Composition operators on Hardy-Orlicz spaces

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Abstract. *We investigate composition operators on Hardy-Orlicz spaces when the Orlicz function Ψ grows rapidly: compactness, weak compactness, to be p-summing, order bounded, ..., and show how these notions behave according to the growth of Ψ. We introduce an adapted version of Carleson measure. We construct various examples showing that our results are essentially sharp. In the last part, we study the case of Bergman-Orlicz spaces.*

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1 Introduction.

Composition operators on the classical Hardy spaces H^p have been widely studied (see [36], and [11], and references therein; see also [19] and [20], and [7], [10], [18], [28], [33], [37], [38] for some more recent works), but it seems that one has not paid much attention to the Hardy-Orlicz spaces (in [40] and [41], J.-O. Strömberg studied Hardy-Orlicz spaces in the case when the Orlicz function Ψ increases smoothly; see also [26] for composition operators). We shall investigate what happens when the Orlicz function grows more rapidly than a power function.

Recall that, given an analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$ of the unit disk $\mathbb{D}$, the *composition operator* associated to ϕ is the map $C_\phi\colon f \mapsto f \circ \phi$. This map may operate on various Banach spaces X of analytic functions on $\mathbb{D}$ (Hardy spaces, Bergman spaces, ..., and their weighted versions (see [42] for instance), Bloch spaces $\mathscr{B}$ and $\mathscr{B}_0$, $BMOA$ and $VMOA$, Dirichlet spaces (see [1]), or some more general spaces as Nevanlinna or Smirnov classes: [8], [9], [17], [21]; see also [3], [4], [14] for composition operators on $\mathscr{H}^p$ spaces of Dirichlet series, though they are not induced by an analytic self-map of $\mathbb{D}$). The goal is to link properties of the composition operator $C_\phi\colon X \to X$ (compactness, strong or weak, for example) to properties of the *symbol* ϕ (essentially its behaviour near the frontier of $\mathbb{D}$). For that study, one can roughly speaking (see [11], Chapter 4, though their notions are different from ours) distinguish two kind of spaces.

1) The *small spaces* X; those spaces are in a sense close to the Hardy space H^∞: the compactness of $C_\phi\colon X \to X$ is very restrictive and it imposes severe restrictions on ϕ. For example, if $X = H^\infty$, a theorem of J. Schwartz ([34]) implies that $\mathbb{C}_\phi\colon H^\infty \to H^\infty$ is compact if and only if $\|\phi\|_\infty < 1$ (weakly compact suffices: see [23]), which in turn implies reinforced compactness properties for C_ϕ. For example, $C_\phi\colon H^\infty \to H^\infty$ is nuclear and 1-summing as soon as it is compact.

2) The *large spaces* X; those spaces are in a sense close to the Hardy space H^1: the compactness of $C_\phi\colon X \to X$ can take place fairly often, and in general implies no self-improvement. For example, for $X = H^2$, $C_\phi\colon H^2 \to H^2$ can be compact without being Hilbert-Schmidt, even if ϕ is injective ([39], Theorem 6.2). Another formulation (which lends better to generalizations in the non-Hilbertian case) is that C_ϕ can be non-order bounded (see [16], and our Section 3) and yet compact.

In this paper, we shall rather be on the small space side, since we shall work in spaces associated to a very large Orlicz function Ψ (typically: $\Psi(x) = \Psi_2(x) = \mathrm{e}^{x^2} - 1$), and the previous situation will not take place: our operators will be *e.g.* order-bounded as soon as they are (weakly) compact, even if the situation is not so extreme as for H^∞. However, for slightly smaller Orlicz functions (for instance $\Psi(x) = \exp\big[\big(\log(x+1)\big)^2\big] - 1$), the situation is closer to the H^2 case: the composition operators may be compact on H^Ψ, but not order-bounded (Theorem 4.22).

This paper is divided into five parts. Section 1 is this Introduction. In Section 2, which is essentially notational, we recall some more or less standard facts on Orlicz functions Ψ, on associated Orlicz spaces L^Ψ, and the "little" Orlicz space M^Ψ, and their banachic properties, associated with various slow growth conditions (indicated by subscripts: Δ_1, Δ_2, $\ldots$) or fast growth conditions (indicated by superscripts: Δ^0, Δ^1, Δ^2, $\ldots$).

In Section 3, we introduce the Hardy-Orlicz space H^Ψ, and its (or his) "little brother" HM^Ψ. These spaces have already been studied (see [15], [31]), but rather for slowly growing functions Ψ (having Δ_2 most of the time), and their definition is not so clearly outlined, so we give a detailed exposition of the equivalence of the two natural definitions that one has in mind (if one wants to extend the case of Hardy spaces H^p associated to $\Psi(x) = x^p$), as well as of the automatic boundedness (through the Littlewood subordination principle and the case of inner self-maps of the disk) of composition operators on those spaces. Two of the main theorems are Theorem 3.24 and Theorem 3.27. Roughly speaking, Theorem 3.24 says the following: if Ψ is very fast growing (having Δ^2 more precisely), H^Ψ is a small space, the (weak) compactness of C_ϕ is very restrictive, and even if the situation is not so extreme as for H^∞ ($\|\phi\|_\infty < 1$), ϕ has to tend to the boundary very slowly, and C_ϕ is automatically order-bounded into M^Ψ. However, Theorem 3.27 shows the limits of this self-improvement: C_ϕ may be order-bounded into M^Ψ (and hence compact), but p-summing for no finite p. We also show that, when Ψ has Δ^2 growth, there are always symbols

ϕ inducing compact composition operators on H^2 (even Hilbert-Schmidt), but not compact on H^Ψ.

Section 4 is devoted to the use of Carleson measures. The usefulness of those measures in the study of composition operators is well-known (see [12], [11], [6]) and, to our knowledge, first explicitly used for compactness in [27]. In particular, we recall the following necessary and sufficient condition for $C_\phi\colon H^2 \to H^2$ to be compact: if $h > 0$ and $w \in \partial\mathbb{D}$, consider the *Carleson window*:

$$W(w,h) = \{z \in \mathbb{D}\,;\ |z| \geq 1-h \ \text{ and } \ |\arg(z\bar{w})| \leq h\}.$$

If ϕ is an analytic self-map of $\mathbb{D}$ with boundary values ϕ^*, and $\mu_\phi = \phi^*(m)$ denotes the image under ϕ^* of the normalized Lebesgue measure (Haar measure) on $\mathbb{T} = \partial\mathbb{D}$, the measure μ_ϕ is always a Carleson measure, *i.e.*:

$$\sup_{w \in \partial\mathbb{D}} \mu_\phi\big(W(w,h)\big) = O\,(h).$$

Now, we can state:

Theorem 1.1 (B. MacCluer [27]) *The composition operator $C_\phi\colon H^2 \to H^2$ is compact if and only if μ_ϕ satisfies the* "little-oh" *condition, i.e. if and only if:*

$$\sup_{w \in \partial\mathbb{D}} \mu_\phi\big(W(w,h)\big) = o\,(h) \ \ \textit{as} \ \ h \to 0. \tag{1.1}$$

There is another famous necessary and sufficient compactness condition, due to J. Shapiro ([35]): Let us denote by N_ϕ the Nevanlinna counting function of ϕ, *i.e.*:

$$N_\phi(w) = \begin{cases} \displaystyle\sum_{\phi(z)=w} \log \frac{1}{|z|} & \text{if } w \neq \phi(0) \text{ and } w \in \phi(\mathbb{D}) \\ 0 & \text{if } w \notin \phi(\mathbb{D}). \end{cases}$$

By Littlewood's inequality, one always has (see [11], page 33):

$$N_\phi(w) = O\,(1-|w|).$$

Now, Shapiro's Theorem reads:

Theorem 1.2 (J. Shapiro [35]) *The composition operator $C_\phi\colon H^2 \to H^2$ is compact if and only if N_ϕ satisfies the* "little-oh" *condition, i.e.:*

$$N_\phi(w) = o\,(1-|w|) \ \ \textit{as} \ \ |w| \overset{<}{\to} 1. \tag{1.2}$$

Theorem 1.2 is very elegant, and probably more "popular" than Theorem 1.1. Yet, it is difficult to apply because the assumption (1.2) is difficult to check. Here, we shall appeal to Theorem 1.1 to prove (Theorem 4.1) that the compactness of $C_\phi\colon H^2 \to H^2$ cannot be read on $|\phi^*|$ when ϕ is not finitely valent; more precisely, there are two analytic self-maps ϕ_1 and $\phi_2\colon \mathbb{D} \to \mathbb{D}$

such that: $|\phi_1^*| = |\phi_2^*|$ m-*a.e.*, but $C_{\phi_1}\colon H^2 \to H^2$ is not compact, though $C_{\phi_2}\colon H^2 \to H^2$ is compact.

We show then that every composition operator which is compact on H^Ψ is necessarily compact on H^p for all $p < \infty$. However, there exist (see above, or Section 3) symbols ϕ inducing compact composition operators on H^2 but which are not compact on H^Ψ, when Ψ has Δ^2 growth. Hence condition (1.1) does not suffice to characterize the compact composition operators on H^Ψ. We have to replace Carleson measures and condition (1.1) by what we may call "Ψ-Carleson measures", and an adaptated "*little-oh*" condition, which allows us to characterize compactness for composition operators. It follows that if $\Psi \in \Delta^0$, then the weak compactness of $C_\phi\colon H^\Psi \to H^\Psi$ implies its compactness.

We also show that the above example ϕ_2 induces, for an Orlicz function Ψ which does not satisfy Δ^2, but which satisfies Δ^1, a composition operator on H^Ψ which is compact, but not order bounded into $M^\Psi(\mathbb{T})$ (Theorem 4.22), showing that the assumption that $\Psi \in \Delta^2$ in Theorem 3.24 is not only a technical assumption.

In Section 5, we introduce the Bergman-Orlicz spaces. Let us remind that, in the Hilbertian case, the study of compactness of composition operators is simpler for the Bergman space $\mathscr{B}^2$ than for the Hardy space H^2. For example, we have the following:

Theorem 1.3 (see [36])
i) $C_\phi\colon \mathscr{B}^2 \to \mathscr{B}^2$ *is compact if and only if*

$$\lim_{|z| \overset{<}{\to} 1} \frac{1 - |\phi(z)|}{1 - |z|} = +\infty. \tag{1.3}$$

ii) (1.3) *is always necessary for* $C_\phi\colon H^2 \to H^2$ *to be compact, and it is sufficient when* ϕ *is injective, or only finitely valent.*

iii) There are Blaschke products ϕ *satisfying* (1.3) *for which* $C_\phi\colon H^2 \to H^2$ *is (in an obvious manner) non-compact.*

We perform here a similar study for the Bergman-Orlicz space $\mathscr{B}^\Psi$, and compare the situation with that of the Hardy-Orlicz space H^Ψ. We are naturally led to a reinforcement of (1.3) under the form:

$$\frac{\Psi^{-1}\left[\dfrac{1}{(1 - |\phi(a)|)^2}\right]}{\Psi^{-1}\left[\dfrac{1}{(1 - |a|)^2}\right]} \underset{|a| \overset{<}{\to} 1}{\longrightarrow} 0 \tag{1.4}$$

(always necessary, and sufficient when Ψ is Δ^2), which reads, in the case $\Psi(x) = \Psi_2(x) = \mathrm{e}^{x^2} - 1$:

$$1 - |\phi(z)| \geq c_\varepsilon (1 - |z|)^\varepsilon \quad \text{for all } \varepsilon > 0. \tag{1.5}$$

In [36], the construction of a Blaschke product satisfying (1.3) is fairly delicate, and appeals to Frostman's Lemma and Julia-Caratheodory's Theorem on non-angular derivatives at the boundary. Here, we can no longer use these tools for the reinforcement (1.5), so we do make a direct construction, using the Parseval formula for finite groups. In passing, the construction gives a simpler proof for (1.3). Otherwise, the theorem which we obain is similar to Shapiro's one, if one ignores some technical difficulties due to the non-separability of $\mathscr{B}^\Psi$: we have to "transit" by the smaller Bergman-Morse-Transue space $\mathscr{B}M^\Psi$, which is the closure of H^∞ in $\mathscr{B}^\Psi$, is separable, and has $\mathscr{B}^\Psi$ as its bidual.

2 Notation

Let $\mathbb{D}$ be the open unit disk of the complex plane, that is the set of complex numbers with modulus strictly less than 1, and $\mathbb{T}$ the unit circle, *i.e.* the set of complex numbers with modulus 1.

We shall consider in this paper Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$, which will be the unit circle $\mathbb{T}$, with its (normalized) Haar measure m (most often identified with the normalized Lebesgue measure $dx/2\pi$ on the interval $[0, 2\pi]$), or the open unit disk $\mathbb{D}$, provided with the normalized area measure $\mathscr{A}$.

By an Orlicz function, we shall understand that $\Psi\colon [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function Ψ having the following additional properties: Ψ is continuous at 0, strictly convex (hence increasing), and such that

$$\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty.$$

This is essentially to exclude the case of $\Psi(x) = ax$.

If Ψ' is the left (or instead, if one prefers, the right) derivative of Ψ, one has $\Psi(x) = \int_0^x \Psi'(t)\, dt$ for every $x > 0$.

The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f\colon \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_\Omega \Psi\Big(\frac{|f(t)|}{C}\Big)\, d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants C such that this integral is ≤ 1. The Morse-Transue space $M^\Psi(\Omega)$ is the subspace generated by $L^\infty(\Omega)$, or, equivalently, the subspace of all functions f for which the above integral is finite for all $C > 0$.

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*\colon [0,\infty] \to [0,\infty]$ defined by:

$$\Phi(x) = \sup_{y\geq 0} \big(xy - \Psi(y)\big),$$

The extra assumptions on Ψ ensure that Φ is itself strictly convex.

When Φ satisfies the Δ_2 condition (see the definition below), L^Ψ is (isomorphically, if L^Φ is itself normed by the Luxemburg norm) the dual space of L^Φ, which is, in turn, the dual of M^Ψ.

2.1 Growth conditions

We shall have to use various growth conditions for the Orlicz function Ψ. These conditions are usually denoted as Δ-conditions. Our interest is in Orlicz functions which have a somewhat fast growth. Usually, some of these conditions are defined through a moderate growth condition on the complementary function Φ of Ψ, and the condition Δ for the Orlicz function is translated as a ∇-condition for the complementary function. So we shall distinguish between *moderate growth* conditions, that we shall define for the complementary Orlicz function, and *fast growth* conditions. To emphasize this distinction, we shall denote, sometimes in changing the usual notation (see [22, 30]), the moderate growth conditions with a subscript, and the fast growth conditions with a superscript.

Moderate growth conditions

- The Orlicz function Φ satifies the Δ_1-*condition* ($\Phi \in \Delta_1$) if, for some constant $c > 0$, one has:

  $$\Phi(xy) \leq c\,\Phi(x)\Phi(y)$$

  for x, y large enough.

  This is equivalent to say that

  $$\Phi(axy) \leq \Phi(x)\Phi(y)$$

  for some constant $a > 0$ and x, y large enough.

  This condition is usually denoted by Δ' (see [30], page 28).

- Φ satisfies the Δ_2-*condition* ($\Phi \in \Delta_2$) if

  $$\Phi(2x) \leq K\,\Phi(x)$$

  for some constant $K > 1$ and x large enough.

One has:

$$\Phi \in \Delta_1 \quad \Rightarrow \quad \Phi \in \Delta_2.$$

Fast growth conditions

- The Orlicz function Ψ satisfies the Δ^0*-condition* ($\Psi \in \Delta^0$) if (see [25]), for some $\beta > 1$:
$$\lim_{x\to+\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$
A typical example is $\Psi(x) = \exp\big[\log(x+2)\log\log(x+2)\big] - 2^{\log\log 2}$; another is $\Psi(x) = \exp\big[\big(\log(x+1)^{3/2}\big)\big] - 1$.

- The Orlicz function Ψ satisfies the Δ^1*-condition* ($\Psi \in \Delta^1$) if there is some $\beta > 1$ such that:
$$x\Psi(x) \leq \Psi(\beta x)$$
for x large enough.

Note that this latter condition is usually written as Δ_3-condition, with a subscript (see [30], §2.5). This notation fits better with our convention, and the superscript 1 agrees with the fact that this Δ^1-condition is between the Δ^0-condition and the following Δ^2-condition. $\Psi \in \Delta^1$ implies that
$$\Psi(x) \geq \exp\big(\alpha\,(\log x)^2\big)$$
for some $\alpha > 0$ and x large enough (see [30], Proposition 2, page 37). A typical example is $\Psi(x) = \mathrm{e}^{(\log(x+1))^2} - 1$.

- The Orlicz function $\Psi\colon [0,\infty) \to [0,\infty)$ is said to satisfy the Δ^2*-condition* ($\Psi \in \Delta^2$) if there exists some $\alpha > 1$ such that:
$$\Psi(x)^2 \leq \Psi(\alpha x)$$
for x large enough.

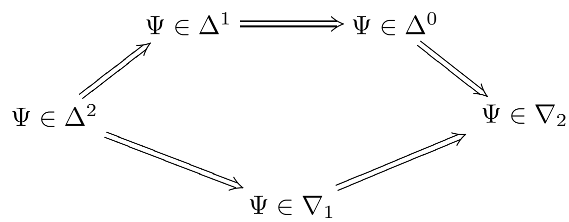
This implies that
$$\Psi(x) \geq \exp(x^\alpha)$$
for some $\alpha > 0$ and x large enough ([30], Proposition 6, page 40). A typical example is $\Psi(x) = \Psi_2(x) = \mathrm{e}^{x^2} - 1$.

Conditions of regularity

- The Orlicz function Ψ satisfies the ∇_2*-condition* ($\Psi \in \nabla_2$) if its complementary function Φ satisfies the Δ_2-condition.
This is equivalent to say that for some constant $\beta > 1$ and some $x_0 > 0$, one has $\Psi(\beta x) \geq 2\beta\,\Psi(x)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty$.
In particular, this excludes the case $L^\Psi = L^1$.

- The Orlicz function Ψ satisfies the ∇_1-*condition* ($\Psi \in \nabla_1$) if its complementary function Φ satisfies the Δ_1-condition.
  This is equivalent to say that
  $$\Psi(x)\Psi(y) \leq \Psi(bxy)$$
  for some constant $b > 0$ and x, y large enough.
  All power functions $\Psi(x) = x^p$ satisfy ∇_1, but $\Psi(x) = x^p \log(x+1)$ does not.

One has (see [30], page 43):



But Δ^1 does not imply ∇_1. That ∇_1 does not even imply Δ^0 is clear since any power function $\Psi(x) = x^p$ $(p \geq 1)$ is in ∇_1.

2.2 Some specific functions

In this paper, we shall make a repeated use of the following functions:

- If Ψ is an Orlicz function, we set, for every $K > 0$:

  (2.1) $$\chi_K(x) = \Psi\big(K\Psi^{-1}(x)\big), \quad x > 0.$$

  For example, if $\Psi(x) = \mathrm{e}^x - 1$, then $\Psi^{-1}(x) = \log(1+x)$, and $\chi_K(x) = (1+x)^K - 1$.

  Note that:

  - $\Psi \in \Delta^0$ means that $\dfrac{\chi_\beta(u)}{u} \underset{u\to\infty}{\longrightarrow} +\infty$, for some $\beta > 1$.
  - $\Psi \in \Delta^1$ means that $\chi_\beta(u) \geq u\Psi^{-1}(u)$ for u large enough, for some $\beta > 1$.
  - $\Psi \in \Delta^2$ means that $\chi_\alpha(u) \geq u^2$ for u large enough, for some $\alpha > 1$.
  - $\Psi \in \nabla_1$ means that $\chi_A(u) \geq (\Psi(A)/b)\, u$ for u large enough and for every A large enough, for some $b > 0$.

- For $|a| = 1$ and $0 \leq r < 1$, $u_{a,r}$ is the function defined on the unit disk $\mathbb{D}$ by:

  (2.2) $$u_{a,r}(z) = \left(\frac{1-r}{1-\bar{a}rz}\right)^2, \quad |z| < 1.$$

  Note that $\|u_{a,r}\|_\infty = 1$ and $\|u_{a,r}\|_1 \leq 1 - r$.

3 Composition operators on Hardy-Orlicz spaces

3.1 Hardy-Orlicz spaces

It is well-known that the classical H^p spaces ($1 \leq p \leq \infty$) can be defined in two equivalent ways:

1) H^p is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ for which, setting $f_r(t) = f(r\mathrm{e}^{it})$:

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \|f_r\|_p$$

is finite (recall that the numbers $\|f_r\|_p$ increase with r). When $f \in H^p$, the Fatou-Riesz Theorem asserts that the boundary limits $f^*(t) = \lim_{r \nearrow 1} f_r(t)$ exist almost everywhere and $\|f\|_{H^p} = \|f^*\|_p$. One has $f^* \in L^p([0, 2\pi])$, and its Fourier coefficients $\widehat{f^*}(n)$ vanish for $n < 0$.

2) Conversely, for every function $g \in L^p([0, 2\pi])$ whose Fourier coefficients $\hat{g}(n)$ vanish for $n < 0$, the analytic extension $P[g]\colon \mathbb{D} \to \mathbb{C}$ defined by $P[g](z) = \sum_{n \geq 0} \hat{g}(n) z^n$ is in H^p and g is the boundary limit $(P[g])^*$ of $P[g]$.

Hardy-Orlicz spaces H^Ψ are defined in a similar way. However, we did not find very satisfactory references, and, though the reasonings are essentially the same as in the classical case, the lack of homogeneity of Ψ and the presence of the two spaces M^Ψ and L^Ψ gives proofs which are not so obvious and therefore we shall give some details.

It should be noted that our definition is not exactly the same as the one given in [30], § 9.1.

We shall begin with the following proposition.

Proposition 3.1 *Let $f\colon \mathbb{D} \to \mathbb{C}$ be an analytic function. For every Orlicz function Ψ, the following assertions are equivalent:*

1) $\sup_{0 \leq r < 1} \|f_r\|_\Psi < +\infty$, where $f_r(t) = f(r\mathrm{e}^{it})$;

2) there exists $f^* \in L^\Psi([0, 2\pi])$ such that $\widehat{f^*}(n) = 0$ for $n < 0$ and for which $f(z) = \sum_{n \geq 0} \widehat{f^*}(n) z^n$, $z \in \mathbb{D}$.

When these conditions are satisfied, one has $\|f^*\|_\Psi = \sup_{0 \leq r < 1} \|f_r\|_\Psi$.

Let us note that, since Ψ is convex and increasing, $\Psi(a|f|)$ is subharmonic on $\mathbb{D}$, and hence the numbers $\int_{\mathbb{T}} \Psi(a|f_r|)\, dm$ increase with r, for every $a > 0$.

This proposition leads to the following definition.

Definition 3.2 *Given an Orlicz function Ψ, the* Hardy-Orlicz *space H^Ψ associated to Ψ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that one of the equivalent conditions of the above proposition is satisfied. The norm of f is defined by $\|f\|_{H^\Psi} = \|f^*\|_\Psi$. We shall denote by HM^Ψ the* Hardy-Morse-Transue space, *i.e. the subspace $\{f \in H^\Psi\, ;\ f^* \in M^\Psi(\mathbb{T})\}$.*

In the sequel, we shall make no distinction between f and f^*, unless there may be some ambiguity, and shall write f instead of f^* for the boundary limit.

Hence we shall allow ourselves to write $f(\mathrm{e}^{it})$ instead of $f^*(t)$, or even $f^*(\mathrm{e}^{it})$. Moreover, we shall write $\|f\|_\Psi$ instead of $\|f\|_{H^\Psi}$.

It follows that H^Ψ becomes a subspace of $L^\Psi(\mathbb{T})$ and $HM^\Psi = H^\Psi \cap M^\Psi(\mathbb{T})$. These two spaces are closed (hence Banach spaces) since Proposition 3.1 gives:

Corollary 3.3 *H^Ψ is weak-star closed in $L^\Psi = (M^\Phi)^*$. When Ψ satisfies ∇_2, it is isometrically isomorphic to the bidual of HM^Ψ.*

Proof. The weak-star closure of H^Ψ is obvious with Proposition 3.1, 2).

