *nor* $N_G(S)$ *controls fusion in* S (see Section 4.2 of [FF08] for details). However, in the quotient group $\overline{G} \cong R$ the normalizer $N_{\overline{G}}(\overline{A})$ does control fusion in $\overline{S}$. This further illuminates why the classification is needed to pass to the "correct" quotient for our fusion arguments.

For a specific example, take $R = PSL_2(q)$ satisfying the conditions at the end of the previous subsection (where it was called G) and take E to be the $\mathbb{F}_pR$-module affording the regular representation of R over the field $\mathbb{F}_p$.

According to Theorem 4.3, the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$ has the homotopy type of the fiber of the composite

$$BG^\wedge_p \longrightarrow BR^\wedge_p \longrightarrow B(N_R(\mathfrak{A}_1(T))/\mathfrak{A}_1(T))^\wedge_p.$$

The structure of G was studied in the [FF08, 4.1], and moreover, Section 4.2 there also describes the shape of the normalizer in a concrete example where R is a special linear group and T is a subgroup of diagonal matrices.

5.3. **Exotic extensions of $G_2(q)$.** We conclude with somewhat exotic examples which show that in Theorem 4.4, the normalizer of the subgroup $\overline{A}$ in $\overline{G}$ cannot be replaced by the normalizer of the corresponding Sylow subgroup. These groups are constructed as "half-split" extensions of the simple group $G_2(q)$, for q coprime to 3, as established in [FF08]. (The simple groups $G_2(q)$ are in family (iii) of Theorem 5.1.) For convenience we restate Theorem 4.4 of [FF08] here (whose proof is constructive):

Proposition 5.4. *Let p be a prime dividing the order of the finite group R and let X be a subgroup of order p in R. Then there is an $\mathbb{F}_pR$-module E and an extension*

$$1 \longrightarrow E \longrightarrow G \longrightarrow R \longrightarrow 1$$

of R by E such that the extension of X by E does not split, but the extension of Z by E splits for every subgroup Z of order p in R that is not conjugate in R to X. In particular, for nonidentity elements $x \in X$ and $z \in Z$ every element in the coset xE has order p^2 whereas zE contains elements of order p in G.

We study the case $R = G_2(q)$ with $p = 3$ and $(q, 3) = 1$, and we let $T \in Syl_3(R)$. The description in [FF08, Proposition 2.7(3)] of the normalizer of T in R implies that $BN_R(T)^\wedge_3$ is not homotopy equivalent to $BG_2(q)^\wedge_3$. Now apply the previous Proposition with $Z = Z(T)$ and $X = \langle\, x \,\rangle$ for any $x \in T - Z$ of order 3 (all such subgroups X are conjugate in G, but none are conjugate to Z). Then if S is the Sylow 3-subgroup of G containing E such that $S/E = T$, then the "half-split" construction forces $E \leq \mathfrak{A}_1(S)$ and $\mathfrak{A}_1(S)/E = ZE/E$ (unlike a split extension of R by E where $\mathfrak{A}_1(S) = S$). Again the considerations in [FF08, Section 4.3] and Theorem 4.3 establish that the $B\mathbb{Z}/3$-cellularization of $(BG)^\wedge_3$ has the homotopy type the homotopy fibre of the map $(BG)^\wedge_3 \to BPSL^*_3(q)^\wedge_3$, but not of the map $(BG)^\wedge_3 \to B(N_R(T)/T)^\wedge_3$ when $9 \mid q^2 - 1$. Here $N_R(Z) \cong PSL^*_3(q)$, where the latter group denotes the projective version of the group $SL_3(q)$ or $SU_3(q)$ together with the outer (graph) automorphism of order 2 inverting its center.

From this computation one can deduce, then, that the object that determines the $\Sigma B\mathbb{Z}/p$-nullification of the cofibre of the map of Theorem 2.1 in the case of finite groups is the normalizer of $\overline{A}$, and not the Sylow normalizer, as might be inferred from the particular cases studied in [FS07]. This example also highlights the importance of having a classification of *all* groups possessing a nontrivial strongly closed p-subgroup that is not Sylow — not just the simple groups having such a subgroup that contains $\Omega_1(S)$ — since the subgroup $\mathfrak{A}_1(S)$ does not pass in a transparent fashion to quotients.

In conclusion, some interesting open questions remain. We have characterized with precision $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for every finite G, and in the course of the proof we have also described $\mathbf{CW}_{B\mathbb{Z}/p}BG^\wedge_p$ when G is generated by order p elements. However we do not address the issue of what happens in general with the cellularization of $BG^\wedge_p$ if we remove the generation hypothesis. There are some cases that can be deduced from the previous developments — for example if G is not equal to $\Omega_1(G)$ but is mod p equivalent to a group that is so — but it would be nice to have a general statement.

The extensions of our techniques and results to more general p-local spaces with a notion of p-fusion seem to be the natural next step of our study; in particular, classifying spaces of p-local finite groups and some families of non-finite groups offer enticing possibilities.

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