Suppose now that Φ satisfies ∇_2, it is plainly seen that $(HM^\Psi)^\perp$ is the closed subspace of $L^\Phi = (M^\Psi)^*$ generated by all characters e_n with $n < 0$, where $e_n(t) = \mathrm{e}^{int}$ (for convenience, we define the duality between $f \in L^\Psi$ and $g \in L^\Phi$ by integrating the product $f\check{g}$, where $\check{g}(t) = g(-t)$). As $H^\Psi \subseteq L^\Psi = (L^\Phi)^* = (M^\Phi)^*$ is the orthogonal of this latter subspace. So, we have $(HM^\Psi)^{\perp\perp} = H^\Psi$. □

Proof of Proposition 3.1. Assume that 1) is satisfied. Since $\|f_r\|_1 \leq C_\Psi \|f_r\|_\Psi$, one has $f \in H^1$, and hence, by Fatou-Riesz Theorem, f has almost everywhere a boundary limit $f^* \in L^1(m)$. If $C = \sup_{0\leq r<1} \|f_r\|_\Psi$, one has:

$$\int_{\mathbb{T}} \Psi\Big(\frac{|f_r|}{C}\Big)\, dm \leq 1$$

for every $r < 1$; hence Fatou's lemma implies:

$$\int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{C}\Big)\, dm \leq 1,$$

i.e. $f^* \in L^\Psi$ and $\|f^*\|_\Psi \leq C$.

Conversely, assume that 2) is satisfied. In particular $f^* \in L^1(m)$; hence $f \in H^1$ and $f^* = \lim_{r\to 1} f_r$ almost everywhere. One has $f_r = f^* * P_r$, where P_r is the Poisson kernel at r. Hence, using Jensen's formula for the probability measure $P_r(\theta - t)\,\frac{dt}{2\pi}$, we get:

$$\begin{aligned}
\int_0^{2\pi} \Psi\Big(\frac{|(f^* * P_r)(\theta)|}{\|f^*\|_\Psi}\Big)\,\frac{d\theta}{2\pi} &\leq \int_0^{2\pi} \Psi\Big(\int_0^{2\pi} \frac{|f^*(t)|}{\|f^*\|_\Psi} P_r(\theta - t)\,\frac{dt}{2\pi}\Big)\,\frac{d\theta}{2\pi} \\
&\leq \int_0^{2\pi} \Big(\int_0^{2\pi} \Psi\Big(\frac{|f^*(t)|}{\|f^*\|_\Psi}\Big)\, P_r(\theta - t)\,\frac{dt}{2\pi}\Big)\,\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \Big(\int_0^{2\pi} P_r(\theta - t)\,\frac{d\theta}{2\pi}\Big)\Psi\Big(\frac{|f^*(t)|}{\|f^*\|_\Psi}\Big)\,\frac{dt}{2\pi} \\
&= \int_0^{2\pi} \Psi\Big(\frac{|f^*(t)|}{\|f^*\|_\Psi}\Big)\,\frac{dt}{2\pi} \leq 1\,,
\end{aligned}$$

so that $\|f_r\|_\Psi \leq \|f^*\|_\Psi$. Hence we have 1), and $\|f\|_{H^\Psi} \leq \|f^*\|_\Psi$.

The two parts of the proof actually give $\|f\|_{H^\Psi} = \|f^*\|_\Psi$. □

Proposition 3.4 *For every $f \in HM^\Psi$, one has $\|f_r - f^*\|_\Psi \underset{r\to 1}{\longrightarrow} 0$. Therefore the polynomials on $\mathbb{D}$ are dense in HM^Ψ.*

Equivalently, on $\mathbb{T} = \partial\mathbb{D}$, the analytic trigonometric polynomial are dense in HM^Ψ.

Proof. Let $f \in HM^\Psi$ and $\varepsilon > 0$. Since $M^\Psi = \overline{C(\mathbb{T})}^{L^\Psi}$, there exists a continuous function h on $\mathbb{T}$ such that $\|f - h\|_\Psi \leq \varepsilon$. We have, for every $r < 1$:

$$\|P_r * f - f\|_\Psi \leq \|P_r * (f-h)\|_\Psi + \|P_r * h - h\|_\Psi + \|h - f\|_\Psi \leq 2\varepsilon + \|P_r * h - h\|_\Psi$$

because $\|P_r * g\|_\Psi \leq \|g\|_\Psi$, for every $r < 1$ and every $g \in L^\Psi$.

But now, $P_r * h \underset{r\to 1}{\longrightarrow} h$ uniformly. The conclusion follows. □

Remark. We do not have to use a maximal function to prove the existence of boundary limits because we use their existence for functions in H^1. However, as in the classical case, the Marcinkiewicz interpolation Theorem, or, rather, its Orlicz space version ensures that the maximal non-tangential function is in L^Ψ. This result is undoubtedly known, but perhaps never stated in the following form. Recall that N_α is defined, for every f, say in $L^1(\mathbb{T})$, as

$$(N_\alpha f)(\mathrm{e}^{i\theta}) = \sup_{r\mathrm{e}^{it} \in S_\theta} |(f * P_r)(\mathrm{e}^{it})| = \sup_{z \in S_\theta} |f(z)|,$$

where S_θ is the Stolz domain at $\mathrm{e}^{i\theta}$ with opening α (see [5], page 177); here f defines a harmonic function in $\mathbb{D}$.

Proposition 3.5 *Assume that the complementary function Φ of the Orlicz function Ψ satisfies the Δ_2 condition (i.e. $\Psi \in \nabla_2$). Then every linear, or sublinear, operator which is of weak-type $(1,1)$ and (strong) type (∞,∞) is bounded from L^Ψ into itself. In particular, for every $f \in L^\Psi(\mathbb{T})$, the maximal non-tangential function $N_\alpha f$ is in $L^\Psi(\mathbb{T})$ $(0 < \alpha < 1)$.*

Proof. If $\Psi \in \nabla_2$, then ([30], Theorem 3, 1 (iii), page 23), there exists some $\beta > 1$ such that $x\Psi'(x) \geq \beta\Psi(x)$ for x large enough. Integrating between u and v, for $u < v$ large enough, we get $\frac{\Psi(u)}{\Psi(v)} \geq \left(\frac{u}{v}\right)^\beta$. Hence, for s, t large enough $\frac{\Psi^{-1}(s)}{\Psi^{-1}(s/t)} \leq t^{1/\beta}$. This means (see [5], Theorem 8.18) that the upper Boyd index of L^Ψ is $\leq 1/\beta < 1$. Hence ([5], Theorem 5.17), N_α is bounded on L^Ψ (it is well-known that $N_\alpha f$ is dominated by the Hardy-Littlewood maximal function Mf). □

The following, essentially well-known, criterion for compactness of operators will be very useful.

Proposition 3.6 *1) Every bounded linear operator $T\colon H^\Psi \to X$ from H^Ψ into a Banach space X which maps every bounded sequence which is uniformly convergent on compact subsets of $\mathbb{D}$ into a norm convergent sequence is compact.*

2) Conversely, if $T\colon H^\Psi \to X$ is compact and weak-star to weak continuous, or if $T\colon H^\Psi \to Y^*$ is compact and weak-star continuous, then T maps every bounded sequence which is uniformly convergent on compact subsets of $\mathbb{D}$ into a norm convergent sequence.

Though well-known (at least for the classical case of H^p spaces), the link with the weak (actually the weak-star) topology is usually not highlighted. Indeed, the criterion is an easy consequence of the following proposition. Note that Proposition 3.6 will apply to the composition operators on H^Ψ since they are weak-star continuous.

Proposition 3.7 *On the unit ball of H^Ψ, the weak-star topology is the topology of uniform convergence on every compact subset of $\mathbb{D}$.*

Proof. First we notice that the topologies are metrizable. Indeed, this is known for the topology of uniform convergence on every compact subset of $\mathbb{D}$ and, on the other hand, M^Φ is separable, so that the weak-star topology is metrizable on the unit ball of its dual space L^Ψ, and *a fortiori* on that of H^Ψ. Now, it is sufficient to prove that the convergent sequences in both topologies are the same.

Let $(f_k)_{k\geq 1}$ be in the unit ball of H^Ψ and weak-star convergent to $f \in H^\Psi$. Let us fix a compact subset K of $\mathbb{D}$. We may suppose that K is the closed ball of center 0 and radius $r < 1$. First, testing the weak-star convergence on characters, we have $\widehat{f_k}(n) \underset{k\to\infty}{\longrightarrow} \hat{f}(n)$ for every $n \in \mathbb{Z}$. Then:

$$\sup_{|z|\leq r} |f_k(z) - f(z)| = \sup_{|z|=r} |f_k(z) - f(z)| \leq \sum_{n\geq 0} r^n |\widehat{f_k}(n) - \hat{f}(n)|.$$

The last term obviously tends to zero when k tends to infinity. The result follows.

Conversely, let $(f_k)_{k\geq 1}$ be in the unit ball of H^Ψ and converging to some holomorphic function f uniformly on every compact subset of $\mathbb{D}$. We first notice that f actually lies in the unit ball of H^Ψ (by Fatou's lemma). Fix $h \in M^\Phi$ and $\varepsilon > 0$. There exists some $r < 1$ such that $\|P_r * h - h\|_\Phi \leq \varepsilon/8$, where P_r is the Poisson kernel with parameter r. Then (see [30], page 58, inequality (3) for the presence of the coefficient 2):

$$\begin{aligned}
|\langle h, f_k - f\rangle| &= |\langle P_r * h - h, f_k - f\rangle| + |\langle P_r * h, f_k - f\rangle| \\
&\leq 2\,\|P_r * h - h\|_\Phi \|f_k - f\|_\Psi + |\langle h, P_r * (f_k - f)\rangle| \\
&\leq \frac{\varepsilon}{2} + 2\,\|h\|_\Phi\,\|[f_k]_r - f_r\|_\Psi \\
&\leq \frac{\varepsilon}{2} + 2\alpha\,\|h\|_\Phi \|[f_k]_r - f_r\|_\infty \\
&= \frac{\varepsilon}{2} + 2\alpha\,\|h\|_\Phi \sup_{|z|=r} |f_k(z) - f(z)|.
\end{aligned}$$

where α is the norm of the injection of L^∞ into L^Ψ.

Now, by uniform convergence on the closed ball of center 0 and radius r, there exists $k_\varepsilon \geq 1$ such that for every integer $k \geq k_\varepsilon$, one has

$$\|h\|_\Phi \sup_{|z|=r} |f_k(z) - f(z)| \leq \varepsilon/4.$$

It follows that $(f_k)_k$ weak-star converges to f. □

However, we shall have to use a similar compactness criterion for Bergman-Orlicz spaces, and it is worth stating and proving a general criterion. We shall say that a Banach space of holomorphic functions on an open subset Ω of the complex plane has the *Fatou property* if X is continuously embedded (though the canonical injection) in $\mathscr{H}(\Omega)$, the space of holomorphic functions on Ω, equipped with its natural topology of compact convergence, and if it has the following property: for every bounded sequence $(f_n)_n$ in X which converges uniformly on compact subsets of Ω to a function f, one has $f \in X$. Then:

Proposition 3.8 (Compactness criterion) *Let X, Y be two Banach spaces of analytic functions on an open set $\Omega \subseteq \mathbb{C}$ which have the Fatou property. Let ϕ be an analytic self-map of Ω such that $C_\phi = f \circ \phi \in Y$ whenever $f \in X$.*

Then $C_\phi \colon X \to Y$ is compact if and only if for every bounded sequence $(f_n)_n$ in X which converges to 0 uniformly on compact subsets of Ω, one has $\|C_\phi(f_n)\|_Y \to 0$.

Note that Hardy-Orlicz H^Ψ and Bergman-Orlicz $\mathscr{B}^\Psi$ (see Section 5) spaces trivially have the Fatou property, because of Fatou's Lemma.

Proof. Assume that the above condition is fulfilled. Let $(f_n)_{n\geq 1}$ be in the unit ball of X. The assumption on X implies that $(f_n)_n$ is a normal family in $\mathscr{H}(\Omega)$. Montel's Theorem allows us to extract a subsequence, that we still denote by $(f_n)_n$ to save notation, which converges to some $f \in \mathscr{H}(\Omega)$, uniformly on compact subsets of Ω. Since X has the Fatou property, one has $f \in X$. Now, since $(f_n - f)_n$ is a bounded sequence in X which converges to 0 uniformly on compact subsets of Ω, one has $\|C_\phi(f_n) - C_\phi(f)\|_Y = \|C_\phi(f_n - f)\|_Y \to 0$. Hence C_ϕ is compact.

Conversely, assume that C_ϕ is compact. Let $(f_n)_n$ be a bounded sequence in X which converges to 0 uniformly on compact subsets of Ω. By the compactness of C_ϕ, we may assume that $C_\phi(f_n) \to g \in Y$. The space Y being continuously embedded in $\mathscr{H}(\Omega)$, $(f_n \circ \phi)_n$ converges pointwise to g. Since $(f_n)_n$ converges to 0 uniformly on compact subsets of Ω, the same is true for $(f_n \circ \phi)_n$. Hence $g = 0$. Therefore, since C_ϕ is compact, we get $\|C_\phi(f_n)\|_Y \to 0$. □

3.2 Preliminary results

Lemma 3.9 *Let $(\Omega, \mathbb{P})$ be any probability space. For every function $g \in L^\infty(\Omega)$, one has:*

$$\|g\|_\Psi \leq \frac{\|g\|_\infty}{\Psi^{-1}(\|g\|_\infty / \|g\|_1)}.$$

Proof. We may suppose that $\|g\|_\infty = 1$.
Since $\Psi(0) = 0$, the convexity of Ψ implies $\Psi(ax) \leq a\Psi(x)$ for $0 \leq a \leq 1$. Hence, for every $C > 0$, one has, since $|g| \leq 1$:

$$\int_\Omega \Psi(|g|/C)\, d\mathbb{P} \leq \int_\Omega |g|\Psi(1/C)\, d\mathbb{P} = \|g\|_1 \Psi(1/C).$$

But $\|g\|_1 \Psi(1/C) \leq 1$ if and only if $C \geq 1/\Psi^{-1}(1/\|g\|_1)$, and that proves the lemma. $\square$

Corollary 3.10 *For $|a| = 1$ and $0 \leq r < 1$, one has:*

$$\|u_{a,r}\|_\Psi \leq \frac{1}{\Psi^{-1}(\frac{1}{1-r})}\,.$$

Proof. One has $\|u_{a,r}\|_\infty = 1$, and:

$$\begin{aligned}\|u_{a,r}\|_1 &= \int_0^{2\pi} \left|\frac{1-r}{1-\bar{a}r\mathrm{e}^{it}}\right|^2 dm(t)\\ &= (1-r)^2 \sum_{n=0}^{+\infty} r^{2n} = \frac{(1-r)^2}{1-r^2} = \frac{1-r}{1+r}\cdot\end{aligned}$$

Hence $\|u_{a,r}\|_\Psi \leq 1/\Psi^{-1}(1+r/1-r)$, by using Lemma 3.9, giving the result since $(1+r)/(1-r) \geq 1/1-r$. $\square$

Remark. We hence get actually $\|u_{a,r}\|_\Psi \leq 1/\Psi^{-1}(1+r/1-r)$; the term $1+r$ has no important meaning, so we omit it in the statement of Corollary 3.10, but sometimes, for symmetry of formulae, or in order to be in accordance with the classical case, we shall use this more precise estimate.

For every $f \in L^1(\mathbb{T})$ and every $z = r\,\mathrm{e}^{i\theta} \in \mathbb{D}$, one has

$$(P[f])(z) = \int_0^{2\pi} f(\mathrm{e}^{it}) P_z(t)\, dm(t),$$

where P_z is the Poisson kernel:

$$P_z(t) = \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} = \frac{1-|z|^2}{|\mathrm{e}^{it} - z|^2}\,,$$

and $f(z) = (P[f])(z)$ when f is analytic on $\mathbb{D}$. Since $P_z \in L^\infty(\mathbb{T}) \subseteq L^\Phi(\mathbb{T})$, it follows that the evaluation in $z \in \mathbb{D}$:

$$\delta_z(f) = f(z)$$

is a continuous linear form on H^Ψ. The following lemma explicits the behaviour of its norm.

Lemma 3.11 *For $|z| < 1$, the norm of the evaluation functional at z is:*

$$\|\delta_z\|_{(HM^\Psi)^*} = \|\delta_z\|_{(H^\Psi)^*} \approx \Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\cdot$$

More precisely:

$$\frac{1}{4}\Psi^{-1}\Big(\frac{1+|z|}{1-|z|}\Big) \leq \|\delta_z\|_{(H^\Psi)^*} \leq 2\Psi^{-1}\Big(\frac{1+|z|}{1-|z|}\Big)\cdot$$

Remark. In particular:

$$\frac{1}{4}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big) \leq \|\delta_z\|_{(H^\Psi)^*} \leq 4\Psi^{-1}\Big(\frac{1}{1-|z|}\Big),$$

which often suffices for our purpose.

Proof. The first equality $\|\delta_z\|_{(HM^\Psi)^*} = \|\delta_z\|_{(H^\Psi)^*}$ comes from the fact that $f_r \in HM^\Psi$, for every $f \in H^\Psi$ and $r < 1$ (thus $f(rz) \underset{r\to 1}{\longrightarrow} f(z)$, when $z \in \mathbb{D}$ and $f \in H^\Psi$).

On the one hand, we have, when $|z| = r$, using [30], inequality (4) page 58, and Lemma 3.9, since $\|P_z\|_1 = 1$ and $\|P_z\|_\infty = \frac{1+r}{1-r}$:

$$\|\delta_z\|_{(H^\Psi)^*} \leq 2\,\|P_z\|_\Phi \leq 2\frac{1+r}{1-r}\,\frac{1}{\Phi^{-1}\big(\frac{1+r}{1-r}\big)}\,,$$

which is less than $2\Psi^{-1}(1+r/1-r)$, by using the inequality (see [30], Proposition 1 (ii), page 14, or [22], pages 12–13):

$$\Psi^{-1}(x)\Phi^{-1}(x) \geq x\,, \qquad x > 0.$$

On the other hand, one has, using Corollary 3.10, with $r = |z|$ and $\bar{a}z = r$:

$$\|\delta_z\|_{(H^\Psi)^*} \geq \frac{|u_{a,r}(z)|}{\|u_{a,r}\|_\Psi} \geq \frac{1/(1+r)^2}{1/\Psi^{-1}\big(\frac{1+r}{1-r}\big)} \geq \frac{1}{4}\Psi^{-1}\Big(\frac{1+r}{1-r}\Big),$$

and that ends the proof. □

3.3 Composition operators

We establish now some estimations for the norm of composition operators.

Proposition 3.12 *1) Every analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator $C_\phi\colon H^\Psi \to H^\Psi$ by setting $C_\phi(f) = f\circ\phi$. More precisely:*

$$\|C_\phi\| \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}\cdot$$

In particular, $\|C_\phi\| \leq 1$ if $\phi(0) = 0$.

2) One has:

$$\|C_\phi\| \geq \frac{1}{8\Psi^{-1}(1)}\Psi^{-1}\Big(\frac{1+|\phi(0)|}{1-|\phi(0)|}\Big).$$

3) When $\Psi \in \nabla_1$ globally: $\Psi(x)\Psi(y) \leq \Psi(bxy)$ for all $x, y \geq 0$, we also have:

$$\|C_\phi\| \leq b\Psi^{-1}\Big(\frac{1+|\phi(0)|}{1-|\phi(0)|}\Big).$$

4) Moreover, C_ϕ maps HM^Ψ into HM^Ψ. Hence, if $\Psi \in \nabla_2$, then $C_\phi\colon H^\Psi \to H^\Psi$ is the bi-adjoint of the composition operator $C_\phi\colon HM^\Psi \to HM^\Psi$.

Note that when $\Psi(x) = x^p$ for $1 \leq p < \infty$, then $\Psi \in \nabla_1$ globally, with $b = 1$.

Proof. 1) Assume first that $\phi(0) = 0$. Let $f \in H^\Psi$, with $\|f\|_\Psi = 1$. Since Ψ is convex and increasing, the function $u = \Psi \circ |f|$ is subharmonic on $\mathbb{D}$, thanks to Jensen's inequality. The condition $\phi(0) = 0$ allows to use Littlewood's subordination principle ([12], Theorem 1.7); for $r < 1$, one has:

$$\int_0^{2\pi} \Psi\big(|(f\circ\phi)(re^{it})|\big)\,\frac{dt}{2\pi} \leq \int_0^{2\pi} \Psi\big(|f(re^{it})|\big)\,\frac{dt}{2\pi} \leq 1.$$

Hence $f\circ\phi \in H^\Psi$ and $\|f\circ\phi\|_\Psi \leq 1$.

Assume now that ϕ is an inner function, and let $a = \phi(0)$. It is known that (see [29], Theorem 1) that

$$\phi^*(m) = P_a.m\,,$$

where $\phi^*(m)$ is the image under ϕ^* (the boundary limit of ϕ) of the normalized Lebesgue measure m, and $P_a.m$ is the measure of density P_a, the Poisson kernel at a. Therefore, for every $f \in H^\Psi$ with $\|f\|_\Psi = 1$, one has for $0 \leq r < 1$, in setting $K_a = \|P_a\|_\infty = \frac{1+|a|}{1-|a|}$:

$$\begin{aligned}
\int_0^{2\pi} \Psi\Big(\frac{|(f\circ\phi)(re^{it})|}{K_a}\Big)\,\frac{dt}{2\pi} &\leq \int_{\mathbb{T}} \Psi\Big(\frac{|(f\circ\phi)^*|}{K_a}\Big)\,dm \\
&= \int_{\mathbb{T}} \Psi\Big(\frac{|f^*\circ\phi^*|}{K_a}\Big)\,dm \quad \text{(recall that } |\phi^*| = 1) \\
&= \int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{K_a}\Big)\,d\phi^*(m) = \int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{K_a}\Big)\,P_a\,dm \\
&\leq \int_{\mathbb{T}} \frac{1}{K_a}\Psi(|f^*|)\,P_a\,dm\,, \quad \text{since } K_a > 1 \\
&\leq \int_{\mathbb{T}} \frac{1}{K_a}\Psi(|f^*|)\,\|P_a\|_\infty\,dm \\
&= \int_{\mathbb{T}} \Psi(|f^*|)\,dm \leq 1.
\end{aligned} \tag{3.1}$$

Hence $\|(f\circ\phi)_r\|_\Psi \leq K_a$, and therefore $\|f\circ\phi\|_\Psi \leq K_a$.

Then, for an arbitrary ϕ, let $a = \phi(0)$ again, and let ϕ_a be the automorphism $z \mapsto \frac{z-a}{1-\bar{a}z}$, whose inverse is ϕ_{-a}. Since $\phi = \phi_{-a} \circ (\phi_a \circ \phi)$, one has $C_\phi = C_{\phi_a \circ \phi} \circ C_{\phi_{-a}}$. But ϕ_{-a} is inner and, on the other hand, $(\phi_a \circ \phi)(0) = 0$; hence parts a) and b) of the proof give:

$$\|C_\phi\| \leq \|C_{\phi_{-a}}\| \leq K_a = \frac{1+|a|}{1-|a|}\,,$$

which gives the first part of the proof.

2) By Lemma 3.11, we have for every $f \in H^\Psi$ with $\|f\|_\Psi \leq 1$:

$$|(f \circ \phi)(0)| \leq \|\delta_0\|_{(H^\Psi)^*} \|f \circ \phi\|_\Psi \leq 2\Psi^{-1}(1)\,\|C_\phi\|.$$

In other words:

$$|f\big(\phi(0)\big)| \leq 2\Psi^{-1}(1)\|C_\phi\|$$

for every such $f \in H^\Psi$. Hence:

$$\|\delta_{\phi(0)}\|_{(H^\Psi)^*} \leq 2\Psi^{-1}(1)\,\|C_\phi\|\,,$$

giving

$$\|C_\phi\| \geq \frac{1}{8\Psi^{-1}(1)}\Psi^{-1}\bigg(\frac{1+|\phi(0)|}{1-|\phi(0)|}\bigg),$$

by using Lemma 3.11 again, but the minoration.

3) When $\Psi \in \nabla_1$ globally, we go back to the proof of 1). We have only to modify inequalities (3.1) in b). Setting $K'_a = \Psi^{-1}(K_a)$, and writing $P_a = \Psi\big(\Psi^{-1}(P_a)\big)$, we get, for every $f \in H^\Psi$ with $\|f\|_\Psi = 1$:

$$\begin{aligned}
\int_0^{2\pi} \Psi\Big(\frac{|(f\circ\phi)(re^{it})|}{bK'_a}\Big)\,\frac{dt}{2\pi} &\leq \int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{bK'_a}\Big)\,P_a\,dm \\
&= \int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{bK'_a}\Big)\,\Psi\big(\Psi^{-1}(P_a)\big)\,dm \\
&\leq \int_{\mathbb{T}} \Psi\Big(\frac{|f^*|}{K'_a}\Psi^{-1}(P_a)\Big)\,dm \\
&\leq \int_{\mathbb{T}} \Psi(|f^*|)\,dm \leq 1,
\end{aligned}$$

since $\Psi^{-1}(P_a) \leq \Psi^{-1}(\|P_a\|_\infty) = K'_a$, giving $\|f\circ\phi\|_\Psi \leq bK'_a$.

4) Suppose now that $f \in HM^\Psi$. As before, when $\phi(0) = 0$, Littlewood's subordination principle gives, for every $C > 0$:

$$\begin{aligned}
\int_0^{2\pi} \Psi\big(C|(f\circ\phi)(e^{it})|\big)\,\frac{dt}{2\pi} &= \sup_{r<1}\int_0^{2\pi} \Psi\big(C|(f\circ\phi)(re^{it})|\big)\,\frac{dt}{2\pi} \\
&\leq \sup_{r<1}\int_0^{2\pi} \Psi\big(C|f(re^{it})|\big)\,\frac{dt}{2\pi} = \int_0^{2\pi} \Psi\big(C|f(e^{it})|\big)\,\frac{dt}{2\pi} < +\infty\,;
\end{aligned}$$

hence $f\circ\phi \in HM^\Psi$. When ϕ is inner, the same computations as in 1) b) above, using that $\phi^*(m) = P_a.m$, where $a = \phi(0)$, give, for every $C > 0$:

$$\int_0^{2\pi} \Psi\big(C|(f\circ\phi)(re^{it})|\big)\,\frac{dt}{2\pi} \leq \int_{\mathbb{T}} \Psi(C|f^*|)\|P_a\|_\infty\,dm = K_a \int_{\mathbb{T}} \Psi(C|f^*|)\,dm < +\infty,$$

and $f\circ\phi \in HM^\Psi$ again. The general case follows, as in 1) c) above, since $f \in HM^\Psi$ implies $f\circ\phi_{-a} \in HM^\Psi$, because ϕ_{-a} is inner, and then $f\circ\phi = (f\circ\phi_{-a})\circ(\phi_a\circ\phi) \in HM^\Psi$ since $(\phi_a\circ\phi)(0) = 0$. □

3.4 Order bounded composition operators

Recall that an operator $T\colon X \to Z$ from a Banach space X into a Banach subspace Z of a Banach lattice Y is order bounded if there is some positive $y \in Y$ such that $|Tx| \leq y$ for every x in the unit ball of X.

Before studying order bounded composition operators, we shall recall the following, certainly well-known, result, which says that order boundedness can be seen as stronger than compactness.

Proposition 3.13 *Let $T\colon L^2(\mu) \to L^2(\mu)$ be a continuous linear operator. Then T is order bounded if and only if it is a Hilbert-Schmidt operator.*

The proof is straightforward: if B is the unit ball of $L^2(\mu)$, and $(e_i)_i$ is an orthonormal basis, one has $\sup_{f\in B}|Tf| = \big(\sum_i |Te_i|^2\big)^{1/2}$. Hence $\sup_{f\in B}|Tf| \in L^2(\mu)$ if and only if $\int \big(\sum_i |Te_i|^2\big)\,d\mu = \sum_i \|Te_i\|^2 < +\infty$, *i.e.* if and only if T is Hilbert-Schmidt.

J. H. Shapiro and P. D. Taylor proved in [39] that there exist composition operators on H^2 which are compact but not Hilbert-Schmidt. We are going to see that, when the Orlicz function Ψ grows fast enough, the compactness of composition operators on H^Ψ is equivalent to their order boundedness.

Proposition 3.14 *The composition operator $C_\phi\colon H^\Psi \to H^\Psi$ is order bounded (resp. order bounded into $M^\Psi(\mathbb{T})$) if and only if $\Psi^{-1}\big(\frac{1}{1-|\phi|}\big) \in L^\Psi(\mathbb{T})$ (resp. $\Psi^{-1}\big(\frac{1}{1-|\phi|}\big) \in M^\Psi(\mathbb{T})$). Equivalently, if and only if*

$$\text{(OB1)} \qquad \chi_A\Big(\frac{1}{1-|\phi|}\Big) \in L^1(\mathbb{T}) \quad \textit{for some } A > 0,$$

respectively:

$$\text{(OB2)} \qquad \chi_A\Big(\frac{1}{1-|\phi|}\Big) \in L^1(\mathbb{T}) \quad \textit{for every } A > 0,$$

In other words (recall that $\chi_A(x) = \Psi\big(A\Psi^{-1}(x)\big)$), if and only if:

$$\int_{\mathbb{T}} \Psi\Big[A\Psi^{-1}\Big(\frac{1}{1-|\phi|}\Big)\Big]\,dm < +\infty \quad \text{for some (resp. every) } A > 0.$$

Remark For a sequence $(g_n)_n$ of elements of $L^\Psi(\Omega)$, one has $\|g_n\|_\Psi \underset{n\to+\infty}{\longrightarrow} 0$ if and only if

$$\int_\Omega \Psi\Big(\frac{|g_n|}{\varepsilon}\Big)\,d\mathbb{P} \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{for every } \varepsilon > 0. \tag{3.2}$$

In fact, if (3.2) holds, then for $n \geq n_\varepsilon$, the above integrals are ≤ 1, and hence $\|g_n\|_\Psi \leq \varepsilon$. Conversely, assume that $\|g_n\|_\Psi \underset{n\to+\infty}{\longrightarrow} 0$, and let $\varepsilon > 0$ be given. Let $0 < \delta \leq 1$. Since $\|g_n/(\varepsilon\delta)\|_\Psi \underset{n\to+\infty}{\longrightarrow} 0$, one has $\|g_n/(\varepsilon\delta)\|_\Psi \leq 1$, and hence $\int_\Omega \Psi\big(\frac{|g_n|}{\varepsilon\delta}\big)\,d\mathbb{P} \leq 1$, for n large enough. Then, using the convexity of Ψ:

$$\int_\Omega \Psi\Big(\frac{|g_n|}{\varepsilon}\Big)\,d\mathbb{P} = \int_\Omega \Psi\Big(\delta\,\frac{|g_n|}{\varepsilon\delta}\Big)\,d\mathbb{P} \leq \delta \int_\Omega \Psi\Big(\frac{|g_n|}{\varepsilon\delta}\Big)\,d\mathbb{P} \leq \delta\,,$$

for n large enough.

Therefore, using Lebesgue's dominated convergence Theorem, it follows that if $C_\phi\colon H^\Psi \to H^\Psi$ is order bounded *into* $M^\Psi(\mathbb{T})$, then the composition operator $C_\phi\colon H^\Psi \to H^\Psi$ is compact.

Proof of Proposition 3.14.

As HM^Ψ is separable, there exists a sequence $(f_n)_{n\geq 1}$ in the unit ball of HM^Ψ such that

$$\sup_n \big|f_n \circ \phi(re^{i\theta})\big| = \|\delta_{\phi(re^{i\theta})}\|_{(HM^\Psi)^*}.$$

Now, suppose that C_ϕ is order bounded into $L^\Psi(\mathbb{T})$ (resp. into $M^\Psi(\mathbb{T})$). Then there exists some g in $L^\Psi(\mathbb{T})$ (resp. in $M^\Psi(\mathbb{T})$) such that $g \geq |C_\phi(f)|$ *a.e.*, for every f in the unit ball of H^Ψ. Using Lemma 3.11, we have *a.e.*

$$\Psi^{-1}\Big(\frac{1}{1-|\phi(re^{i\theta})|}\Big) \leq 4\|\delta_{\phi(re^{i\theta})}\|_{(H^\Psi)^*} = 4\sup_n \big|f_n\circ\phi(re^{i\theta})\big| \leq 4g.$$

The result hence follows letting $r \uparrow 1$.

The converse is obvious. □

Theorem 3.15 *If the composition operator $C_\phi\colon H^\Psi \to H^\Psi$ is order bounded, then:*

(OB3) $$m(1-|\phi| < \lambda) = O\Big(\frac{1}{\chi_A(1/\lambda)}\Big), \quad \textit{as } \lambda \to 0, \quad \textit{for some } A > 0.$$

and if it is order bounded into $M^\Psi(\mathbb{T})$, then:

(OB4) $$m(1-|\phi| < \lambda) = O\Big(\frac{1}{\chi_A(1/\lambda)}\Big), \quad \textit{as } \lambda \to 0, \quad \textit{for every } A > 0.$$

Under the hypothesis $\Psi \in \Delta^1$, the converse holds.

Proof. The necessary condition follows from Proposition 3.14 and Markov's inequality:

$$m\Big(\frac{1}{1-|\phi|} > t\Big) \leq \frac{1}{\chi_A(t)} \int_{\mathbb{T}} \chi_A\Big(\frac{1}{1-|\phi|}\Big)\, dm.$$

For the converse, we shall prove a stronger result, and for that, we define the *weak-L^Ψ space* as follows:

Definition 3.16 *The weak-L^Ψ space $L^{\Psi,\infty}(\Omega)$ is the space of measurable functions $f\colon \Omega \to \mathbb{C}$ such that, for some constant $c > 0$, one has, for every $t > 0$:*

$$\mathbb{P}(|f| > t) \leq \frac{1}{\Psi(ct)}\,\cdot$$

For subsequent references, we shall state separately the following elementary result.

Lemma 3.17 *For every $f \in L^\Psi(\Omega)$, one has, for every $t > 0$:*

$$\|f\|_\Psi \geq \frac{t}{\Psi^{-1}\big(\frac{1}{\mathbb{P}(|f|>t)}\big)}\,\cdot$$

Proof. By Markov's inequality, one has, for $t > 0$:

$$\Psi\Big(\frac{t}{\|f\|_\Psi}\Big)\mathbb{P}(|f| > t) \leq \int_\Omega \Psi\Big(\frac{|f|}{\|f\|_\Psi}\Big)\, d\mathbb{P} \leq 1;$$

and that gives the lemma. □

Since Lemma 3.17 can be read:

$$\mathbb{P}(|f| > t) \leq \frac{1}{\Psi(t/\|f\|_\Psi)}\,,$$

we get that $L^\Psi(\Omega) \subseteq L^{\Psi,\infty}(\Omega)$.

The converse of Theorem 3.15 is now an immediate consequence of the following proposition.

Proposition 3.18
1) *a) If $\Psi \in \Delta^1$, then $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$.*
b) If $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$, then $\Psi \in \Delta^0$.
2) *If $L^\Psi(\mathbb{T}) = L^{\Psi,\infty}(\mathbb{T})$, then condition (OB3) implies that $C_\phi\colon H^\Psi \to H^\Psi$ is order bounded, and condition (OB4) implies that it is order bounded into $M^\Psi(\mathbb{T})$.*

Lemma 3.19 *The following assertions are equivalent*
i) $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$.
ii) $\displaystyle\int_1^\infty \frac{\Psi'(u)}{\Psi(Bu)}\, du \equiv \int_{\Psi(1)}^{+\infty} \frac{1}{\chi_B(x)}\, dx < +\infty$, for some $B > 1$.

Proof of the Lemma. Assume first that $1/\chi_B$ is integrable on $(\Psi(1), \infty)$. For every $f \in L^{\Psi,\infty}(\Omega)$, there is a $c > 0$ such that $\mathbb{P}(|f| > t) \leq 1/\Psi(ct)$. Then

$$\begin{aligned}\int_\Omega \Psi\Big(c\frac{|f|}{B}\Big)\, d\mathbb{P} &= \int_0^{+\infty} \mathbb{P}(|f| > Bt/c)\, \Psi'(t)\, dt \\ &\leq \Psi(1) + \int_1^{+\infty} \mathbb{P}(|f| > Bt/c)\, \Psi'(t)\, dt \\ &\leq \Psi(1) + \int_1^{+\infty} \frac{\Psi'(t)}{\Psi(Bt)}\, dt = \Psi(1) + \int_{\Psi(1)}^{+\infty} \frac{du}{\chi_B(u)} < +\infty,\end{aligned}$$

so that $f \in L^\Psi(\Omega)$.

Conversely, assume that $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$. Since $1/\Psi$ is decreasing, there is a measurable function $f\colon \Omega \to \mathbb{C}$ such that $\mathbb{P}(|f| > t) = 1/\Psi(t)$, when $t \geq \Psi^{-1}(1)$. Such a function is in $L^{\Psi,\infty}(\Omega)$; hence it is in $L^\Psi(\Omega)$, by our hypothesis. Therefore, there is a $B > 1$ such that

$$\int_\Omega \Psi(|f|/B)\, d\mathbb{P} < +\infty.$$

But

$$\begin{aligned}\int_\Omega \Psi(|f|/B)\, d\mathbb{P} = \int_0^{+\infty} \mathbb{P}(|f| > Bt)\, \Psi'(t)\, dt &\geq \int_{\frac{1}{B}\Psi^{-1}(1)}^{+\infty} \frac{\Psi'(t)}{\Psi(Bt)}\, dt \\ &\geq \int_{\Psi^{-1}(1)}^{+\infty} \frac{\Psi'(t)}{\Psi(Bt)}\, dt = \int_1^{+\infty} \frac{du}{\chi_B(u)}\,,\end{aligned}$$

and hence $1/\chi_B$ is integrable on $(1, \infty)$. □

Proof of Proposition 3.18. 1)a) We first remark that for every Orlicz function Ψ, one has $\Psi(2x) \geq x\Psi'(x)$ for every $x > 0$, because, since Ψ' is positive and increasing:

$$\Psi(2x) = \int_0^{2x} \Psi'(t)\, dt \geq \int_x^{2x} \Psi'(t)\, dt \geq x\Psi'(x).$$

Assume now that $\Psi \in \Delta^1$: $x\Psi(x) \leq \Psi(\beta x)$ for some $\beta > 0$ and for $x \geq x_0 > 0$. By Lemma 3.19, it suffices to show that $1/\chi_{2\beta}$ is integrable on $(\Psi(x_0), +\infty)$. But

$$\int_{\Psi(x_0)}^{+\infty} \frac{dx}{\chi_{2\beta}(x)} = \int_{x_0}^{+\infty} \frac{\Psi'(u)}{\Psi(2\beta u)}\, du \leq \int_{x_0}^{+\infty} \frac{\Psi(2u)/u}{2u\Psi(2u)}\, du = \int_{x_0}^{+\infty} \frac{du}{2u^2} < +\infty.$$

1)b) Suppose now that $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$. By the preceding lemma, there exists some $B > 1$ such that

$$\lim_{x\to+\infty} \int_x^{2x} \frac{\Psi'(u)}{\Psi(Bu)}\, du = 0.$$

By convexity, $\Psi(2x) \geq 2\Psi(x)$ and Ψ' is nonnegative, so that

$$\int_x^{2x} \frac{\Psi'(u)}{\Psi(Bu)}\, du \geq \frac{1}{\Psi(2Bx)} \int_x^{2x} \Psi'(u)\, du \geq \frac{\Psi(2x) - \Psi(x)}{\Psi(2Bx)} \geq \frac{\Psi(x)}{\Psi(2Bx)} \cdot$$

Therefore Ψ satisfies Δ^0.

2) Assume that $L^\Psi(\mathbb{T}) = L^{\Psi,\infty}(\mathbb{T})$ and that condition (OB3) (resp. (OB4)) holds. By Lemma 3.19, there is a $B > 0$ such that $1/\chi_B$ is integrable on $(1, +\infty)$. We get, using condition (OB3) (resp. (OB4)) and setting $C = A/B$:

$$\begin{aligned}
\int_{\mathbb{T}} \chi_C\Big(\frac{1}{1-|\phi|}\Big)\, dm &= \int_0^{+\infty} m\Big(\frac{1}{1-|\phi|} > t\Big) \chi_C'(t)\, dt \\
&= \int_0^{+\infty} m(1 - |\phi| < 1/t)\, \chi_C'(t)\, dt \\
&\leq \chi_C(1) + K \int_1^{+\infty} \frac{\chi_C'(t)}{\chi_A(t)}\, dt.
\end{aligned}$$

But, if we set $u = \chi_C(t)$, *i.e.* $u = \Psi\big(C\Psi^{-1}(t)\big)$, then $\Psi^{-1}(u) = C\Psi^{-1}(t)$, and hence

$$\chi_A(t) = \Psi\big(A\Psi^{-1}(t)\big) = \Psi\big(B\Psi^{-1}(u)\big) = \chi_B(u).$$

Therefore:

$$\int_{\mathbb{T}} \chi_C\Big(\frac{1}{1-|\phi|}\Big)\, dm \leq \chi_C(1) + K \int_{\chi_C(1)}^{+\infty} \frac{du}{\chi_B(u)}\, du < +\infty.$$

It follows from Proposition 3.14 that C_ϕ is order bounded (resp. into $M^\Psi(\mathbb{T})$). □

Remark. The condition $\Psi \in \Delta^1$ is not equivalent to $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$. For example, we can take:

$$\Psi(x) = \exp\big[\big(\log(x+1)\big)^{3/2}\big] - 1.$$

Then, as x tends to infinity,

$$\Psi(Kx) \sim \Psi(x)\, \exp\Big(\frac{3}{2}(\log K)(\log x)^{1/2}\Big),$$

and hence $\Psi \notin \Delta^1$. On the other hand, $\Psi'(x) \sim \frac{3}{2}\frac{(\log x)^{1/2}}{x}\Psi(x)$; hence:

$$\begin{aligned}
\int_{\Psi(1)}^{+\infty} \frac{du}{\chi_K(u)} &= \int_1^{+\infty} \frac{\Psi'(t)}{\Psi(Kt)}\, dt \\
&\sim \int_0^{+\infty} \frac{3u^2}{\exp(\frac{3}{2}(\log K)u)}\, du = \int_0^{+\infty} \frac{3u^2}{K^{3u/2}}\, du < +\infty
\end{aligned}$$

for $K > 1$. Therefore $L^\Psi(\Omega) = L^{\Psi,\infty}(\Omega)$ by Lemma 3.19.

3.5 Weakly compact composition operators

We saw in Lemma 3.10 that

$$\|u_{a,r}\|_\Psi \leq \frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\,\cdot$$

The next result shows that the weak compactness of C_ϕ transforms the "***big-oh***" into a "*little-oh*", when Ψ grows fast enough.

Theorem 3.20 *Assume that $\Psi \in \Delta^0$. Then the weak compactness of the composition operator $C_\phi\colon H^\Psi \to H^\Psi$ implies that:*

$$\text{(W)} \qquad \sup_{a\in\mathbb{T}} \|C_\phi(u_{a,r})\|_\Psi = o\left(\frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\right), \quad \textit{as } r \to 1.$$

Proof. Actually, we only need to use that the restriction of $C_\phi\colon H^\Psi \to H^\Psi$ to HM^Ψ is weakly compact. We proved in [25], Theorem 4, that, under the hypothesis $\Psi \in \Delta^0$, the operator $C_\phi\colon HM^\Psi \to HM^\Psi$ is weakly compact if and only if for every $\varepsilon > 0$, we can find $K_\varepsilon > 0$ such that, for every $f \in HM^\Psi$, one has:

$$\|C_\phi(f)\|_\Psi \leq K_\varepsilon \|f\|_1 + \varepsilon\, \|f\|_\Psi.$$

Using Corollary 3.10, we get, for every $\varepsilon > 0$, since $\|u_{a,r}\|_1 \leq 1 - r$:

$$\|C_\phi(u_{a,r})\|_\Psi \leq K_\varepsilon(1-r) + \varepsilon\, \frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\,\cdot$$

But $\frac{\Psi(x)}{x} \underset{x\to+\infty}{\longrightarrow} +\infty$. Hence

$$1 - r = o\Big(\frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\Big) \quad \text{as } r \to 1,$$

and that proves the theorem. □

We shall see in Section 4 that the converse holds when Ψ satisfies Δ^0 and moreover that C_ϕ is then compact. That will use some other techniques. Nevertheless, we can prove, from now on, the following result.

Theorem 3.21 *If $\Psi \in \Delta^2$, then condition:*

$$\text{(W)} \qquad \sup_{a\in\mathbb{T}} \|C_\phi(u_{a,r})\|_\Psi = o\left(\frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\right), \quad \textit{as } r \to 1.$$

in Theorem 3.20 implies condition:

$$\text{(OB4)} \qquad m(1 - |\phi| < \lambda) = O\Big(\frac{1}{\chi_A(1/\lambda)}\Big), \quad \textit{as } \lambda \to 0, \quad \textit{for every } A > 0.$$

in Theorem 3.15.

Remark that when $\Psi \in \Delta^0$ (and in particular when $\Psi \in \Delta^2$), one has, for any $B > \beta A$ (where β is given by the definition of Δ^0): $\frac{\Psi(Bx)}{\Psi(Ax)} \to +\infty$ as $x \to +\infty$; hence $1/\chi_A(x) = o\big(1/\chi_B(x)\big)$ and the "*big-oh*" condition in (OB4) may be replaced by a "*little-oh*" condition.

Before proving this theorem, let us note that:

Proposition 3.22 *Condition*

$$\text{(W)} \qquad \sup_{a\in\mathbb{T}} \|C_\phi(u_{a,r})\|_\Psi = o\left(\frac{1}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\right) \quad \text{as } r \to 1$$

implies that $m(|\phi| = 1) = 0$.

Proof. Otherwise, one has $m(|\phi| = 1) = \delta > 0$. Splitting the unit circle $\mathbb{T}$ into N parts, we get some $a \in \mathbb{T}$ such that:

$$m(|\phi - a| \leq \pi/N) \geq \delta/N.$$

But, the inequality $|\phi - a| \leq \pi/N$ implies, with $r = 1 - 1/N$ (since $|\phi| \leq 1$):

$$|1 - \bar{a}r\phi| \leq |1 - \bar{a}\phi| + (1 - r)|\bar{a}\phi| = |a - \phi| + (1 - r)|\phi| < 5/N = 5(1 - r).$$

Hence:

$$\begin{aligned} m\big(|C_\phi(u_{a,r})| > 1/25\big) &= m\big(|1 - \bar{a}r\phi| < 5(1 - r)\big) \\ &\geq m\big(|\phi - a| \leq \pi(1 - r)\big) \geq \delta(1 - r)\,, \end{aligned}$$

and therefore, by Lemma 3.17:

$$\|C_\phi(u_{a,r})\|_\Psi \geq \frac{1/25}{\Psi^{-1}\big(1/\delta(1 - r)\big)} \geq \frac{\delta/25}{\Psi^{-1}\big(1/(1 - r)\big)}\,.$$

Since r can be taken arbitrarily close to 1, that proves the proposition. □

Proof of Theorem 3.21. Assume that condition (W) is satisfied, and fix $A > 1$. Let $\varepsilon > 0$ to be adjusted later. We can find $r_\varepsilon < 1$ such that $r_\varepsilon \leq r < 1$ implies:

$$\|C_\phi(u_{a,r})\|_\Psi \leq \frac{\varepsilon}{\Psi^{-1}\big(\frac{1}{1-r}\big)}\,, \quad \forall a \in \mathbb{T}.$$

Now, Lemma 3.17 also reads:

$$m(|f| > t) \leq \frac{1}{\Psi\big(\frac{t}{\|f\|_\Psi}\big)}\,,$$

so that, if one sets $B = 1/9\varepsilon$:

$$m\big(|1 - \bar{a}r\phi| < 3(1 - r)\big) = m\big(|C_\phi(u_{a,r})| > 1/9\big) \leq \frac{1}{\Psi\big[B\Psi^{-1}\big(\frac{1}{1-r}\big)\big]}\,.$$

We claim that this implies a good upper bound on $m(|\phi| > r)$, even if we loose a factor $1/(1 - r)$, due to the effect of a rotation on ϕ. For that, we shall use the following lemma.

Lemma 3.23 *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be an analytic function. Then, for every r with $0 < r < 1$, there exists $a \in \mathbb{T}$ such that:*

$$m\big(|1 - \bar{a} r\, \phi| < 3(1 - r)\big) \geq \frac{1 - r}{8}\, m(|\phi| > r).$$

Admitting this for a while, we are going to finish the proof.

Fix an r such that $r_\varepsilon \leq r < 1$, and take an $a \in \mathbb{T}$ as in Lemma 3.23. We get, from the preceding, in setting $\lambda = 1 - r$:

$$\begin{aligned} m(1 - |\phi| < \lambda) &= m(|\phi| > r) \\ &\leq \frac{8}{1 - r} m\big(|1 - \bar{a} r \phi| < 3(1 - r)\big) \\ &\leq \frac{8}{1 - r} \frac{1}{\Psi\big[B \Psi^{-1}\big(\frac{1}{1 - r}\big)\big]}\,, \end{aligned}$$

i.e., setting $x = \Psi^{-1}(1/1 - r) = \Psi^{-1}(1/\lambda)$:

$$m(1 - |\phi| < \lambda) \leq 8\, \frac{\Psi(x)}{\Psi(Bx)} \cdot$$

But Ψ satifies the Δ^2-condition: $\big[\Psi(y)\big]^2 \leq \Psi(\alpha y)$ for some $\alpha > 1$ and y large enough. Then, adjusting now $\varepsilon > 0$ as $\varepsilon = 1/9\alpha A$, in order that $B = \alpha A$, we get, for x large enough, since $A > 1$:

$$\Psi(x)\Psi(Ax) \leq [\Psi(Ax)]^2 \leq \Psi(Bx).$$

Therefore, for r close enough to 1:

$$m(1 - |\phi| < \lambda) \leq \frac{8}{\Psi(Ax)} = \frac{8}{\chi_A(1/\lambda)} \cdot$$

We hence get condition (OB4), and that proves Theorem 3.21. □

Proof of Lemma 3.23. Let $\lambda = 1 - r$, and let $\delta > 0$ be a number which we shall specify later. Consider the set:

$$C_\delta = \{z \in \mathbb{D}\,;\ |z| \geq 1 - \lambda \ \text{ and } \ |\arg z| \leq \delta\}$$

(for $\delta = \lambda$, C_δ is a closed Carleson window). It is geometrically clear that $(1 - \lambda) C_\delta$ is contained in the closed disk of center 1 and whose edge contains $(1 - \lambda)^2 \mathrm{e}^{i\delta}$; hence, for every $z \in C_\delta$, one has:

$$\begin{aligned} |1 - (1 - \lambda) z|^2 \leq |1 - (1 - \lambda)^2 \mathrm{e}^{i\delta}|^2 &= 2(1 - \lambda)^2 (1 - \cos\delta) + \lambda^2 (2 - \lambda)^2 \\ &\leq (1 - \lambda)^2 \delta^2 + \lambda^2 (2 - \lambda)^2 \leq 9\lambda^2 \end{aligned}$$

if $\delta \leq \lambda$.

By rotation, one has, for every $a \in \mathbb{T}$:

$$|z| \geq 1 - \lambda \quad \text{and} \quad |\arg(\bar{a}z)| \leq \delta \quad \Rightarrow \quad |1 - (1-\lambda)\bar{a}z| \leq 3\lambda.$$

Let now $N \geq 2$ be the integer such that:

$$\frac{\pi}{N} \leq \lambda < \frac{\pi}{N-1}\,,$$

and take $\delta = \pi/N$.

One has, by the previous inequalities, setting $a_k = \mathrm{e}^{2ik\delta}$:

$$\{z \in \mathbb{D}\,;\ |z| \geq 1-\lambda\} = \bigcup_{1\leq k\leq N} \bar{a}_k C_\delta \subseteq \bigcup_{1\leq k\leq N} \{z \in \mathbb{D}\,;\ |1-(1-\lambda)\bar{a}_k z| \leq 3\lambda\}.$$

Hence, with $z = \phi(\mathrm{e}^{it})$ (remark that, by Proposition 3.22, we have only to consider the values of e^{it} for which $|\phi(\mathrm{e}^{it})| < 1$; however, in this lemma, we may replace $\mathbb{D}$ by $\overline{\mathbb{D}}$), and get:

$$m(|\phi| \geq 1-\lambda) \leq N \sup_{1\leq k\leq N} m(|1-(1-\lambda)\bar{a}_k\phi| \leq 3\lambda).$$

Therefore, we can find some k such that:

$$m(|1-(1-\lambda)\bar{a}_k\phi| \leq 3\lambda) \geq \frac{1}{N} m(|\phi| \geq 1-\lambda) \geq \frac{\lambda}{8} m(|\phi| \geq 1-\lambda),$$

since $\lambda \leq 2\pi/N \leq 8/N$. That proves Lemma 3.23. □

Since the Δ^2-condition implies the Δ^1-condition, which, in its turn, implies the Δ^0-condition, we get, from Theorem 3.15, Theorem 3.20 and Theorem 3.21 that the weak compactness of C_ϕ implies its order boundedness into $M^\Psi(\mathbb{T})$, and thanks to the Remark after Proposition 3.14, its compactness. We get

Theorem 3.24 *If Ψ satisfies the Δ^2-condition, then the following assertions for composition operator $C_\phi\colon H^\Psi \to H^\Psi$ are equivalent:*

1) *C_ϕ is order bounded into $M^\Psi(\mathbb{T})$;*

2) *C_ϕ is compact;*

3) *C_ϕ is weakly compact;*

4) *$\Psi^{-1}\big(\frac{1}{1-|\phi|}\big) \in M^\Psi(\mathbb{T})$ (i.e.: $\chi_B(1/1-|\phi|) \in L^1(\mathbb{T})$ for every $B > 0$);*

5) *$m(1-|\phi| < \lambda) = O\left(\frac{1}{\chi_A(1/\lambda)}\right)$ as $\lambda \to 0$, for every $A > 0$;*

6) *$\sup_{a\in\mathbb{T}} \|C_\phi(u_{a,r})\|_\Psi = o\left(\frac{1}{\Psi^{-1}\left(\frac{1}{1-r}\right)}\right)$ as $r \to 1$ $\quad (W)$*

Remark. We shall see in the next section (Theorem 4.22) that the assumption $\Psi \in \Delta^2$ cannot be removed in general: Theorem 3.24 is not true for the Orlicz function $\Psi(x) = \exp\big[\big((\log(x+1)\big)^2\big] - 1$ (which nevertheless satisfies Δ^1).

If one specializes this corollary to the case where $\Psi(x) = \Psi_2(x) = \mathrm{e}^{x^2} - 1$, which verifies the Δ^2-condition, we get, using Stirling's formula:

Corollary 3.25 *The following assertions are equivalent:*

1) $C_\phi \colon H^{\Psi_2} \to H^{\Psi_2}$ *is order bounded into* $M^{\Psi_2}(\mathbb{T})$*;*

2) $C_\phi \colon H^{\Psi_2} \to H^{\Psi_2}$ *is compact;*

3) $C_\phi \colon H^{\Psi_2} \to H^{\Psi_2}$ *is weakly compact;*

4) $\frac{1}{1-|\phi|} \in L^p(\mathbb{T}), \forall p \geq 1$*;*

5) $\forall q \geq 1\ \exists C_q > 0$*:* $\quad m(1-|\phi| < \lambda) \leq C_q \lambda^q$*;*

6) $\forall q \geq 1\ \|\phi^n\|_1 = o\,(n^{-q})$*;*

7) $\|\phi^n\|_{\Psi_2} = o\,(1/\sqrt{\log n})$.

As a consequence of Theorem 3.24, we obtain the following:

Corollary 3.26 *Assume that* $\Psi \in \Delta^2$*. Then there exist compact composition operators* $C_\phi \colon H^p \to H^p$ *for* $1 \leq p < \infty$ *which are not compact as operators* $C_\phi \colon H^\Psi \to H^\Psi$*.*

Remark. We shall see in Theorem 4.3 that compactness on H^Ψ implies compactness on H^p for $p < \infty$. Note that this shows that, though H^Ψ is an interpolation space between H^1 and H^∞ (see [5], Theorem V.10.8), the compactness of $C_\phi \colon H^1 \to H^1$ with the continuity of $C_\phi \colon H^\infty \to H^\infty$ does not suffice to have compactness for $C_\phi \colon H^\Psi \to H^\Psi$.

Proof. As the Δ^2 condition implies the Δ^0 condition, we have $x = o\big(\chi_\beta(x)\big)$ as $x \to \infty$, for some $\beta > 1$. It follows that we can find a positive function $a \colon \mathbb{T} \to \mathbb{R}_+$ such that $a \geq 2$, $a \in L^1$ but $\chi_\beta(a) \notin L^1$. Set $h = 1 - \frac{1}{a}$. One has $1/2 \leq h \leq 1$ and in particular $\log h \in L^1(\mathbb{T})$. Then the outer function $\phi \colon \mathbb{D} \to \mathbb{C}$ defined for $z \in \mathbb{D}$ by:

$$\phi(z) = \exp\left[\int_{\mathbb{T}} \frac{u+z}{u-z}\, \log h(u)\, dm(u)\right]$$

is analytic on $\mathbb{D}$ and its boundary limit verifies $|\phi| = h \leq 1$ on $\mathbb{T}$. By [39], Theorem 6.2, $C_\phi \colon H^2 \to H^2$ is Hilbert-Schmidt, and hence compact, since $\int_{\mathbb{T}} \frac{1}{1-|\phi|}\, dm = \int_{\mathbb{T}} a\, dm < +\infty$. It is then compact from H^p to H^p for every $p < \infty$ ([39], Theorem 6.1). However, $\int_{\mathbb{T}} \chi_\beta\big(\frac{1}{1-|\phi|}\big)\, dm = \int_{\mathbb{T}} \chi_\beta(a)\, dm = +\infty$, and hence, by our Theorem 3.24, C_ϕ is not compact on H^Ψ. $\square$

3.6 p-summing operators.

Recall that an operator $T\colon X \to Y$ between two Banach spaces is said to be p-summing ($1 \leq p < +\infty$) if there is a constant $C > 0$ such that, for every choice of $x_1, \ldots, x_n \in X$, one has:

$$\sum_{k=1}^{n} \|Tx_k\|^p \leq C^p \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} \sum_{k=1}^{n} |x^*(x_k)|^p.$$

In other terms, T maps weakly unconditionaly p-summable sequences into norm p-summable sequences. When $X \subseteq Y = L^\Psi$, this implies that whenever $\sum_{n\geq 1} |g_n| \in L^\Psi$, then $\sum_{n\geq 1} \|Tg_n\|_\Psi^p < +\infty$.

For $1 \leq p < +\infty$, J. H. Shapiro and P. D. Taylor proved in [39], Theorem 6.2, that the condition:

$$\int_{\mathbb{T}} \frac{dm}{1 - |\phi|} < +\infty \tag{3.3}$$

implies that the composition operator $C_\phi\colon H^p \to H^p$ is p-summing (and condition (3.3) is necessary for $1 \leq p \leq 2$; in particular, for $p = 2$, it is equivalent to say that C_ϕ is Hilbert-Schmidt). Actually, they proved that (3.3) is equivalent to the fact that C_ϕ is order bounded on H^p, and (acknowledging to A. Shields, L. Wallen, and J. Williams) every order bounded operator into an L^p-space is p-summing. The counterpart of (3.3) in our setting, are conditions (OB1) and (OB2)

$$\int_{\mathbb{T}} \chi_A\Big(\frac{1}{1 - |\phi|}\Big)\, dm < +\infty$$

in Proposition 3.14. We are going to see that, if Ψ grows fast enough, order boundedeness does not imply that C_ϕ is p-summing. Note that, for composition operators on H^∞, being p-summing is equivalent to being compact ([23], Theorem 2.6), but H^∞ corresponds to the very degenerate Orlicz function $\Psi(x) = 0$ for $0 \leq x \leq 1$ and $\Psi(x) = +\infty$ for $x > 1$, which does not match in the proof below.

Theorem 3.27 *If $\Psi \in \Delta^2$, then there exists a composition operator $C_\phi\colon H^\Psi \to H^\Psi$ which is order bounded into $M^\Psi(\mathbb{T})$, and hence compact, but which is p-summing for no $p \geq 1$.*

Note that every p-summing operator is Dunford-Pettis (it maps the weakly convergent sequences into norm convergent sequences); therefore, when it starts from a reflexive space, it is compact. However, when $\Psi \in \Delta^2$, being Dunford-Pettis implies compactness for composition operators on H^Ψ, though H^Ψ is not reflexive, thanks to the next proposition and Theorem 3.24. Later (see Theorem 4.21), we shall see that, under condition Δ^0, every Dunford-Pettis composition operator is compact.

Proposition 3.28 *When $\Psi \in \nabla_2$, every Dunford-Pettis composition operator satisfies condition* (W).

Proof. Let $g_{a,r} = \Psi^{-1}\big(1/(1-r)\big)u_{a,r}$. If condition (W) were not satisfied, , we could find a sequence $(a_n)_{n\geq 1}$ in $\mathbb{T}$ and a sequence of numbers $(r_n)_{n\geq 1}$ tending to 1 such that $\|C_\phi(g_{a_n,r_n})\|_\Psi \geq \delta > 0$ for all $n \geq 1$. But $(1-r)^2\Psi^{-1}\big(1/(1-r)\big) \underset{r\to 1}{\longrightarrow} 0$. Therefore $g_{a_n,r_n}(z) = (1-r_n)^2\Psi^{-1}\big(1/(1-r_n)\big)/(1-\bar{a}_n r_n z)$ tends to 0 uniformly on compact sets of $\mathbb{D}$. Hence, by Proposition 3.7, $(g_{a_n,r_n})_{n\geq 1}$ tends weakly to 0 (because $g_{a_n,r_n} \in HM^\Psi$ and, on HM^Ψ, the weak-star topology of H^Ψ is the weak topology). Since C_ϕ is Dunford-Pettis, $\big(C_\phi(g_{a_n,r_n})\big)_{n\geq 1}$ tends in norm to 0, and we get a contradiction, proving the proposition. □

Proof of Theorem 3.27. We shall begin with some preliminaries. First, since $\Psi \in \Delta^2$, there exists $\alpha > 1$ such that $\big[\Psi(x)\big]^2 \leq \Psi(\alpha x)$ for x large enough. Hence:

$$\frac{\Psi(x)}{\Psi(x/\alpha)} \geq \Psi\Big(\frac{x}{\alpha}\Big) \underset{x\to+\infty}{\longrightarrow} +\infty.$$

Therefore, there exists, for every $n \geq 1$, some $x_n > 0$ such that:

$$\frac{\Psi(x)}{\Psi(x/\alpha)} \geq 2^n \qquad \forall x \geq x_n.$$

Then

$$\Psi\Big(\frac{x}{\alpha}\Big) \leq \frac{1}{2^n}\Psi(x) + \Psi\Big(\frac{x_n}{\alpha}\Big) \leq \frac{1}{2^n}\Psi(x) + \Psi(x_n) \quad \forall x > 0.$$

For convenience, we shall assume, as we may, that $\Psi(x_n) \geq 1$.
Remark also that, setting $a = \Psi^{-1}(1)$, one has, for every $f \in L^\infty$:

$$\int_{\mathbb{T}} \Psi\Big(a\,\frac{|f|}{\|f\|_\infty}\Big)\,dm \leq 1,$$

so that:

$$\|f\|_\Psi \leq \frac{1}{a}\|f\|_\infty.$$

We are now going to start the construction.

For $n \geq 1$, let $M_n = \log(n+1)$. Choose positive numbers β_n which tend to 0 fast enough to have:

$$\sum_{k>n} \beta_k \leq \beta_n\,, \quad \forall n \geq 1,$$

and

$$t_n = \frac{\Psi^{-1}(8/\beta_n)}{M_n} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Set:

$$r_n = 1 - \frac{1}{\Psi(t_n)}\,.$$

One has $r_n \underset{n\to+\infty}{\longrightarrow} 1$ and:

$$\chi_{M_n}\Big(\frac{1}{1-r_n}\Big) = \frac{8}{\beta_n} \cdot$$

Actually, for the end of the proof, we shall have to choose the β_n's decreasing so fast that:

$$\Big[\Psi\Big(\frac{t_1+\cdots+t_{n-1}}{\alpha}\Big) + \Psi(x_n)\Big]\frac{2^n t_n}{\Psi(t_n)} \leq \frac{1}{2^n} \cdot$$

This is possible, by induction, since $t/\Psi(t) \underset{t\to+\infty}{\longrightarrow} 0$. Note that, since $\Psi(x_n) \geq 1$, one has, in particular: $\sum_{n=1}^{+\infty} \frac{2^n t_n}{\Psi(t_n)} < +\infty$

Let B_n be disjoint measurable subsets of $\mathbb{T}$ with measure $m(B_n) = c\beta_n$ (where $c \geq 1$ is such that $\sum_{n\geq 1} \beta_n = 1/c$), and whose union is $\mathbb{T}$. Define $h\colon \mathbb{T} \to \mathbb{C}$ by:

$$h = \sum_{n\geq 1} r_n \mathbb{I}_{B_n}.$$

One has $\log h \in L^1(\mathbb{T})$, since h does not vanish, $r_n \geq 1/2$ for n large enough, and $\sum_n m(B_n) = 1 < +\infty$. We can define the outer function:

$$\phi(z) = \exp\Big[\int_{\mathbb{T}} \frac{u+z}{u-z}\,\log h(u)\,dm(u)\Big], \quad |z|<1.$$

ϕ is analytic on $\mathbb{D}$ and its boundary limit verifies $|\phi| = h \leq 1$ on $\mathbb{T}$. Hence ϕ defines a composition operator on H^Ψ.

For any $A > 0$, one has, when n is large enough to ensure $M_n \geq A$, and when $r_n \leq r < r_{n+1}$:

$$\begin{aligned} m(|\phi| > r) &= \sum_{k>n} m(|\phi| = r_k) = \sum_{k>n} c\,\beta_k \leq c\,\beta_n = \frac{8\,c}{\chi_{M_n}\big(1/(1-r_n)\big)} \\ &\leq \frac{8\,c}{\chi_A\big(1/(1-r_n)\big)} \leq \frac{8\,c}{\chi_A\big(1/(1-r)\big)} \cdot \end{aligned}$$

Since $r_n \underset{n\to+\infty}{\longrightarrow} 1$, it follows from Theorem 3.15 that $C_\phi\colon H^\Psi \to H^\Psi$ is order-bounded into $M^\Psi(\mathbb{T})$ (and hence is compact).

We are now going to construct a sequence of functions $g_n \in H^\Psi$ such that $\sum_n |g_n| \in L^\Psi$, but $\sum_n \|C_\phi(g_n)\|_\Psi^p = +\infty$ for all $p \geq 1$. That will prove that C_ϕ is p-summing for no $p \geq 1$.

Since

$$m(|\phi| \geq r_n) \geq m(|\phi| = r_n) = c\,\beta_n \geq \beta_n,$$

we can apply Lemma 3.17 and Lemma 3.23 (which remain valid with non-strict inequalities instead of strict ones), and we are able to find, for every $n \geq 1$, some $a_n \in \mathbb{T}$ such that:

$$\|C_\phi(u_{a_n,r_n})\|_\Psi \geq \frac{1/9}{\Psi^{-1}\Big(\dfrac{8}{(1-r_n)\beta_n}\Big)} \cdot$$

But:

$$\frac{8}{(1-r_n)\beta_n} = \Psi(t_n)\Psi(M_n t_n).$$

Since now Ψ satisfies Δ^2: $[\Psi(x)]^2 \leq \Psi(\alpha x)$, for x large enough and one has, for n large enough, since $M_n \geq 1$ (for $n \geq 2$):

$$\Psi(t_n)\Psi(M_n t_n) \leq \big[\Psi(M_n t_n)\big]^2 \leq \Psi(\alpha M_n t_n).$$

Therefore:

$$\|C_\phi(u_{a_n,r_n})\|_\Psi \geq \frac{1/9}{\alpha M_n t_n} \cdot$$

Taking now:

$$g_n = \Psi^{-1}\Big(\frac{1}{1-r_n}\Big) u_{a_n,r_n} = t_n u_{a_n,r_n} \,,$$

one has $\|g_n\|_\Psi \leq 1$ (by Corollary 3.10), and

$$\|C_\phi(g_n)\|_\Psi \geq \frac{1/9}{\alpha M_n} = \frac{1/9}{\alpha \log(n+1)} \cdot$$

Therefore

$$\sum_{n=1}^{+\infty} \|C_\phi(g_n)\|_\Psi^p = +\infty$$

for every $p \geq 1$.

It remains to show that $g = \sum_n |g_n| \in L^\Psi$.

We shall follow the lines of proof of Theorem II.1 in [24].

By Markov's inequality, one has:

$$m(|g_n| > 2^{-n}) \leq 2^n t_n \|u_{a_n,r_n}\|_1 \leq 2^n t_n (1-r_n) = \frac{2^n t_n}{\Psi(t_n)} \cdot$$

Set:

$$A_n = \{|g_n| > 2^{-n}\} \, ; \qquad \tilde{A}_n = A_n \setminus \bigcup_{j>n} A_j,$$

and

$$\breve{g}_n = g_n \mathbb{1}_{\{|g_n| > 2^{-n}\}}.$$

Since:

$$\sum_{n=1}^{+\infty} \|g_n - \breve{g}_n\|_\Psi \leq \frac{1}{a} \sum_{n=1}^{+\infty} \|g_n - \breve{g}_n\|_\infty \leq \frac{1}{a} \sum_{n=1}^{+\infty} \frac{1}{2^n} = \frac{1}{a} < +\infty,$$

it suffices to show that $\breve{g} = \sum_n |\breve{g}_n| \in L^\Psi$.

But $\breve{g}$ vanishes out of $\bigcup_{n\geq 1} \tilde{A}_n \cup \big(\limsup_n A_n\big)$, and $m\big(\limsup_n A_n\big) = 0$, since

$$\sum_{n=1}^{+\infty} m(A_n) \leq \sum_{n=1}^{+\infty} \frac{2^n t_n}{\Psi(t_n)} < +\infty \, .$$

Therefore:

$$\int_{\mathbb{T}} \Psi\Big(\frac{|\breve{g}|}{2\alpha}\Big)\, dm = \sum_{n=1}^{+\infty} \int_{\tilde{A}_n} \Psi\Big(\frac{|\breve{g}|}{2\alpha}\Big)\, dm.$$

Since $\breve{g}_j = 0$ on $\tilde{A}_n$ for $j > n$, we get:

$$\int_{\mathbb{T}} \Psi\Big(\frac{|\breve{g}|}{2\alpha}\Big)\, dm = \sum_{n=1}^{+\infty} \int_{\tilde{A}_n} \Psi\Big(\frac{|\breve{g}_1| + \cdots + |\breve{g}_n|}{2\alpha}\Big)\, dm.$$

Now, by the convexity of Ψ:

$$\Psi\Big(\frac{|\breve{g}_1| + \cdots + |\breve{g}_n|}{2\alpha}\Big) \leq \frac{1}{2}\Big[\Psi\Big(\frac{|\breve{g}_1| + \cdots + |\breve{g}_{n-1}|}{\alpha}\Big) + \Psi\Big(\frac{|\breve{g}_n|}{\alpha}\Big)\Big]\,.$$

But:

$$\Psi\Big(\frac{|\breve{g}_n|}{\alpha}\Big) \leq \frac{1}{2^n}\Psi(|\breve{g}_n|) + \Psi(x_n)\,,$$

and

$$\Psi\Big(\frac{|\breve{g}_1| + \cdots + |\breve{g}_{n-1}|}{\alpha}\Big) \leq \Psi\Big(\frac{t_1 + \cdots + t_{n-1}}{\alpha}\Big)\,;$$

therefore, using that $\int_{\mathbb{T}} \Psi(|\breve{g}_n|)\, dm \leq \int_{\mathbb{T}} \Psi(|g_n|)\, dm \leq 1$:

$$\begin{aligned}
\int_{\mathbb{T}} \Psi\Big(\frac{|\breve{g}|}{2\alpha}\Big)\, dm &\leq \sum_{n=1}^{+\infty} \frac{1}{2}\Big[\Psi\Big(\frac{t_1 + \cdots + t_{n-1}}{\alpha}\Big) m(\tilde{A}_n) \\
&\qquad\qquad + \frac{1}{2^n}\int_{\mathbb{T}} \Psi(|\breve{g}_n|)\, dm + \Psi(x_n) m(\tilde{A}_n)\Big] \\
&\leq \sum_{n=1}^{+\infty} \frac{1}{2}\Big[\Big[\Psi\Big(\frac{t_1 + \cdots + t_{n-1}}{\alpha}\Big) + \Psi(x_n)\Big]\frac{2^n t_n}{\Psi(t_n)} + \frac{1}{2^n}\Big] \\
&\leq \sum_{n=1}^{+\infty} \frac{1}{2}\Big[\frac{1}{2^n} + \frac{1}{2^n}\Big] = 1\,,
\end{aligned}$$

which proves that $\breve{g} \in L^\Psi$, and $\|\breve{g}\|_\Psi \leq 2\alpha$.

The proof is fully achieved. □

Remark. In the above proof, we chose $M_n = \log(n+1)$. This choice was only used to conclude that $\sum_n \|C_\phi(g_n)\|_\Psi^p = +\infty$ for every $p < \infty$. Therefore, the above proof shows that, given any increasing function $\Upsilon\colon (0,\infty) \to (0,\infty)$ tending to ∞, we can find, with a suitable choice of a slowly increasing sequence $(M_n)_{n\geq 1}$, a symbol ϕ and a sequence $(g_n)_{n\geq 1}$ in H^Ψ such that $\sum_n |g_n| \in L^\Psi$, although $\sum_n \Upsilon(\|C_\phi(g_n)\|_\Psi) = +\infty$.

4 Carleson measures

4.1 Introduction

B. MacCluer ([27]; see also [11], Theorem 3.12) has characterized compact composition operators on Hardy spaces H^p ($p < \infty$) in term of Carleson measures. In this section, we shall give an analogue of this result for Hardy-Orlicz spaces H^Ψ, but in terms of "Ψ-Carleson measures". Indeed, Carleson measures do not characterize the compactness of composition operators when Ψ grows too quickly, as it follows from Corollary 3.26.

Before that, we shall recall some definitions (see for example [11], pages 37–38, or [12], page 157).

Let $\xi \in \mathbb{T}$ and $h \in (0,1)$. Define

$$S(\xi, h) = \{z \in \overline{\mathbb{D}}\,;\ |\xi - z| < h\}. \tag{4.1}$$

The *Carleson window* $W(\xi, h)$ is the following subset of $\overline{\mathbb{D}}$:

$$W(\xi, h) = \{z \in \overline{\mathbb{D}}\,;\ 1 - h < |z| \leq 1 \quad \text{and} \quad |\arg(z\bar{\xi})| < h\}. \tag{4.2}$$

It is easy to show that we have for every $\xi \in \mathbb{T}$ and $h \in (0,1)$:

$$S(\xi, h/2) \subseteq W(\xi, h) \qquad \text{and} \qquad W(\xi, h/2) \subseteq S(\xi, h)\,,$$

so that, in the sequel, we may work equivalently with either $S(\xi, h)$ or $W(\xi, h)$. Recall that a positive Borel measure μ on $\mathbb{D}$ (or $\overline{\mathbb{D}}$) is called a *Carleson measure* if there exists some constant $K > 0$ such that:

$$\mu\big(S(\xi, h)\big) \leq Kh\,, \qquad \forall \xi \in \mathbb{T}\,,\ \forall h \in (0,1).$$

Carleson's Theorem (see [11], Theorem 2.33, or [12], Theorem 9.3) asserts that, for $0 < p < \infty$, the Hardy space H^p is continuously embedded into $L^p(\mu)$ if and only if μ is a Carleson measure.

Given an analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$, we define the *pullback measure* μ_ϕ on the closed unit disk $\overline{\mathbb{D}}$ (which we shall denote simply μ when this is unambiguous) as the image of the Haar measure m of $\mathbb{T} = \partial\mathbb{D}$ under the map ϕ^* (the boundary limit of ϕ):

$$\mu_\phi(E) = m\big({\phi^*}^{-1}(E)\big), \tag{4.3}$$

for every Borel subset E of $\overline{\mathbb{D}}$.

The automatic continuity of composition operators C_ϕ on the Hardy space H^p, combined with Carleson's Theorem means that μ_ϕ is always a Carleson measure.

B. MacCluer ([27], [11], Theorem 3.12) showed that:

(MC) *The composition operator C_ϕ is compact on H^2 if and only if:*

$$\mu_\phi\big(S(\xi, h)\big) = o\,(h) \ \text{ as } h \to 0, \ \text{ uniformly for } \xi \in \mathbb{T}.$$

While the Shapiro's compactness criterion, *via* the Nevanlinna counting function ([36]), deals with the behavior of ϕ *inside* the open unit disk, the characterization (MC) deals with its boundary values ϕ^*. It is natural to wonder whether the modulus of ϕ^* on $\mathbb{T} = \partial\mathbb{D}$ suffices to characterize the compactness of C_ϕ. This leads to the following question: if two functions ϕ_1 and ϕ_2 have the same modulus on $\mathbb{T}$, are the compactness of the two associated composition operators equivalent? We have seen in Theorem 3.24 that the answer is positive on H^Ψ when $\Psi \in \Delta^2$. However, on H^2 it turns out to be negative. We give the following counterexample.

Theorem 4.1 *There exist two analytic functions ϕ_1 and ϕ_2 from $\mathbb{D}$ into itself such that $|\phi_1^*| = |\phi_2^*|$ on $\mathbb{T}$ but for which the composition operator $C_{\phi_2}\colon H^2 \to H^2$ is compact, though $C_{\phi_1}\colon H^2 \to H^2$ is not compact.*

Remark. Let Ψ be an Orlicz function which satisfies Δ^2. We shall see in Theorem 4.3 that every composition operator $C_\phi\colon H^2 \to H^2$ is compact as soon as $C_\phi\colon H^\Psi \to H^\Psi$ is compact. Hence, in the above theorem, $C_{\phi_1}\colon H^\Psi \to H^\Psi$ is not compact. It follows hence from Theorem 3.24, since ϕ_1^* and ϕ_2^* have the same modulus, that $C_{\phi_2}\colon H^\Psi \to H^\Psi$ is not compact (and, even, not weakly compact), though $C_{\phi_2}\colon H^2 \to H^2$ is compact. We have already seen such a phenomenon in Corollary 3.26. However, the results of the next subsection will allow us to conclude (Theorem 4.22) that, when $\Psi(x) = \exp\big[\big(\log(x+1)^2\big] - 1$, which does not satisfy condition Δ^2, but satisfies conditions Δ^1 and ∇_1, the composition operator $C_{\phi_2}\colon H^\Psi \to H^\Psi$ *is* compact, but not order bounded into $M^\Psi(\mathbb{T})$. That will show that our assumption that $\Psi \in \Delta^2$ in Theorem 3.24 is not only a technical one.

Proof. We start simply with $\phi_1(z) = \frac{1+z}{2}$. It is well known that C_{ϕ_1} is not compact on H^2 (this was first observed in H. J. Schwartz's thesis [34]: see [39], page 471). Now, let:

$$M(z) = \exp\Big(-\frac{1+z}{1-z}\Big)$$

and

$$\phi_2(z) = \phi_1(z)\, M(z).$$

For simplicity, we shall write $\phi = \phi_2$, and we are going to show that C_ϕ is a compact operator on H^2, using the criterion (MC).

Let $\xi = \mathrm{e}^{i\alpha} \in \mathbb{T}$, with $|\alpha| \leq \pi$. We are going to prove that:

$$\mu_\phi\big(S(\xi,h)\big) = O\,(h^{3/2}).$$

First, observe that, for $h \in (0,1)$:

$$S(\xi,h) \subseteq \{z \in \overline{\mathbb{D}}\,;\ 1-h < |z| \leq 1 \text{ and } |\arg(\bar{z}\xi)| \leq 2h\}.$$

Hence for h small enough,

$$\mu_\phi\big(S(\xi,h)\big) \leq m\big(\{\theta \in (-\pi,\pi)\,;\ 1-h < |\phi(\mathrm{e}^{i\theta})| \leq 1 \text{ and } |\arg(\mathrm{e}^{-i\alpha}\phi(\mathrm{e}^{i\theta}))| \leq 2h\}\big).$$

For $\theta \in (-\pi, \pi)$, one has $|\phi(\mathrm{e}^{i\theta})| = |\phi_1(\mathrm{e}^{i\theta})| = \cos(\theta/2)$, and so the condition $1 - h < |\phi(\mathrm{e}^{i\theta})| \leq 1$ is equivalent to $1 - h < \cos(\theta/2) < 1$, which implies, since $\cos t = 1 - 2\sin^2(t/2) \leq 1 - 2t^2/\pi^2 \leq 1 - t^2/5$ for $0 \leq t \leq \pi/2$ (because $\sin t \geq \frac{2}{\pi} t$), that $1 - h < 1 - (\theta/2)^2/5$, *i.e.* $\theta^2 \leq 20h$ and so $|\theta| \leq 6\sqrt{h}$. On the other hand, $M(\mathrm{e}^{i\theta}) = \exp\big(-i\cot(\theta/2)\big)$; hence $\arg\phi(\mathrm{e}^{i\theta}) = \theta/2 - \cot(\theta/2)$, modulo 2π. Therefore, for h small enough:

$$\begin{aligned}\mu_\phi\big(S(\xi, h)\big) &\leq m(\{|\theta| \leq 6\sqrt{h}\,;\ |-\alpha + \theta/2 - \cot(\theta/2)| \leq 2h, \bmod 2\pi\}) \\ &\leq 2\sum_{n\in\mathbb{Z}} m(\{|t| \leq 3\sqrt{h}\,;\ |-\alpha + t - \cot t + 2\pi n| \leq 2h\}).\end{aligned}$$

We have to majorize both $\sum_{n\in\mathbb{Z}} m(\{0 < t \leq 3\sqrt{h}\,;\ |-\alpha + t - \cot t + 2\pi n| \leq 2h\})$ and $\sum_{n\in\mathbb{Z}} m(\{0 < t \leq 3\sqrt{h}\,;\ |\alpha + t - \cot t + 2\pi n| \leq 2h\})$.

The function

$$F(t) = t - \cot(t)$$

is increasing , and we define $a_n, b_n \in (0, \pi)$ by:

$$F(a_n) = \alpha - 2\pi(n_h + n) - 2h \quad \text{and} \quad F(b_n) = \alpha - 2\pi(n_h + n) + 2h,$$

where the integer n_h is given large enough to ensure that $a_0 \leq 3\sqrt{h}$. Of course, $a_n < b_n < a_{n-1}$. Observe that $4h = F(b_0) - F(a_0) \geq b_0 - a_0$, and then $b_0 \leq 3\sqrt{h} + 4h \leq 4\sqrt{h}$ for h small enough. One has:

$$\sum_{n\in\mathbb{Z}} m(\{0 < t \leq 3\sqrt{h}\,;\ |-\alpha + t - \cot t + 2\pi n| \leq 2h\}) \leq \sum_{n=0}^{\infty}(b_n - a_n).$$

Since $F'(t) = 1 + \dfrac{1}{\sin^2 t} \geq \dfrac{1}{t^2}$, one has, on the one hand:

$$4h = F(b_n) - F(a_n) = \int_{a_n}^{b_n} F'(t)\,dt \geq \frac{b_n - a_n}{a_n b_n}; \tag{4.4}$$

hence:

$$b_n - a_n \leq 4h a_n b_n \leq 4h b_n^2 \quad \text{, for all } n \geq 0. \tag{4.5}$$

On the other hand, let us first point out that, for $0 \leq t \leq 1$,

$$F'(t) = 1 + \frac{1}{\sin^2 t} \leq 1 + \frac{\pi^2/4}{t^2} \leq \frac{4}{t^2}.$$

Hence, for h small enough:

$$2\pi = F(b_n) - F(b_{n+1}) = \int_{b_{n+1}}^{b_n} F'(t)\,dt \leq 4\frac{b_n - b_{n+1}}{b_{n+1} b_n}\,,$$

and we get:

$$b_{n+1}^2 \leq b_{n+1} b_n \leq \frac{2}{\pi}(b_n - b_{n+1}).$$

Hence, using the fact that (4.4) gives $b_0 - a_0 \leq 4ha_0b_0 \leq 48\,h^2$, we get, from (4.5):

$$\begin{aligned}\sum_{n=0}^{\infty}(b_n - a_n) &\leq (b_0 - a_0) + \frac{8h}{\pi}\sum_{n=0}^{\infty}(b_n - b_{n+1}) \\ &\leq 48\,h^2 + \frac{8h}{\pi}b_0 \leq 48\,h^2 + \frac{8h}{\pi}4\sqrt{h} \leq \Big(48\sqrt{h} + \frac{32}{\pi}\Big)h^{3/2} \leq 11\,h^{3/2},\end{aligned}$$

for h small enough.

In the same way, we have

$$\sum_{n=0}^{\infty} m(\{0 < t \leq 3\sqrt{h}\,;\ |\alpha + t - \cot t + 2\pi n| \leq 2h\}) \leq 11\,h^{3/2}.$$

We can hence conclude that $\mu_\phi\big(S(\xi,h)\big) \leq Ch^{3/2}$, where C is a numerical constant. □

Remark. C_ϕ actually maps continuously H^2 into H^3, and compactly H^2 into H^p, for any $p < 3$ (see [16] or Theorem 4.10 and 4.11).

However, in some cases, the behaviour of $|\phi^*|$ on the boundary $\partial\mathbb{D}$ suffices.

Proposition 4.2 *Let ϕ_1 and ϕ_2 be two analytic self-maps of $\mathbb{D}$ such that $|\phi_1^*| \leq |\phi_2^*|$ on $\partial\mathbb{D}$. Assume that they are both one-to-one on $\mathbb{D}$, and that there exists $a \in \mathbb{D}$ such that $\phi_2(a) = 0$. Then the compactness of $C_{\phi_2}\colon H^2 \to H^2$ implies that of $C_{\phi_1}\colon H^2 \to H^2$.*

Proof. By composing ϕ_1 and ϕ_2 with the automorphism of $\mathbb{D}$ which maps 0 into a, we may assume that $a = 0$. We can hence write $\phi_2(z) = z\phi(z)$, with $\phi\colon \mathbb{D} \to \mathbb{C}$ analytic in $\mathbb{D}$. ϕ does not vanish in $\mathbb{D}$ because of the injectivity of ϕ_2 (this is obvious for $z \neq 0$, and for $z = 0$, follows from the fact that the injectivity of ϕ_2 implies $\phi_2'(0) \neq 0$).

Then there is some $\delta > 0$ such that $|\phi(z)| \geq \delta$ for every $z \in \mathbb{D}$. In fact, by continuity, there is some $\alpha > 0$ and some $0 < r < 1$ such that $|\phi(z)| \geq \alpha$ and $|\phi_2(z)| \leq \alpha$ for $|z| \leq r$. But being analytic and non constant, ϕ_2 is an open map, so there is some $\rho > 0$ such that $\rho\mathbb{D} \subseteq \phi_2(r\overline{\mathbb{D}})$. Injectivity of ϕ_2 shows that $\phi_2(\mathbb{D} \setminus r\overline{\mathbb{D}}) \cap \rho\mathbb{D} = \emptyset$, that is to say that $|\phi_2(z)| \geq \rho$ for $|z| > r$. *A fortiori* $|\phi(z)| \geq \rho$ for $|z| > r$. The claim is proved with $\delta = \min(\alpha,\rho)$.

Then $\frac{1}{\phi} \in H^\infty$, as well as $\frac{\phi_1}{\phi}$. Since $\big|\frac{\phi_1^*}{\phi^*}\big| = \big|\frac{\phi_1^*}{\phi_2^*}\big| \leq 1$, one has $\big|\frac{\phi_1(z)}{\phi(z)}\big| \leq 1$ for every $z \in \mathbb{D}$. Hence:

$$\frac{1 - |\phi_1(z)|}{1 - |z|} \geq \frac{1 - |\phi(z)|}{1 - |z|} = \frac{1 - |\phi_2(z)|}{1 - |z|} - |\phi(z)|.$$

Now ([36], Theorem 3.5), the compactness of $C_{\phi_2}\colon H^2 \to H^2$ implies that

$$\lim_{z \overset{<}{\to} 1} \frac{1 - |\phi_2(z)|}{1 - |z|} = +\infty;$$

we get:

$$\lim_{z \overset{<}{\to} 1} \frac{1 - |\phi_1(z)|}{1 - |z|} = +\infty,$$

which implies the compactness of $C_{\phi_1}\colon H^2 \to H^2$, thanks to the injectivity of ϕ_1 ([36], Theorem 3.2). $\square$

4.2 Compactness on H^Ψ versus compactness on H^2

The equivalence (MC) holds actually for every H^p space (with $p < \infty$) instead of H^2. We are going to see in this section that for Hardy-Orlicz spaces H^Ψ, one needs a new notion of Carleson measures, which one may call *Ψ-Carleson measures*. Before that, we are going to see that condition (MC) allows to get that the compactness of composition operators on H^Ψ always implies that on H^p for $p < \infty$. Recall that, when $\Psi \in \Delta^2$, we have seen in Corollary 3.26 that the converse is not true.

Theorem 4.3 *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be an analytic function. If one of the following conditions:*

i) C_ϕ is a compact operator on H^Ψ

ii) $\Psi \in \Delta^0$ and C_ϕ is a weakly compact operator on H^Ψ

is satisfied, the composition operator C_ϕ is compact on H^2.

Note that we have proved in Theorem 3.24 that the weak compactness of $C_\phi\colon H^\Psi \to H^\Psi$ implies its compactness only when Ψ satisfies the Δ^2 condition. Nevertheless, we shall show in Theorem 4.21 that when $\Psi \in \Delta^0$, the weak compactness of C_ϕ is equivalent to its compactness. This is obviously false (in particular when L^Ψ is reflexive) without any assumption on Ψ.

Proof. We are going to use the characterization (MC) for compact composition operators on H^2.

Suppose that the condition on μ_ϕ is not fulfilled. Then there exist $\beta \in (0,1)$, $\xi_n \in \mathbb{T}$, and $h_n \in (0,1)$, with $h_n \underset{n\to+\infty}{\longrightarrow} 0$, such that:

$$\mu_\phi\big(S(\xi_n, h_n)\big) \geq \beta h_n.$$

We are now going to use the function:

$$v_n(z) = \frac{h_n^2}{(1 - \overline{a}_n z)^2}\,,$$

where:

$$a_n = (1 - h_n)\xi_n.$$

Of course, v_n is actually nothing but $u_{\xi_n, 1-h_n}$. We have by Corollary 3.10:

$$\|v_n\|_\Psi \leq \frac{1}{\Psi^{-1}(1/h_n)}.$$

Define $g_n = \Psi^{-1}(1/h_n)v_n$, which is in the unit ball of HM^Ψ. We have assumed at the beginning of the paper that $x = o\big(\Psi(x)\big)$ as $x \to \infty$; hence $\Psi^{-1}(x) = o\,(x)$ as $x \to \infty$, and so $h_n^2 \Psi^{-1}(1/h_n) \to 0$. Therefore $(g_n)_n$ converges uniformly to zero on compact subsets of $\mathbb{D}$ and $\|g_n\|_1 \to 0$, because $\|g_n\|_1 \leq h_n \Psi^{-1}(1/h_n)$. Then, in both cases, we should have $\|C_\phi(g_n)\|_\Psi \to 0$. Indeed, in case *i)*, this follows from Proposition 3.6, and in case *ii)*, this follows from [25], Theorem 4.

We are going to show that this is not true. Indeed:

$$\begin{aligned}\int_{\mathbb{T}} \Psi\Big(\frac{4}{\beta}|g_n \circ \phi|\Big)\, dm &\geq \int_{\mathbb{D}} \Psi\Big(\frac{4}{\beta}\Psi^{-1}(1/h_n)|v_n(z)|\Big)\, d\mu_\phi \\ &\geq \int_{S(\xi_n, h_n)} \Psi\Big(\frac{4}{\beta}\Psi^{-1}(1/h_n)|v_n(z)|\Big)\, d\mu_\phi.\end{aligned}$$

But when $z \in S(\xi_n, h_n)$, one has $|v_n(z)| \geq 1/4$, because

$$|1 - \overline{a}_n z| \leq |1 - \overline{a}_n \xi_n| + |\overline{a}_n(\xi_n - z)| = h_n + (1 - h_n)h_n \leq 2h_n.$$

We obtain that by convexity (since $\beta < 1$):

$$\begin{aligned}\int_{\mathbb{T}} \Psi\Big(\frac{4}{\beta}|g_n \circ \phi|\Big)\, dm &\geq \int_{\mathbb{T}} \frac{1}{\beta}\Psi\big(4\Psi^{-1}(1/h_n)|v_n(z)|\big)\, dm \\ &\geq \frac{1}{\beta h_n}\mu_\phi\big(S(\xi_n, h_n)\big) \geq 1.\end{aligned}$$

This implies that $\|C_\phi(g_n)\|_\Psi \geq \beta/4$ and proves the theorem. □

4.3 General measures

We used several times the criterion (MC) for compactness on H^2 via Carleson measures. The fact that this provides such a useful tool leads to wonder if the boundedness and the compactness on Hardy-Orlicz spaces can be expressed in such a pleasant manner. Theorem 4.10 and Theorem 4.11 are the Orlicz version of Carleson's Theorem for H^p spaces ([11], Theorem 2.35).

The key for our general characterization is the use of the following functions (see (4.2) for the definition of the Carleson's window $W(\xi, h)$).

Definition 4.4 *For any positive finite Borel measure μ on the unit disk $\mathbb{D}$ (or on $\overline{\mathbb{D}}$), we set, for $h \in (0,1]$:*

$$\rho_\mu(h) = \sup_{\xi \in \mathbb{T}} \mu\big(W(\xi, h)\big), \tag{4.6}$$

$$K_\mu(h) = \sup_{0 < t \leq h} \frac{\rho_\mu(t)}{t} \tag{4.7}$$

Hence $\mu\big(W(\xi, t)\big) \leq K_\mu(h) t$ for $t \leq h$.

The measure μ is a Carleson measure if and only if $K_\mu(h)$ is bounded by a constant K, for $h \in (0,1)$ and this happens as soon as $K_\mu(h_0)$ is finite for some $h_0 \in (0,1)$.

Definition 4.5 *We say that Ψ satisfies the ∇_0 condition if for some $x_0 > 0$, $C \geq 1$ and every $x_0 \leq x \leq y$, one has:*

$$\frac{\Psi(2x)}{\Psi(x)} \leq \frac{\Psi(2Cy)}{\Psi(y)}. \tag{4.8}$$

This is a condition on the regularity of Ψ. It is satisfied if

$$\frac{\Psi(2x)}{\Psi(x)} \leq C\, \frac{\Psi(2y)}{\Psi(y)}.$$

Proposition 4.6 *the following assertions are equivalent*

i) *Ψ satisfies the ∇_0 condition.*

ii) *There exists some $x_0 > 0$ satisfying: for every $\beta > 1$, there exists $C_\beta \geq 1$ such that*

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi(\beta C_\beta y)}{\Psi(y)}\ , \text{ for every } x_0 \leq x \leq y.$$

iii) *There exist $x_0 > 0$, $\beta > 1$ and $C_\beta \geq 1$ such that*

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi(\beta C_\beta y)}{\Psi(y)}\ , \text{ for every } x_0 \leq x \leq y.$$

Proof. We only have to prove $i) \Rightarrow ii)$, since $iii) \Rightarrow i)$ is similar and $ii) \Rightarrow iii)$ is trivial.

If $\beta \in (1,2]$, it is easy, taking $C_\beta = 2C/\beta$. Now, if $\beta \in (2^b, 2^{b+1}]$ for some integer $b \geq 1$, we write for every $x_0 \leq x \leq y$:

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi(2^{b+1} x)}{\Psi(x)} = \frac{\Psi(2^{b+1} x)}{\Psi(2^b x)} \cdots \frac{\Psi(2x)}{\Psi(x)}.$$

But we have for every integer $j \geq 1$: $2^{j-1}x \leq (2C)^{j-1}x \leq (2C)^{j-1}y$, so:

$$\frac{\Psi(2^j x)}{\Psi(2^{j-1}x)} \leq \frac{\Psi((2C)^j y)}{\Psi((2C)^{j-1}y)},$$

and we obtain:

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi((2C)^{b+1}y)}{\Psi(y)} \leq \frac{\Psi(\beta C_\beta y)}{\Psi(y)},$$

where $C_\beta = 2C^{b+1}$. □

Examples. It is immediately seen that the following functions satisfy ∇_0: $\Psi(x) = x^p$, $\Psi(x) = \exp\big[\big(\log(x+1)\big)^\alpha\big] - 1$, $\Psi(x) = \mathrm{e}^{x^\alpha} - 1$, $\alpha \geq 1$.

Note that when $\Psi \in \nabla_0$, with constant $C = 1$, *i.e.* $\Psi(\beta x)/\Psi(x)$ is increasing for x large enough, then we have the dichotomy: either $\Psi \in \Delta_2$, or $\Psi \in \Delta^0$. We shall say that ∇_0 is *uniformly satisfied* if there exist $C \geq 1$ and $x_0 > 0$ such that, for every $\beta > 1$:

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi(C\beta y)}{\Psi(y)} \quad \text{for} \quad x_0 \leq x \leq y, \tag{4.9}$$

One has:

Proposition 4.7

1) Condition Δ^2 implies condition ∇_0 uniformly.

2) If $\Psi \in \nabla_0$ uniformly, then $\Psi \in \nabla_1$.

3) The function $\kappa(x) = \log \Psi(\mathrm{e}^x)$ is convex on $(x_0, +\infty)$ if and only if ∇_0 is satisfied with constant $C = 1$.

We shall say that Ψ *is κ-convex* when κ is convex at infinity. Note that Ψ is κ-convex whenever Ψ is log-convex. In the above examples Ψ is κ-convex; it also the case of $\Psi(x) = x^2/\log x$, $x \geq \mathrm{e}$; but, on the other hand, if $\Psi(x) = x^2 \log x$ for $x \geq \mathrm{e}$, then Ψ is not κ-convex. Nevertheless, for $\beta^2 \leq x \leq y$, one has:

$$\frac{\Psi(\beta x)}{\Psi(x)} = \beta^2\Big(1 + \frac{\log\beta}{\log x}\Big) \leq \frac{3\beta^2}{2} \leq \frac{3}{2}\,\frac{\Psi(\beta y)}{\Psi(y)}\,,$$

and hence $\Psi \in \nabla_0$.

We do not know whether $\Psi \in \nabla_0$ uniformly implies that Ψ is equivalent to an Orlicz function for which the associated function κ is convex.

Proof. 1) Since $\Psi \in \Delta^2$, one has $\big[\Psi(u)\big]^2 \leq \Psi(\alpha u)$ for some $\alpha > 1$ and $x \geq x_0$. We may assume that $\Psi(x_0) \geq 1$. Then, for $y \geq x \geq x_0$ and every $\beta > 1$:

$$\Psi(\beta x)\Psi(y) \leq \big[\Psi(\beta y)\big]^2 \leq \Psi(\alpha\beta y) \leq \Psi(\alpha\beta y)\Psi(x),$$

which is (4.9).

2) Suppose that Ψ satisfies condition ∇_0 uniformly. We may assume that $\Psi(x_0) \geq 1$. Let $x_0 \leq u \leq v$; we can write $u = \beta x_0$ for some $\beta \geq 1$. Then condition (4.9) gives:

$$\Psi(u)\Psi(v) = \Psi(\beta x_0)\Psi(v) \leq \Psi(x_0)\Psi(C\beta v) \leq \Psi\big(\Psi(x_0)C\beta v\big) = \Psi(buv),$$

with $b = C\Psi(x_0)/x_0$.

3) Assume that κ is convex on $(x_0, +\infty)$. For every $\beta > 1$, let $\kappa_\beta(t) = \kappa(t \log \beta) = \log(\Psi(\beta^t))$, which is convex on $(x_0/\log(\beta), +\infty)$. Taking $y \geq x \geq \mathrm{e}^{x_0}$, write $x = \beta^\theta$ and $y = \beta^{\theta'}$ with $\theta \leq \theta'$. Convexity of κ gives, since $\theta' \geq \theta \geq x_0/\log(\beta)$:

$$\kappa_\beta(\theta + 1) - \kappa_\beta(\theta) \leq \kappa_\beta(\theta' + 1) - \kappa_\beta(\theta'),$$

which means that:

$$\frac{\Psi(\beta x)}{\Psi(x)} \leq \frac{\Psi(\beta y)}{\Psi(y)}.$$

Assume that (4.9) is fulfilled for every $\beta > 1$, with $C = 1$. Then, taking $y = \beta x$, one has:

$$\big[\Psi(\beta x)\big]^2 \leq \Psi(x)\Psi(\beta^2 x).$$

Let $u < v$ be large enough. Taking $x = u^2$ and $\beta = v/u$, we get:

$$\big[\Psi(uv)\big]^2 \leq \Psi(u^2)\Psi(v^2),$$

which means that κ is convex. □

Remark. The growth and regularity conditions for Ψ can be expressed in the following form:

- $\Psi \in \Delta^0$ iff $\kappa(x + \beta') - \kappa(x) \underset{x\to\infty}{\longrightarrow} +\infty$, for some $\beta' > 0$.
- $\Psi \in \Delta^1$ iff for some $\beta' > 0$, one has $x + \kappa(x) \leq \kappa(x + \beta')$, for x large enough.
- $\Psi \in \Delta^2$ iff for some $\alpha' > 0$, one has $2\kappa(x) \leq \kappa(x + \alpha')$, for x large enough.
- $\Psi \in \nabla_1$ iff $\kappa_B(x) + \kappa_B(y) \leq \kappa_B(x + y)$ for x, y large enough, with $B = \mathrm{e}^{-b}$.
- $\Psi \in \nabla_0$ iff for some $c' \geq 1$ and $A > 1$, one has $\kappa_A(\theta + 1) - \kappa_A(\theta) \leq \kappa_A(\theta' + c') - \kappa_A(\theta')$ for $\theta \leq \theta'$ large enough.

Before proving the main results of this section, let us collect some basic facts on the compactness of the embedding of H^{Ψ_1} into $L^{\Psi_2}(\mu)$. First:

Lemma 4.8 *Let Ψ_1, Ψ_2 be two Orlicz functions and μ a finite Borel measure on $\overline{\mathbb{D}}$. Assume that the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ compactly. Then, $\mu(\mathbb{T}) = 0$.*

Proof. The sequence $(z^n)_{n\geq 1}$ is weakly null in M^{Ψ_1} by Riemann-Lebesgue's Lemma. Its image by a compact operator is then norm null. This implies that for every $\varepsilon \in (0,1)$, we have, for n large enough,

$$\int_{\overline{\mathbb{D}}} \Psi_2\big(|z|^n\big)\,d\mu \leq \varepsilon.$$

Fatou's Lemma yields $\Psi_2(1)\mu(\mathbb{T}) \leq \varepsilon$. □

Now, we summarize what is true in full generality about compactness for canonical embeddings.

Proposition 4.9 *Let Ψ_1, Ψ_2 be two Orlicz functions and μ a finite Borel measure on $\overline{\mathbb{D}}$. The following assertions are equivalent*

i) The identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ compactly.

ii) Every sequence in the unit ball of H^{Ψ_1}, which is convergent to 0 uniformly on every compact subset of $\mathbb{D}$, is norm-null in $L^{\Psi_2}(\mu)$.

iii) The identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ continuously and $\lim\limits_{r\to 1^-} \big\|I_r\big\| = 0$, where $I_r(f) = f\,\mathbb{1}_{\overline{\mathbb{D}}\setminus r\mathbb{D}}$.

Proof. $i) \Rightarrow ii)$: let $(f_n)_{n\geq 1}$ be a sequence in the unit ball of H^{Ψ_1}, which is uniformly convergent to 0 on every compact subset of $\mathbb{D}$. In particular, $f_n(z)$ converges to 0 for every $z \in \mathbb{D}$. This means that $(f_n)_{n\geq 1}$ converges to 0 μ-almost everywhere, since $\mu(\mathbb{T}) = 0$ by the preceding lemma. If the conclusion did not hold, we could assume, up to an extraction, that $\underline{\lim}\|f_n\|_{\Psi_2} > 0$. Thus by compactness of the embedding, up to a new extraction, $(f_n)_{n\geq 1}$ is norm-convergent to some $g \in L^{\Psi_2}(\mu)$. Necessarily $g \neq 0$. A subsequence of $(f_n)_{n\geq 1}$ would be convergent to g μ-almost everywhere. This gives a contradiction.

$ii) \Rightarrow iii)$: if not, there exist a sequence $(f_n)_{n\geq 1}$ in the unit ball of H^{Ψ_1} and $\delta > 0$ with $\|f_n \mathbb{1}_{\overline{\mathbb{D}}\setminus(1-\frac{1}{n})\mathbb{D}}\|_{\Psi_2} > \delta$, for every $n \geq 1$. Let us introduce the sequence $g_n(z) = z^n f_n(z)$ for $z \in \mathbb{D}$. The sequence $(g_n)_{n\geq 1}$ lies in the unit ball of H^{Ψ_1} and is convergent to 0 uniformly on every compact subset of $\mathbb{D}$. But

$$\|g_n\|_{\Psi_2} \geq \|z^n f_n \mathbb{1}_{\overline{\mathbb{D}}\setminus(1-\frac{1}{n})\mathbb{D}}\|_{\Psi_2} \geq \Big(1-\frac{1}{n}\Big)^n \|f_n \mathbb{1}_{\overline{\mathbb{D}}\setminus(1-\frac{1}{n})\mathbb{D}}\|_{\Psi_2} \geq \Big(1-\frac{1}{n}\Big)^n \delta.$$

This contradicts (ii).

$iii) \Rightarrow ii)$ is very easy.

$ii) \Rightarrow i)$ follows from Proposition 3.6. □

We can now state some deeper characterizations

Theorem 4.10 *Let μ be a finite Borel measure on the closed unit disk $\overline{\mathbb{D}}$ and let Ψ_1 and Ψ_2 be two Orlicz functions. Then:*

1) If the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ continuously, there exists some $A > 0$ such that:

$$(R) \qquad \rho_\mu(h) \leq \frac{1}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)} \quad \textit{for every } h \in (0,1].$$

2) If there exists some $A > 0$ such that:

$$(K) \qquad K_\mu(h) \leq \frac{1/h}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)} \quad \textit{for every } h \in (0,1],$$

then the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ continuously.

Theorem 4.11 *Let μ be a finite Borel measure on the closed unit disk $\overline{\mathbb{D}}$ and let Ψ_1 and Ψ_2 be two Orlicz functions. Then:*

1) If the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ compactly, then

$$(R_0) \qquad \rho_\mu(h) = o\left(\frac{1}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)}\right) \quad \textit{as } h \to 0,\ \textit{for every } A > 0.$$

2) If $\mu(\mathbb{T}) = 0$ and

$$(K_0) \qquad K_\mu(h) = o\left(\frac{1/h}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)}\right) \quad \textit{as } h \to 0\,,\ \textit{for every } A > 0,$$

then the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ compactly.

3) When $\Psi_1 = \Psi_2 = \Psi$ satisfies condition ∇_0, then the above conditions are equivalent: the identity maps H^{Ψ_1} into $L^{\Psi_2}(\mu)$ compactly if and only if condition (R_0) is satisfied and if and only if condition (K_0) is satisfied.

Remarks.

1. a) If $\rho_\mu(h) \leq C/\Psi_2\big(A\Psi_1^{-1}(1/h)\big)$ with $C > 1$, then the convexity of Ψ_2 gives $\Psi_2(t/C) \leq \Psi_2(t)/C$ and hence:

$$\rho_\mu(h) \leq \frac{1}{\Psi_2\big(\frac{A}{C}\Psi_1^{-1}(1/h)\big)}.$$

b) If $\rho_\mu(h) \leq 1/\Psi_2\big(A\Psi_1^{-1}(1/h)\big)$ only for $h \leq h_A$, one can find some $C = C_A > 0$ such that $\rho_\mu(h) \leq C/\Psi_2\big(A\Psi_1^{-1}(1/h)\big)$ for every $h \in (0,1]$. In fact, $\rho_\mu(h) \leq \mu(\overline{\mathbb{D}})$ and $1/\Psi_2\big(A\Psi_1^{-1}(1/h)\big) \geq 1/\Psi_2\big(A\Psi_1^{-1}(1/h_A)\big)$ for $h \geq h_A$; hence $\rho_\mu(h) \leq C/\Psi_2\big(A\Psi_1^{-1}(1/h)\big)$ with $C = \mu(\overline{\mathbb{D}})/\Psi_2\big(A\Psi_1^{-1}(1/h_A)\big)$.

The same remark applies for K_μ.

2. a) In the case where $\Psi_1 = \Psi_2 = \Psi$, one has: $\Psi\big(A\Psi^{-1}(t)\big) \leq \Psi\big(\Psi^{-1}(t)\big) = t$ when $A \leq 1$ and $\Psi\big(A\Psi^{-1}(t)\big) \geq \Psi\big(\Psi^{-1}(t)\big) = t$ when $A \geq 1$. On the other hand, if $\Psi \in \Delta_2$, one has, for some constant $C = C_A > 0$: $\Psi(Ax) \leq C\Psi(x)$, when $A \geq 1$ and $\Psi(Ax) \geq (1/C)\Psi(x)$ when $A \leq 1$. Hence, when $\Psi \in \Delta_2$, one has, for every $A > 0$:

$$\frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)} = \frac{1}{\chi_A(1/h)} \approx h$$

and Theorem 4.10 is nothing but Carleson's Theorem.

b) If $\Psi_1(x) = x^p$ and $\Psi_2(x) = x^q$ with $p < q < \infty$, then:

$$\Psi_2\big(A\Psi_1^{-1}(t)\big) = A^q t^{q/p},$$

and condition (R) means that μ is a β-Carleson measure, with $\beta = q/p$ (see [12], Theorem 9.4).

c) If, for fixed $A > 0$, the function $x \mapsto \dfrac{\Psi_1(x)}{\Psi_2(Ax)}$ is non increasing, at least for x large enough, conditions (R) and (K) (resp. conditions (R_0) and (K_0) below) are clearly equivalent. This is the case in the framework of classical Hardy spaces: $\Psi_1(x) = x^p$ and $\Psi_2(x) = x^q$, with $q \geq p$.

When $\Psi_1 = \Psi_2 = \Psi$, this is equivalent, if $A > 1$, to the convexity of the function $\kappa(x) = \log \Psi(\mathrm{e}^x)$ (see Proposition 4.7).

3. a) When $\Psi_1 = \Psi_2 = \Psi$, the condition $\mu(\mathbb{T}) = 0$ is automatically fulfilled (and so can be removed from (K_0)). This follows on one hand from the majorization in (K_0) , which implies that $K_\mu(h) \to 0$ (when $h \to 0$); and on the other hand from the inequality:

$$\mu\big(\overline{\mathbb{D}} \setminus r\mathbb{D}\big) \leq \frac{2\pi}{1-r}\rho_\mu(1-r) \leq 2\pi K_\mu(1-r).$$

Indeed, $\overline{\mathbb{D}} \setminus r\mathbb{D}$ can be covered by less than $\frac{2\pi}{1-r}$ Carleson's windows of "size" $1-r$.

b) Nevertheless, the condition $\mu(\mathbb{T}) = 0$ cannot be removed in full generality in Theorem 4.11. Indeed, if we consider the identity j from H^4 into $L^2(\overline{\mathbb{D}}, \tilde{m})$, where $\tilde{m}$ is 0 on $\mathbb{D}$ and its restriction to the torus is the normalized Lebesgue measure. It is easily seen that $K(h)$ is bounded and so less than $\dfrac{1}{A^2 h^{1/2}}$, for h small enough. But j is not compact.

4. In the case where $\Psi_1 = \Psi_2 = \Psi$ and μ is a Carleson measure, then K_μ is bounded, by say $K \geq 1$, and condition (K) is satisfied for $A = 1/K$, since $A \leq 1$ implies, by the convexity of Ψ: $\Psi\big(A\Psi^{-1}(1/h)\big) \leq A\Psi\big(\Psi^{-1}(1/h)\big) = A/h$. Hence the canonical embedding $H^\Psi \hookrightarrow L^\Psi(\mu)$ is continuous. We get hence, by Carleson's Theorem ([12], Theorem 9.3):

Proposition 4.12 *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Assume that the canonical embedding $j_\mu \colon H^p \to L^p(\mu)$ is continuous for some $0 < p < \infty$. Then $j_\mu \colon H^\Psi \to L^\Psi(\mu)$ is continuous.*

Note that this is actually a consequence of the fact that H^Ψ is an interpolation space for H^1 and H^∞ (see [5], Theorem V.10.8).

When $\mu = \mu_\phi$ is the image of the Haar measure m under ϕ^*, where ϕ is an analytic self-map of $\mathbb{D}$, we know (Proposition 3.12) that the composition operator $C_\phi \colon H^\Psi \to H^\Psi$ is always continuous. This can be read as the continuity of $H^\Psi \hookrightarrow L^\Psi(\mu_\phi)$. Hence condition (R) must be satisfied, for some $A > 0$. Note that for $A \leq 1$, $1/\chi_A(1/h) \geq h$, and so condition (R) is implied by the fact that μ_ϕ is a Carleson measure.

5. To have the majorization in condition (K_0), it suffices that: for every $A > 0$, there exists $h_A \in (0, 1]$ such that

$$K_\mu(h) \leq \frac{1/h}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)}\,, \quad \text{for every } h \in (0, h_A]. \tag{4.10}$$

In fact, fixing $A > 0$ and $\varepsilon \in (0, 1)$, we have, by convexity:

$$\Psi_2(Ax) \leq \varepsilon \Psi_2(\tilde{A}x),$$

with $\tilde{A} = A/\varepsilon$. Since we have (4.10) with $\tilde{A}$, when h is small enough (depending on A and ε), we get, for $x = \Psi_1^{-1}(1/h)$, condition (K_0).

To prove both Theorem 4.10 and Theorem 4.11, we shall need some auxiliary results. The following is actually the heart of the classical Theorem of Carleson, though it is not usually stated in this form. The maximal (non-tangential) function M_f (which is essentially the same as $N_\alpha f$ in the previous section) will be defined by:

$$M_f\big(e^{i\theta}\big) = \sup\{|f(z)|;\ z \in G_\theta\},$$

where

$$G_\theta = \{z \in \mathbb{D};\ |e^{i\theta} - z| < 3(1 - |z|)\}.$$

Theorem 4.13 (Carleson's Theorem) *For every $f \in H^1$ and every finite positive measure μ on the closed unit disk $\overline{\mathbb{D}}$, one has, for every $h \in (0, 1]$ and every $t > 0$:*

$$\mu\big(\{z \in \overline{\mathbb{D}};\ |z| > 1 - h \text{ and } |f(z)| > t\}\big) \leq 2\pi K_\mu(h)\, m(\{M_f > t\}).$$

As this theorem is not usually stated in such a way, we shall give a few words of explanations.

Proof. For convenience , we shall denote, when I is a subarc of $\mathbb{T}$:

$$W(I) = \{z \in \overline{\mathbb{D}};\ |z| > \max(0, 1 - |I|/2) \text{ and } \frac{z}{|z|} \in I\}.$$

Obviously, when $|I| \leq 2$, we have $W(I) = W(\xi, |I|/2)$, where ξ is the center of I.

We shall begin by being somewhat sketchy, and refer to [11], Theorem 2.33, or [12], Theorem 9.3, for the details. Following the lines of [11], page 39, proof of Theorem 2.33, 1) $\Rightarrow$ 2): $\{M_f > t\}$ is the disjoint union of a countable family of open arcs I_j of $\mathbb{T}$, and on the other hand (see [11], page 39), $|f(z)| > t$ implies that $z \in W(I_j)$ for some j.

Now, when $\alpha_j > h$, we have to cover I_j in such a way that we can write $I_j \subset J_{j,1} \cup \cdots \cup J_{j,N_j}$ with the arcs $J_{j,1}, \ldots, J_{j,N_j}$ satisfying: $W(J_{j,k}) = W(\xi_{j,k}, h)$ for every $k = 1, \ldots, N_j$ and $2|I_j| \geq \sum_{k=1}^{N_j} |J_{j,k}|$, and some $\xi_{j,k} \in \mathbb{T}$.

We then notice that:

$$\mu\big(W(J_{j,k})\big) \leq \frac{1}{2} K_\mu(h)\, |J_{j,k}|\,. \tag{4.11}$$

In fact, $2\alpha = |I|$ if $W(\xi, \alpha) = W(I)$; hence:

$$\mu\big(W(J_{j,k})\big) \leq \rho_\mu(h) \leq h K_\mu(h) = \frac{1}{2} |J_{j,k}|\, K_\mu(h).$$

Denoting, for $E \subseteq \overline{\mathbb{D}}$, by E_h the set of points $z \in E$ such that $|z| > 1 - h$, we therefore have, since:

$$W(I_j)_h \subseteq \bigcup_{1 \leq k \leq N_j} W(J_{j,k}),$$

using (4.11):

$$\mu\big(W(I_j)_h\big) \leq \sum_{k=1}^{N_j} \mu\big(W(J_{j,k})\big) \leq \sum_{k=1}^{N_j} \frac{1}{2} K_\mu(h)\, |J_{j,k}| \leq K_\mu(h)\, |I_j|.$$

It follows that:

$$\begin{aligned} \mu\big(\{z \in \overline{\mathbb{D}}\,;\ |z| > 1 - h \text{ and } |f(z)| > t\}\big) &\leq \sum_j \mu\big((W_j)_h\big) \leq K_\mu(h) \sum_j |I_j| \\ &= 2\pi K_\mu(h) \sum_j m(I_j) = 2\pi K_\mu(h)\, m(\{M_f > t\}), \end{aligned}$$

as announced. □

The following estimation will be useful for the study both of boundedness and compactness.

Lemma 4.14 *Let μ be a finite Borel measure on $\overline{\mathbb{D}}$. Let Ψ_1 and Ψ_2 be two Orlicz functions. Suppose that there exists $A > 0$ and $h_A \in (0,1)$ such that*

$$K_\mu(h) \leq \frac{1/h}{\Psi_2\big(A \Psi_1^{-1}(1/h)\big)}, \text{ for every } h \in (0, h_A).$$

Then, for every $f \in H^{\Psi_1}$ such that $\|f\|_{\Psi_1} \leq 1$ and every Borel subset E of $\overline{\mathbb{D}}$, we have:

$$\int_E \Psi_2\Big(\frac{A}{8}|f|\Big)\, d\mu \leq \mu(E) \Psi_2(x_A) + \frac{\pi}{2} \int_{\mathbb{T}} \Psi_1(M_f)\, dm$$

where $x_A = \dfrac{A}{2}\Psi_1^{-1}(1/h_A)$.

Proof. For every $s > 0$, the inequality $|f(z)| > s$ implies that the norm of the evaluation at z is greater than s; hence by Lemma 3.11:

$$s < 4\Psi_1^{-1}\Big(\frac{1}{1-|z|}\Big),$$

i.e.:

$$|z| > 1 - \frac{1}{\Psi_1(s/4)}.$$

Carleson's Theorem (Theorem 4.13) gives:

$$\mu\big(\{|f(z)| > s\}\big) \leq 2\pi K_\mu\Big(\frac{1}{\Psi_1(s/4)}\Big)\, m(\{M_f > s\})$$

when $\Psi_1(s/4) \geq 1$. Hence:

$$\int_E \Psi_2\Big(\frac{A}{8}|f|\Big)\, d\mu = \int_0^\infty \Psi_2'(t)\, \mu(\{|f| > 8t/A\} \cap E)\, dt\,.$$

But our hypothesis means that, when $\Psi_1(s/4) > 1/h_A$:

$$K_\mu\Big(\frac{1}{\Psi_1(s/4)}\Big) \leq \frac{\Psi_1(s/4)}{\Psi_2(As/4)}\,.$$

We have then

$$\mu\big(\{|f(z)| > 8t/A\}\big) \leq 2\pi\frac{\Psi_1(2t/A)}{\Psi_2(2t)}\, m(\{M_f > 8t/A\}).$$

So:

$$\begin{aligned}
\int_E \Psi_2\Big(\frac{A}{8}|f|\Big)\, d\mu \leq & \int_0^{x_A} \Psi_2'(t)\mu(E)\, dt \\
& \qquad + 2\pi\int_{x_A}^{+\infty} \Psi_2'(t)\frac{\Psi_1(2t/A)}{\Psi_2(2t)}\, m(\{M_f > 8t/A\})\, dt \\
\leq & \ \Psi_2(x_A)\mu(E) \\
& \qquad + 2\pi\int_{x_A}^{+\infty} \frac{\Psi_2'(t)}{\Psi_2(2t)}\, \Psi_1(2t/A)\, m(\{M_f > 8t/A\})\, dt\,.
\end{aligned}$$

For the second integral, note that one has $\Psi(x) \leq x\Psi'(x) \leq \Psi(2x)$, for any Orlicz function Ψ. This leads to:

$$\int_{x_A}^{\infty} \frac{\Psi_2'(t)}{\Psi_2(2t)}\, \Psi_1(2t/A)\, m(\{M_f > 8t/A\})\, dt$$

$$\begin{aligned}
&\leq \int_0^{\infty} \frac{\Psi_1(2t/A)}{t}\, m(\{M_f > 8t/A\})\, dt \\
&\leq \frac{2}{A} \int_0^{\infty} \Psi_1'(2t/A)\, m(\{M_f > 8t/A\})\, dt \\
&= \int_0^{\infty} \Psi_1'(x)\, m(\{M_f > 4x\})\, dx = \int_{\mathbb{T}} \Psi_1\Big(\frac{1}{4} M_f\Big)\, dm \\
&\leq \frac{1}{4} \int_{\mathbb{T}} \Psi_1(M_f)\, dm.
\end{aligned}$$

which leads to the desired result. □

For the proofs of Theorem 4.10 and Theorem 4.11, we may restrict ourselves to the case of functions Ψ_1 and Ψ_2 satisfying ∇_2. Indeed, suppose that Ψ_1 and Ψ_2 are Orlicz functions and define $\widetilde{\Psi_j}(t) = \Psi_j(t^2)$, for $j \in \{1, 2\}$. The functions $\widetilde{\Psi_1}$ and $\widetilde{\Psi_2}$ are Orlicz functions satisfying ∇_2 since, with $\beta = 2$, we have for every $t \geq 0$:

$$\widetilde{\Psi_j}(\beta t) = \Psi_j(4t^2) \geq 4\Psi_j(t^2) = 2\beta \widetilde{\Psi_j}(t).$$

Now, we claim that μ satisfies (R), (K), (R_0) or (K_0) for the couple $\big(\Psi_1, \Psi_2\big)$ if and only if μ satisfies it for the couple $\big(\widetilde{\Psi_1}, \widetilde{\Psi_2}\big)$. This is simply due to the fact that for every $A > 0$ and $t \geq 0$, we have

$$\widetilde{\Psi_2}\Big(A \widetilde{\Psi_1}^{-1}(t)\Big) = \Psi_2\Big(A^2 \Psi_1^{-1}(t)\Big).$$

Moreover, notice that, writing $f = Bg^2$ (where B is a Blaschke product), we have $f \in H^{\Psi}$ if and only if $g \in H^{\widetilde{\Psi}}$; thus $\|g\|_{L^{\tilde{\Psi}}} = \sqrt{\|f\|_{L^{\Psi}}}$. It is then clear that

$$\Big\| Id : H^{\Psi_1} \longrightarrow L^{\Psi_2}(\mu) \Big\| = \Big\| Id : H^{\widetilde{\Psi_1}} \longrightarrow L^{\widetilde{\Psi_2}}(\mu) \Big\|^2,$$

so that the canonical embedding is bounded (resp. compact) for the couple $\big(\Psi_1, \Psi_2\big)$ if and only if it is so for the couple $\big(\widetilde{\Psi_1}, \widetilde{\Psi_2}\big)$, thanks to Proposition 4.9.

Proof of Theorem 4.10. 1) Let C be the norm of the canonical embedding $j\colon H^{\Psi_1} \hookrightarrow L^{\Psi_2}(\mu)$, and let $\xi \in \mathbb{T}$ and $h \in (0, 1)$. It suffices to test the continuity of j on $f = \Psi_1^{-1}(1/h) u_{\xi, 1-h}$, which is in the unit ball of HM^{Ψ_1}, by Corollary 3.10.

But, when $z \in W(\xi, h)$ one has, with $a = (1-h)\xi$:

$$\begin{aligned}
|1 - \bar{a}z| &\leq |1 - \bar{a}\xi| + |\bar{a}\xi - \bar{a}z| = h + (1-h)\Big[\Big|\xi - \frac{z}{|z|}\Big| + \Big|\frac{z}{|z|} - z\Big|\Big] \\
&\leq h + (1-h)[h + (1 - |z|)] \leq h + (1-h)[h + h] \leq 3h;
\end{aligned}$$

hence $|u_{\xi,1-h}(z)| \geq 1/9$ and $|f(z)| \geq (1/9)\Psi_1^{-1}(1/h)$; therefore:

$$1 \geq \int_{\mathbb{D}} \Psi_2\Big(\frac{|f|}{C}\Big)\, d\mu \geq \Psi_2\Big(\frac{1}{9C}\Psi_1^{-1}(1/h)\Big)\, \mu\big(W(\xi,h)\big),$$

which is (R).

2) By Proposition 3.5, the maximal (non-tangential) function M is bounded on $L^{\Psi_1}(\mathbb{T})$: there exists a constant $C \geq 1$ such that $\|M_f\|_{\Psi_1} \leq C\|f\|_{\Psi_1}$ for every $f \in L^{\Psi_1}(\mathbb{T})$. We fix f in the unit ball of H^{Ψ_1} (note that $\|f/C\|_{\Psi_1}$ remains ≤ 1) and use Lemma 4.14, with $E = \overline{\mathbb{D}}$ and f replaced by f/C (here $h_A = 1$)). Writing $\tilde{C} = \dfrac{\pi}{2} + \mu(\overline{\mathbb{D}})\Psi_2(x_A)$, we get:

$$\begin{aligned}\int_{\overline{\mathbb{D}}} \Psi_2\Big(\frac{A}{8C\tilde{C}}|f|\Big)\, d\mu &\leq \frac{1}{\tilde{C}}\int_{\overline{\mathbb{D}}} \Psi_2\Big(\frac{A}{8C}|f|\Big)\, d\mu \\ &\leq \frac{1}{\tilde{C}}\bigg(\mu(\overline{\mathbb{D}})\Psi_2(x_A) + \frac{\pi}{2}\int_{\mathbb{T}} \Psi_1\Big(\frac{1}{C}M_f\Big)\, dm\bigg) \\ &\leq \frac{1}{\tilde{C}}\Big(\mu(\overline{\mathbb{D}})\Psi_2(x_A) + \frac{\pi}{2}\Big) = 1,\end{aligned}$$

which means that $\|f\|_{L^{\Psi_2}(\mu)} \leq \dfrac{8C\tilde{C}}{A}\cdot$ □

Proof of Theorem 4.11. 1) Suppose that the embedding is compact, but that condition (R_0) is not satisfied. Then there exist $\varepsilon_0 \in (0,1)$, $A > 0$, a sequence of positive numbers $(h_n)_n$ decreasing to 0, and a sequence of $\xi_n \in \mathbb{T}$, such that:

$$\mu\big(W(\xi_n,h_n)\big) \geq \frac{\varepsilon_0}{\Psi_2\big(A\Psi_1^{-1}(1/h_n)\big)}\cdot$$

Consider the sequence of functions

$$f_n(z) = {\Psi_1}^{-1}(1/h_n)\frac{h_n^2}{(1-\bar{a}_n z)^2} = {\Psi_1}^{-1}(1/h_n)\, u_{\xi_n,|a_n|},$$

where $a_n = (1-h_n)\xi_n$. By Corollary 3.10, f_n is in the unit ball of HM^{Ψ_1}. Moreover, it is plain that $(f_n)_n$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. By the compactness criterion (Proposition 3.6), $(f_n)_n$ is norm-converging to 0 in $L^{\Psi_2}(\mu)$.

But, as above (proof of Theorem 4.10), for every $n \geq 1$, one has $|f_n(z)| \geq (1/9)\Psi_1^{-1}(1/h_n)$ when $z \in W(\xi_n,h_n)$; hence:

$$\begin{aligned}\int_{\mathbb{D}} \Psi_2\Big(\frac{9A}{\varepsilon_0}|f_n|\Big)\, d\mu &\geq \Psi_2\Big(\frac{A}{\varepsilon_0}\Psi_1^{-1}(1/h_n)\Big)\, \mu\big(W(\xi_n,h_n)\big) \\ &\geq \Psi_2\Big(\frac{A}{\varepsilon_0}\Psi_1^{-1}(1/h_n)\Big)\, \frac{\varepsilon_0}{\Psi_2\big(A\Psi_1^{-1}(1/h_n)\big)} \geq 1,\end{aligned}$$

by the convexity of Ψ_2. This implies that $\|f_n\|_{L^{\Psi_2(\mu)}} \geq \varepsilon_0/9A$ and gives a contradiction.

2) We have to prove that for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that the norm of the injection $I_r : H^{\Psi_1} \longrightarrow L^{\Psi_2}(\overline{\mathbb{D}} \setminus r\mathbb{D}, \mu)$ is smaller than ε (see Proposition 4.9).

Let $C \geq 1$ be the norm of the maximal operator, as in the proof of Theorem 4.10: $\|M_f\|_{\Psi_1} \leq C\|f\|_{\Psi_1}$ for every $f \in L^{\Psi_1}(\mathbb{T})$, and set $A = 16C/\varepsilon$. Condition (K_0) gives us $h_A \in (0,1)$ such that:

$$K_\mu(h) \leq \frac{1}{2} \frac{1/h}{\Psi_2\big(A\Psi_1^{-1}(1/h)\big)}$$

when $h \leq h_A$.

Let f in the unit ball of H^{Ψ_1} and $r \in (0,1)$. By Lemma 4.14:

$$\begin{aligned}\int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} \Psi_2\Big(\frac{|f|}{\varepsilon}\Big)\, d\mu &= \int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} \Psi_2\Big(\frac{A}{16C}|f|\Big)\, d\mu \leq \frac{1}{2}\int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} \Psi_2\Big(\frac{A}{8C}|f|\Big)\, d\mu \\ &\leq \frac{1}{2}\left(\mu\big(\overline{\mathbb{D}}\setminus r\mathbb{D}\big)\Psi_2(x_A) + \frac{\pi}{2}\int_{\mathbb{T}} \Psi_1\left(\frac{M_f}{C}\right) dm\right) \\ &\leq \frac{\pi}{4} + \Psi_2(x_A)\mu\big(\overline{\mathbb{D}}\setminus r\mathbb{D}\big).\end{aligned}$$

As $\mu(\mathbb{T}) = 0$, there exists some $r_0 \in (0,1)$ such that $\frac{\pi}{4} + \Psi_2(x_A)\mu\big(\overline{\mathbb{D}}\setminus r\mathbb{D}\big) \leq 1$, for every $r \in (r_0, 1)$. This ends the proof of 2).

3) Assume that Ψ satisfies condition ∇_0 and that condition (R_0) is fulfilled: for every $A > 0$ and every $h \in (0, h_A)$ (h_A small enough), we have:

$$\rho_\mu(h) \leq \frac{1}{\Psi\big[A\Psi^{-1}(1/h)\big]}\,.$$

This implies that:

$$K_\mu(h) = \sup_{0<s\leq h} \frac{\rho_\mu(s)}{s} \leq \sup_{0<s\leq h} \frac{1/s}{\Psi\big[A\Psi^{-1}(1/s)\big]} = \sup_{x\geq \Psi^{-1}(1/h)} \frac{\Psi(x)}{\Psi(A\,x)}.$$

Fix an arbitrary $\beta > 1$ and choose $A = \beta C_\beta > 1$, where C_β is given by the ∇_0 condition for Ψ and Proposition 4.6. We have, for h small enough and $x \geq \Psi^{-1}(1/h)$:

$$\frac{\Psi\big[\beta\Psi^{-1}(1/h)\big]}{\Psi\big[\Psi^{-1}(1/h)\big]} \leq \frac{\Psi(\beta C_\beta\, x)}{\Psi(x)} = \frac{\Psi(A\,x)}{\Psi(x)}\,;$$

we get hence, for h small enough:

$$K_\mu(h) \leq \frac{\Psi\big[\Psi^{-1}(1/h)\big]}{\Psi\big[\beta\Psi^{-1}(1/h)\big]} = \frac{1/h}{\Psi\big[\beta\Psi^{-1}(1/h)\big]}\,,$$

and condition (K_0) is fulfilled.

With 1) and 2) previously shown, this finishes the proof. $\square$

Remark. Actually, the proof of Theorem 4.10, 1) gives, for every measure μ on $\overline{\mathbb{D}}$ and every Orlicz function Ψ:

$$1 \geq \int_{\overline{\mathbb{D}}} \Psi\Big(\frac{|u_{\xi,1-h}|}{\|u_{\xi,1-h}\|_{L^\Psi(\mu)}}\Big)\, d\mu \geq \Psi\Big(\frac{1}{9\, \|u_{\xi,1-h}\|_{L^\Psi(\mu)}}\Big)\, \mu\big(W(\xi,h)\big),$$

and hence:

$$\mu\big(W(\xi,h)\big) \leq \frac{1}{\Psi\Big(\frac{1}{9\, \|u_{\xi,1-h}\|_{L^\Psi(\mu)}}\Big)} \cdot \tag{4.12}$$

In particular, if $\mu = \mu_\phi$ is the image of the Haar measure m under a self-map ϕ of $\mathbb{D}$, one has:

$$\mu\big(W(\xi,h)\big) \leq \frac{1}{\Psi\Big(\frac{1}{9\, \|C_\phi(u_{\xi,1-h})\|_\Psi}\Big)} \cdot \tag{4.13}$$

Condition (4.12) allows to have an upper control for the μ-measure of Carleson windows, with $\|u_{\xi,1-h}\|_{L^\Psi(\mu)}$. It is possible, conversely, to majorize these norms.

Definition 4.15 *We shall say that a measure μ on $\overline{\mathbb{D}}$ is a* Ψ-Carleson measure *if there exists some $A > 0$ such that*

$$\mu\big(W(\xi,h)\big) \leq \frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)}\,, \quad \textit{for every } \xi \in \mathbb{T} \textit{ and every } h \in (0,1).$$

We shall say that a measure μ on $\overline{\mathbb{D}}$ is a vanishing Ψ-Carleson measure *if, for every $A > 0$,*

$$\lim_{h\to 0} \Psi\big(A\Psi^{-1}(1/h)\big).\rho_\mu(h) = 0.$$

Equivalently if, for every $A > 0$, there exists $h_A \in (0,1)$ such that

$$\mu\big(W(\xi,h)\big) \leq \frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)}\,, \quad \textit{for every } \xi \in \mathbb{T} \textit{ and every } h \in (0,h_A).$$

We have the following characterizations:

Proposition 4.16 1) *μ is a Ψ-Carleson measure on $\overline{\mathbb{D}}$ if and only if there exists some constant $C \geq 1$ such that :*

$$\|u_{\xi,1-h}\|_{L^\Psi(\mu)} \leq \frac{C}{\Psi^{-1}(1/h)}, \textit{ for every } \xi \in \mathbb{T} \textit{ and every } h \in (0,1).$$

2) *μ is a vanishing Ψ-Carleson measure on $\overline{\mathbb{D}}$ if and only if*

$$\lim_{h\to 0} \sup_{\xi\in\mathbb{T}} \Psi^{-1}(1/h)\|u_{\xi,1-h}\|_{L^\Psi(\mu)} = 0.$$

Proof. The sufficiency (both for 1) and 2)) follows easily from (4.12) in the preceding remark.

The converse is an obvious consequence of the following lemma.

Lemma 4.17 *Suppose that there exist $A > 0$ and $h_0 \in (0,1)$ such that:*

$$\rho_\mu(h) \leq \frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)}, \text{ for every } h \in (0, h_0).$$

Then there exists $h_1 \in (0,1)$ such that:

$$\|u_{\xi,1-h}\|_{L^\Psi(\mu)} \leq \frac{24}{A\Psi^{-1}(1/h)},$$

for every $\xi \in \mathbb{T}$ and every $h \in (0, h_1)$.

Proof of the lemma. This is inspired from [13], Chapter VI, Lemma 3.3, page 239. We may assume that $h \leq h_0/4 \leq 1/4$.

First, writing $a = (1-h)\xi$ (where $\xi \in \mathbb{T}$), we observe that, when $|z-\xi| \geq bh$ for a $b > 0$, we have:

$$\begin{aligned}
|1-\bar{a}z|^2 &= 1 + |a|^2|z|^2 - 2|a|\Re(\bar{\xi}z) \\
&= |a||\xi - z|^2 + (1-|a|) + |a|^2|z|^2 - |a||z|^2 \\
&\geq |a|b^2h^2 + (1-|a|)^2 \geq (|a|b^2+1)h^2.
\end{aligned}$$

So, we have $|u_{\xi,1-h}(z)| \leq \dfrac{1}{|a|b^2+1} \leq \min(1, 2/b^2)$, when $|z-\xi| \geq bh$.

Now, define, for every $n \in \mathbb{N}$ and $\xi \in \mathbb{T}$:

$$S_n = S(\xi, 2^{n+1}h) = \{z \in \overline{\mathbb{D}};\ |z-\xi| < 2^{n+1}h\} \subset W(\xi, 2.2^{n+1}h).$$

Our observation implies that $|u_{\xi,1-h}(z)| \leq \min(1, 2/4^n)$, for every $z \in \overline{\mathbb{D}} \setminus S_{n-1}$. For $z \in S_0$, one has simply $|u_{\xi,1-h}(z)| \leq 1$.

There exists an integer N such that $2^{N+2}h \leq h_0 < 2^{N+3}h$.

Let us compute:

$$\begin{aligned}
\int_{\overline{\mathbb{D}}} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu &= \int_{S_0} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu + \sum_{n=1}^{N} \int_{S_n\setminus S_{n-1}} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu \\
&\qquad + \int_{\overline{\mathbb{D}}\setminus S_N} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu \\
&\leq \Psi(M)\mu(S_0) + \sum_{n=1}^{N} \Psi\Big(\frac{2M}{4^n}\Big)\mu(S_n) + \Psi\Big(\frac{2M}{4^N}\Big)\mu(\overline{\mathbb{D}}) \\
&\leq \sum_{n=0}^{N} \frac{1}{2^{n+1}}\Psi\Big(\frac{4M}{2^n}\Big)\mu(S_n) + \Psi\Big(\frac{2M}{4^N}\Big)\mu(\overline{\mathbb{D}}).
\end{aligned}$$

But for $n \leq N$, we have $2.2^{n+1}h \leq 2^{N+2}h \leq h_0$, so the hypothesis gives:

$$\mu(S_n) \leq \frac{1}{\Psi\big(A\Psi^{-1}(1/2^{n+2}h)\big)}\cdot$$

Take now:

$$M = \frac{A}{24}\Psi^{-1}\Big(\frac{1}{h}\Big)\cdot \tag{4.14}$$

We have, using that $\Psi^{-1}\big(\frac{1}{2^{n+2}h}\big) \geq \frac{1}{2^{n+2}}\Psi^{-1}\big(\frac{1}{h}\big)$,

$$\begin{aligned}\int_{\mathbb{D}} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu &\leq \sum_{n=0}^{N} \frac{1}{2^{n+1}} + \mu(\overline{\mathbb{D}})\Psi\bigg(\frac{A}{12.4^N}\Psi^{-1}\Big(\frac{1}{h}\Big)\bigg)\\ &\leq \frac{1}{2} + \mu(\overline{\mathbb{D}})\Psi\bigg(\frac{16A.h^2}{3h_0^2}\Psi^{-1}\Big(\frac{1}{h}\Big)\bigg),\end{aligned}$$

because $\dfrac{1}{4^N} \leq \Big(\dfrac{8h}{h_0}\Big)^2$.

We can choose h_1 small enough to have:

$$\mu(\overline{\mathbb{D}})\Psi\bigg(\frac{16A.h^2}{3h_0^2}\Psi^{-1}\Big(\frac{1}{h}\Big)\bigg) \leq \frac{1}{2}$$

for every $h \in (0, h_1)$, since $\lim\limits_{h\to 0} h^2\Psi^{-1}\Big(\frac{1}{h}\Big) = 0$.

We get for such h

$$\int_{\mathbb{D}} \Psi\big(M\,|u_{\xi,1-h}|\big)\,d\mu \leq 1,$$

so that

$$\|u_{\xi,1-h}\|_{L^\Psi(\mu)} \leq \frac{1}{M} = \frac{24}{A}\frac{1}{\Psi^{-1}(1/h)}\,,$$

as it was announced. □

Examples and counterexamples.

We are going to give some examples showing that we do not have the reverse implications in general in Theorem 4.10 and Theorem 4.11.

1. *Condition (R) is not sufficient in general to have a continuous embedding.* Let $\Psi(x) = \mathrm{e}^x - 1$ (note that this Orlicz function even fulfills the Δ^1 condition !). Note that $\Psi(A\Psi^{-1}(1/h)) \sim h^{-A}$, when $h \to 0$.

a. Let ν be a probability measure on $\mathbb{T}$, supported by a compact set L of Lebesgue measure zero, such that $\nu(I) \leq |I|^{1/2}$, for each I. We can associate to ν the measure on $\overline{\mathbb{D}}$ defined by $\tilde{\nu}(E) = \nu(E \cap \mathbb{T})$. Then the identity map from H^Ψ to $L^\Psi(\tilde{\nu})$ is not even defined. Nevertheless the condition (R) is clearly fulfilled with $A = 1/2$.

b. Now, we exhibit a similar example (less artificial) on the open disk. Let ν be as previously. By a standard argument: for every integer n, there exists a function g_n in the unit ball of the disk algebra such that $|g_n| = 1$ on L and $\|g_n\|_\Psi \leq 4^{-n}$. As L is compact, there exists some $r_n \in (1/2, 1)$ such that $|g_n(r_n z)| \geq 1/2$ for every $z \in L$. Now, define the measure μ by:

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(E),$$

where:

$$\nu_n(E) = \nu\big(\{z \in \mathbb{T} |\, r_n z \in E\}\big).$$

If W is a Carleson window of "size" h then, for each $n \geq 1$, we have:

$$\nu\big(\{z \in \mathbb{T} |\, r_n z \in W\}\big) \leq \nu(W \cap \mathbb{T}) \leq (2h)^{1/2}.$$

Hence, $\mu(W) \leq (2h)^{1/2}$ and the condition (R) is fulfilled.

Nevertheless, the identity from H^Ψ to $L^1(\mu)$ is not continuous: $\|g_n\|_\Psi \leq 4^{-n}$; but

$$\|g_n\|_1 \geq \frac{1}{2^n} \int_{r_n \mathbb{T}} |g_n| \, d\nu_n \geq \frac{1}{2^n} \int_L |g_n(r_n w)| \, d\nu(w) \geq \frac{1}{2^{n+1}} \cdot$$

2. *Condition (K) is not necessary in general to have a continuous embedding.* When Ψ satisfies Δ_2, the identity from H^Ψ to $L^\Psi(\mu)$ is continuous if and only if μ is a Carleson measure. So the conditions (R), (K) and the continuity are equivalent in this case. Actually, when Ψ does not satisfy Δ_2, we construct below a measure μ on $\mathbb{D}$ such that the identity from H^Ψ to $L^\Psi(\mu)$ is continuous and order bounded , but μ is not a Carleson measure (*a fortiori* does not verify (K)). Note that the measure μ is then Ψ-Carleson but not Carleson. Here is the example:

We have assumed that Ψ does not satisfy Δ_2; so there exists an increasing sequence $(a_n)_{n\geq 1}$ such that $\dfrac{\Psi(a_n)}{n}$ is increasing and $\dfrac{\Psi(2a_n)}{\Psi(a_n)} \geq n2^n$. Now, define the discrete measure

$$\mu = \sum_{n=1}^{\infty} \left(\frac{n}{\Psi(2a_n)} - \frac{n+1}{\Psi(2a_{n+1})} \right) \delta_{x_n},$$

where:

$$x_n = 1 - \frac{1}{\Psi(2a_n)} \cdot$$

As $\mu\big([x_N, 1]\big) = \dfrac{N}{\Psi(2a_N)}$, the measure μ is not Carleson: it should be bounded by $c(1 - x_N) = \dfrac{c}{\Psi(2a_N)}$, where c is some constant.

We know that for every f in the unit ball of H^Ψ and every $x \in (0,1)$, we have $|f(x)| \leq 4\Psi^{-1}\Big(\dfrac{1}{1-x}\Big)$: see Lemma 3.11. So we only have to see that $g \in L^\Psi(\mu)$, where $g(x) = \Psi^{-1}\Big(\dfrac{1}{1-x}\Big)$. Indeed, we have:

$$\begin{aligned}
\int_{\mathbb{D}} \Psi\Big(\frac{|g|}{2}\Big)\, d\mu &= \sum_{n=1}^{\infty} \Big(\frac{n}{\Psi(2a_n)} - \frac{n+1}{\Psi(2a_{n+1})}\Big)\Psi\Big(\frac{|g(x_n)|}{2}\Big) \\
&\leq \sum_{n=1}^{\infty} \frac{n}{\Psi(2a_n)}\Psi\Big(\frac{1}{2}\Psi^{-1}\Big(\frac{1}{1-x_n}\Big)\Big) \\
&\leq \sum_{n=1}^{\infty} n\frac{\Psi(a_n)}{\Psi(2a_n)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&\leq 1
\end{aligned}$$

so $\|g\|_\Psi \leq 2$.

3. *Condition (K_0) is not necessary in general to have a compact embedding.* We can find, for every Orlicz function Ψ not satisfying ∇_0, a measure μ such that the identity from H^Ψ to $L^\Psi(\mu)$ is compact but (K_0) is not satisfied. Indeed: since $\Psi \notin \nabla_0$, we can select two increasing sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$, with $1 \leq x_n \leq y_n \leq x_{n+1}$ and $\Psi(x_n) > 1$, and such that $\lim x_n = +\infty$ and:

$$\frac{\Psi(2x_n)}{\Psi(x_n)} \geq \frac{\Psi(2^n y_n)}{\Psi(y_n)}\cdot$$

Define $r_n = 1 - \dfrac{1}{\Psi(y_n)}$ and the discrete measure:

$$\mu = \sum_{n=1}^{\infty} \frac{1}{\Psi(2^n y_n)}\, \delta_{r_n}.$$

The series converge since $\Psi(2^n y_n) \geq 2^n$.

Thanks to Lemma 3.11, we have, for every f in the unit ball of H^Ψ and every $n \geq 1$:

$$|f(r_n)| \leq 4\Psi^{-1}\Big(\frac{1}{1-r_n}\Big) = 4y_n.$$

Given $r > r_1$, there exists an integer $N \geq 1$ such that $r_N < r \leq r_{N+1}$. Then, for every f in the unit ball of H^Ψ, we have $\|f\|_{L^\Psi(\overline{\mathbb{D}}\setminus r\mathbb{D},\mu)} \leq 2^{-N+2}$, since:

$$\begin{aligned}\int_{\overline{\mathbb{D}}\setminus r\mathbb{D}} \Psi\Big(\frac{|f|}{2^{-N+2}}\Big)\,d\mu &= \sum_{n=N+1}^{\infty} \frac{1}{\Psi(2^n y_n)}\Psi\Big(\frac{|f(r_n)|}{2^{-N+2}}\Big)\\ &\leq \sum_{n=N+1}^{\infty} \frac{\Psi(2^N y_n)}{\Psi(2^n y_n)}\\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{2^{n-N}} = 1.\end{aligned}$$

This implies that $\lim\limits_{r\to 1} \sup\limits_{\|f\|_\Psi\leq 1} \|f\|_{L^\Psi(\overline{\mathbb{D}}\setminus r\mathbb{D},\mu)} = 0$. By Proposition 4.9, the identity from H^Ψ to $L^\Psi(\mu)$ is compact.

On the other hand, writing $h_n = \dfrac{1}{\Psi(x_n)}$ and $t_n = \dfrac{1}{\Psi(y_n)}$, we have:

$$\begin{aligned}K_\mu(h_n) \geq \frac{\mu([1-t_n,1])}{t_n} &= \Psi(y_n)\sum_{m=n}^{\infty}\frac{1}{\Psi(2^m y_m)} \geq \frac{\Psi(y_n)}{\Psi(2^n y_n)}\\ &\geq \frac{\Psi(x_n)}{\Psi(2x_n)} = \frac{1/h_n}{\Psi\big(2\Psi^{-1}(1/h_n)\big)},\end{aligned}$$

and this shows that (K_0) is not satisfied.

4. *Condition (R_0) is not sufficient in general to have a compact embedding.* We can find an Orlicz function Ψ and a vanishing Ψ-Carleson measure (*i.e.* (K_0) is satisfied) μ such that the identity from H^Ψ to $L^\Psi(\mu)$ is not compact.

We shall use the Orlicz function introduced in [25]. The key properties of this function Ψ are:

i) For every $x > 0$, $\Psi(x) \geq x^3/3$.

ii) For every integer $k \geq 1$, $\Psi(k!) \leq (k!)^3$.

iii) For every integer $k \geq 1$, $\Psi(3(k!)) > k.(k!)^3$.

Once again, the job is done by a discrete measure.

Define $x_k = k!$; $y_k = \dfrac{(k+1)!}{k^{1/3}}$; $r_k = 1 - \dfrac{1}{\Psi(y_k)}$ and $\rho_k = 1 - \dfrac{1}{\Psi(x_k)}$. Of course, $x_2 < y_2 < x_3 < \cdots$.

Let ν be the discrete measure defined by:

$$\nu = \sum_{k=2}^{\infty} \nu_k,$$

where:

$$\nu_k = \frac{1}{\Psi\big((k+1)!\big)} \sum_{a^{k^2}=1} \delta_{r_k a}.$$

Observe that $\|\nu_k\| \leq \dfrac{k^2}{\Psi\big((k+1)!\big)} \leq \dfrac{3.k^2}{(k+1)!^3}$ so that the series converges. Note that ν is supported in the union of the circles of radii r_k and not in a subset of the segment $[0,1]$ as in the preceding counterexamples.

In order to show that (R_0) is satisfied, it is clearly sufficient to prove that, when $\dfrac{1}{\Psi(y_k)} \leq h < \dfrac{1}{\Psi(y_{k-1})}$ (with $k \geq 3$), we have:

$$\rho_\nu(h) \leq \frac{1}{\Psi\Big(\dfrac{k^{1/3}}{2}\Psi^{-1}(1/h)\Big)}\cdot$$

Supposing then $\dfrac{1}{\Psi(y_k)} \leq h < \dfrac{1}{\Psi(y_{k-1})}$, we have $\Psi^{-1}(1/h) \leq y_k$ so

$$\Psi\Big(\frac{k^{1/3}}{2}\Psi^{-1}(1/h)\Big) \leq \frac{1}{2}\Psi\big((k+1)!\big).$$

Therefore, it is sufficient to establish that $\rho_\nu(h) \leq \dfrac{2}{\Psi\big((k+1)!\big)}\cdot$

A Carleson window $W(\xi,h)$ (where $\xi \in \mathbb{T}$) can contain at most one k^2-root of the unity, since $2h < \dfrac{2}{\Psi(y_{k-1})} \leq \dfrac{6}{y_{k-1}^3} \leq \dfrac{6(k-1)}{(k!)^3} \leq \dfrac{2\pi}{k^2}\cdot$ This implies that

$$\nu_k\big(W(\xi,h)\big) \leq \frac{1}{\Psi\big((k+1)!\big)}\cdot$$

Nevertheless, when $j<k$, the window $W(\xi,h)$ cannot meet any circle of radius r_j (centered at the origin), so $\nu_j\big(W(\xi,h)\big)=0$. We obtain:

$$\begin{aligned}
\nu\big(W(\xi,h)\big) &= \sum_{j=k}^{\infty}\nu_j\big(W(\xi,h)\big) \leq \frac{1}{\Psi\big((k+1)!\big)} + \sum_{j>k}\frac{j^2}{\Psi\big((j+1)!\big)}\\
&\leq \frac{1}{\Psi\big((k+1)!\big)} + \sum_{j=k+1}^{\infty}\frac{3j^2}{(j+1)!^3}\\
&\leq \frac{1}{\Psi\big((k+1)!\big)} + \frac{3}{(k+1)!^3}\sum_{s=1}^{\infty}\Big(\frac{1}{k+1}\Big)^s\\
&\leq \frac{2}{\Psi\big((k+1)!\big)}\cdot
\end{aligned}$$

This proves that (R_0) is satisfied.

Let us introduce the function $f_k(z) = x_k u_{1,\rho_k}\big(z^{k^2}\big) = x_k\Big(\dfrac{1-\rho_k}{1-\rho_k z^{k^2}}\Big)^2$.

It lies in the unit ball of H^Ψ by Corollary 3.10:

$$\|f_k\|_\Psi = x_k \|u_{1,\rho_k}\|_\Psi \leq \frac{x_k}{\Psi^{-1}\big(\frac{1}{1-\rho_k}\big)} = 1.$$

An easy computation gives $r_k^{k^2} \geq \rho_k$, for every $k \geq 2$. So, for every $a \in \mathbb{T}$ with $a^{k^2} = 1$, we have:

$$f_k(ar_k) \geq x_k \left(\frac{1-\rho_k}{1-\rho_k^2}\right)^2 \geq \frac{1}{4} x_k.$$

So

$$\begin{aligned}\int_{\overline{\mathbb{D}}\setminus r_{k-1}\mathbb{D}} \Psi(12|f_k|)\, d\nu &\geq \int_{\overline{\mathbb{D}}\setminus r_{k-1}\mathbb{D}} \Psi(12|f_k|)\, d\nu_k \geq \frac{k^2}{\Psi((k+1)!)} \Psi(3x_k) \\ &> \frac{k^2}{\Psi((k+1)!)} \big(k.(k!)^3\big) \geq 1.\end{aligned}$$

Therefore, we conclude that $\sup_{\|f\|_\Psi \leq 1} \|f\|_{L^\Psi(\overline{\mathbb{D}}\setminus r_k\mathbb{D},\mu)} \geq \dfrac{1}{12}$, though $r_k \to 1$. By Proposition 4.9, the identity from H^Ψ to $L^\Psi(\mu)$ is not compact.

4.4 Characterization of the compactness of composition operators

For composition operators, compactness can be characterized in terms of Ψ-Carleson measures, as stated in the following result.

Theorem 4.18 *For every analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$ and every Orlicz function Ψ, the composition operator $C_\phi\colon H^\Psi \to H^\Psi$ is compact if and only if one has:*

$$(R_0) \qquad \rho_\mu(h) = o\left(\frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)}\right) \ \text{ as } h \to 0, \text{ for every } A > 0.$$

In other words, if and only if μ_ϕ is a *vanishing Ψ-Carleson measure.*

In order to get this result, we shall show that in Theorem 4.11, conditions (R_0) and (K_0) are equivalent for the pull-back measure μ_ϕ induced by ϕ. This is the object of the following theorem.

Theorem 4.19 *There exists a constant $k_1 > 0$ such that, for every analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$, one has:*

$$(4.15) \qquad \mu_\phi\big(S(\xi, \varepsilon h)\big) \leq k_1\, \varepsilon\, \mu_\phi\big(S(\xi, h)\big)\,,$$

for every $h \in (0, 1 - |\phi(0)|)$, and every $\varepsilon \in (0,1)$.

Note that we prefer here to work with the sets:

$$S(\xi, h) = \{z \in \overline{\mathbb{D}};\ |z - \xi| < h\}\,, \quad \xi \in \mathbb{T},\ 0 < h < 1,$$

instead of the Carleson windows $W(\xi, h)$. Recall also that the pull-back measure μ_ϕ is defined by (4.3).

We are going to postpone the proof of Theorem 4.19, and shall give before some consequences.

4.4.1 Some consequences

An immediate consequence of Proposition 4.16 and Theorem 4.18 is the following

Theorem 4.20 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic and Ψ be an Orlicz function.*
The operator C_ϕ on H^Ψ is compact if and only if

$$\text{(W)} \qquad \sup_{\xi \in \mathbb{T}} \|C_\phi(u_{\xi,1-h})\|_\Psi = o\left(\frac{1}{\Psi^{-1}(1/h)}\right), \quad \textit{as } h \to 0.$$

We deduce:

Theorem 4.21 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic.*
1) *Assume that the Orlicz function Ψ satisfies condition Δ^0. Then, the operator C_ϕ on H^Ψ is weakly compact if and only if it is compact.*
2) *Assume that the Orlicz function Ψ satisfies condition ∇_2. Then, the operator C_ϕ on H^Ψ is a Dunford-Pettis operator if and only if it is compact.*

Recall that (see Theorem 3.24), under condition Δ^2 for Ψ, the weak compactness of the composition operator C_ϕ is equivalent to its compactness, and even to C_ϕ being order bounded into $M^\Psi(\mathbb{T})$. However, we shall see below, in Theorem 4.22, that there exist Orlicz functions $\Psi \in \Delta^0$ (and even $\Psi \in \Delta^1$) for which C_ϕ is compact, but not order bounded into $M^\Psi(\mathbb{T})$.

Proof. In both cases, the result follows from Theorem 4.20, since condition (W) is satisfied. Indeed, if $C_\phi \colon H^\Psi \to H^\Psi$ is weakly compact and $\Psi \in \Delta^0$, we use Theorem 3.20. If $C_\phi \colon H^\Psi \to H^\Psi$ is a Dunford-Pettis operator, this is due to Proposition 3.28. □

Now, we have:

Theorem 4.22 *There exist an Orlicz function Ψ satisfying Δ^1, and an analytic self-map $\phi \colon \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\phi \colon H^\Psi \to H^\Psi$ is not order bounded into $M^\Psi(\mathbb{T})$, though it is compact.*

Remark. It follows that our assumption that $\Psi \in \Delta^2$ in Theorem 3.24 is not only a technical one, though it might perhaps be weakened.

Proof. Let:

$$\Psi(x) = \begin{cases} \exp\big((\log x)^2\big) & \text{if} \quad x \geq \sqrt{\mathrm{e}}\,, \\ \mathrm{e}^{-1/4} x & \text{if} \quad 0 \leq x \leq \sqrt{\mathrm{e}}. \end{cases}$$

It is plain that $\Psi \in \Delta^1 \cap \nabla_0$.

Moreover, for every $A > 0$, one has, for h small enough:

$$\frac{1/h}{\Psi\big(A\Psi^{-1}(1/h)\big)} = \exp\Big[-(\log A)^2 - 2(\log A)\sqrt{\log(1/h)}\Big].$$

Consider now $\phi = \phi_2$, the analytic self-map of $\mathbb{D}$ constructed in Theorem 4.1. Then $C_\phi \colon H^\Psi \to H^\Psi$ is not order bounded into $M^\Psi(\mathbb{T})$, by Theorem 3.15, since, otherwise, $C_{\phi_1} \colon H^\Psi \to H^\Psi$ would also be order bounded into $M^\Psi(\mathbb{T})$, which is easily seen to be not the case (we may also argue as follows: $C_{\phi_1} \colon H^\Psi \to H^\Psi$ would be compact, and hence, by Theorem 4.3, C_{ϕ_1} would be compact from H^2 into H^2, which is false).

On the other hand, we have proved that $\rho_{\mu_\phi}(h) = O\left(h^{3/2}\right)$. So the conclusion follows from Theorem 4.11, 3) and the fact that for every $c > 0$:

$$-(\log A)^2 - 2(\log A)\sqrt{\log(1/h)} \geq -c\frac{1}{2}\log(1/h)$$

when h is small enough. □

4.4.2 Preliminary results

We shall use the *radial maximal function* N, defined for every harmonic function u on $\mathbb{D}$ by:

$$(Nu)(\xi) = \sup_{0 \leq r < 1} |u(r\xi)|\,, \quad \xi \in \mathbb{T}. \tag{4.16}$$

Recall that for every positive harmonic function $u \colon \mathbb{D} \to \mathbb{C}$ whose boundary values u^* are in $L^1(\mathbb{T})$ (*i.e.* $u \in h^1$), one has, for every $\xi \in \mathbb{T}$:

$$Nu(\xi) \leq Mu^*(\xi) \leq \pi Nu(\xi)\,, \tag{4.17}$$

where Mu^* is the Hardy-Littlewood maximal function of u^* (see [2], Theorem 6.31, and [32], Theorem 11.20 and Exercise 19, Chapter 11).

We shall denote by Π the right half-plane:

$$\Pi = \{z \in \mathbb{C}\,;\ \mathrm{Re}\, z > 0\}$$

and by $\mathscr{C}$ the cone:

$$\mathscr{C} = \{z \in \mathbb{C}\,;\ -\pi/6 < \mathrm{Arg}\, z < \pi/6\}.$$

The next result follows from Kolgomorov's Theorem, saying that the Hilbert transform is a weak $(1, 1)$ operator (in applying this theorem to the positive harmonic function $2\,\mathrm{Re}\, g = g + \bar{g}$, noting that $\|\mathrm{Re}\, g\|_1 = \mathrm{Re}\, g(0) = g(0)$).

Lemma 4.23 *There exists a constant $c > 0$ such that, if $g\colon \mathbb{D} \to \Pi$ is an analytic function with $g(0) > 0$, then:*

$$m(\{|g^*| > \lambda\}) \le c\,\frac{g(0)}{\lambda}, \quad \text{for all } \lambda > 0.$$

Applying Lemma 4.23 to $g(z) = \big(f(z) + \overline{f(0)}\big)^3$, and taking into account that $|w_1 + w_2| \ge |w_1|$, if $w_1, w_2 \in \mathscr{C}$, we get:

Lemma 4.24 *Let $f\colon \mathbb{D} \to \mathscr{C}$ be an analytic function with values in the cone $\mathscr{C}$, and write $f = u + iv$, with u, v real-valued. Then:*

$$m(\{|f^*| > \lambda\}) \le 8c \left(\frac{u(0)}{\lambda}\right)^3, \quad \text{for all } \lambda > 0.$$

The next proposition is one of the keys. We postpone its proof.

Proposition 4.25 *There exists a constant $k_2 > 0$ such that, for every analytic function $f = u + iv\colon \mathbb{D} \to \mathscr{C}$ with values in the cone $\mathscr{C}$, one has:*

$$m(\{|f^*| > \lambda\} \cap I) \le k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(I), \quad \text{for all } \lambda > 0, \tag{4.18}$$

where I is the arc $I = \{\mathrm{e}^{it}\,;\ a < t < b\}$, with $a, b \in \mathbb{R}$ and $\alpha > 0$ satisfying $b - a < \pi/2$, $\alpha \ge Nu(\mathrm{e}^{ia})$, and $\alpha \ge Nu(\mathrm{e}^{ib})$.

As a corollary we obtain:

Proposition 4.26 *Let $f\colon \mathbb{D} \to \mathscr{C}$ be an analytic function, and write $f = u+iv$, as in Proposition 4.25. If $\alpha > 0$ satisfies $m(\{Nu > \alpha\}) < 1/4$, then:*

$$m(\{|f^*| > \lambda\}) \le k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(\{Nu > \alpha\}), \quad \text{for all } \lambda \ge \frac{2\alpha}{\sqrt{3}}\,. \tag{4.19}$$

Proof. The set $\{Nu > \alpha\}$ is open, and one can decompose it into a disjoint union of open arcs $\{I_j\}_j$. Each arc has measure $m(I_j) \le m(\{Nu > \alpha\}) < 1/4$, and so is an arc of length less than $\pi/2$. We can then apply Proposition 4.25 and we obtain:

$$m(\{|f^*| > \lambda\} \cap I_j) \le k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(I_j), \quad \text{for every } j.$$

Summing up all these inequalities we get:

$$m\big(\{|f^*| > \lambda\} \cap \{Nu > \alpha\}\big) \le k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(\{Nu > \alpha\})\,.$$

The proposition follows since $|f^*| \le \frac{2}{\sqrt{3}} u^* \le \frac{2}{\sqrt{3}} Nu$, and then $\{|f^*| > \lambda\}$ is contained in $\{Nu > \alpha\}$, for $\lambda \ge \frac{2\alpha}{\sqrt{3}}$. $\square$

We shall need one more result.

Proposition 4.27 *There exists a constant $k_3 > 0$ such that for every analytic function $f\colon \mathbb{D} \to \mathscr{C}$ with values in the cone $\mathscr{C}$, one has, writing $f = u + iv$:*

$$m(\{Mu^* > \alpha\} \leq k_3\, m(\{u^* > \alpha/2\}, \quad \textit{for every } \alpha > 0.$$

In order to prove it, we shall first prove the following lemma.

Lemma 4.28 *There exists a constant $k_4 > 0$ such that for every analytic function $f\colon \mathbb{D} \to \mathscr{C}$ with values in the cone $\mathscr{C}$, one has, writing $f = u + iv$:*

$$M\big((u^*)^2\big)(\xi) \leq k_4 \big(Mu^*(\xi)\big)^2, \quad \textit{for all } \xi \in \mathbb{T}.$$

Proof. Observe that $u^2 - v^2$ is a positive harmonic function since it is the real part of f^2. f^2 belongs to H^1 (see [12], Theorem 3.2), so $u^2 - v^2 \in h^1$ and we can use inequalities (4.17). We also have, since $-\pi/3 < \operatorname{Arg}(f^2) < \pi/3$, that $u^2 \geq 3v^2$, and so $u^2 - v^2 \geq 2u^2/3$ and $|f| \leq \frac{2}{\sqrt{3}} u$. We get:

$$\begin{aligned} M\big((u^*)^2\big) &\leq \frac{3}{2} M\big((u^*)^2 - (v^*)^2\big) \leq \frac{3\pi}{2} N(u^2 - v^2) \\ &\leq \frac{3\pi}{2} N(f^2) = \frac{3\pi}{2} (Nf)^2 \leq \frac{3\pi}{2} \left(\frac{2}{\sqrt{3}} Nu\right)^2 \\ &\leq 2\pi\, (Mu^*)^2 . \end{aligned}$$

□

Proof of Proposition 4.27. Write $A = \{Mu^* > \alpha\}$ and $B = \{u^* > \alpha/2\}$. For every $\xi \in A$, there exists an open arc I_ξ centered at ξ such that:

$$p = \frac{1}{m(I_\xi)} \int_{I_\xi} u^*\, dm > Mu^*(\xi)/2\,, \quad \text{and} \quad p > \alpha\,.$$

We have, using Lemma 4.28:

$$\frac{1}{m(I_\xi)} \int_{I_\xi} (u^*)^2\, dm \leq M\big((u^*)^2\big)(\xi) \leq k_4 \big(Mu^*(\xi)\big)^2 \leq 4k_4\, p^2.$$

Let L be the set $L = \{u^* > p/2\} \cap I_\xi$. We have:

$$\frac{1}{m(I_\xi)} \int_{I_\xi \setminus L} \frac{u^*}{p}\, dm \leq \frac{1}{2}\,;$$

hence, using the Cauchy-Schwarz inequality:

$$\begin{aligned} \frac{1}{2} &\leq \frac{1}{m(I_\xi)} \int_{I_\xi} \frac{u^*}{p}\, dm - \frac{1}{m(I_\xi)} \int_{I_\xi \setminus L} \frac{u^*}{p}\, dm = \frac{1}{m(I_\xi)} \int_L \frac{u^*}{p}\, dm \\ &\leq \sqrt{\frac{m(L)}{m(I_\xi)}} \left(\frac{1}{m(I_\xi)} \int_{I_\xi} \left(\frac{u^*}{p}\right)^2 dm\right)^{1/2} \leq 2\sqrt{\frac{k_4 m(L)}{m(I_\xi)}}\,. \end{aligned}$$

Therefore, $16k_4\, m(L) \geq m(I_\xi)$, and, since $L \subseteq B \cap I_\xi$, we have, for every $\xi \in A$, an arc I_ξ containing ξ such that $16k_4\, m(B \cap I_\xi) \geq m(I_\xi)$. Applying the Hardy-Littlewood covering lemma, we then obtain:

$$m(A) \leq 3 \sum_{j=1}^{n} m(I_{\xi_j}) \leq 3 \times 16 \times k_4 \times \sum_{j=1}^{n} m(I_{\xi_j} \cap B) \leq k_3 m(B). \qquad \square$$

Proof of Proposition 4.25. Composing f with a suitable rotation, we can suppose that $a = -\delta$ and $b = \delta$, for $0 < \delta < \pi/4$. Let us call I^- and I^+ the arcs

$$I^- = \{e^{it};\ -\delta < t < 0\}, \qquad I^+ = \{e^{it};\ 0 < t < \delta\}.$$

We shall prove that:

$$m(\{|f^*| > \lambda\} \cap I^+) \leq k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(I^+), \tag{4.20}$$

using just the fact that $Nu(e^{i\delta}) \leq \alpha$.

In the same way one can prove:

$$m(\{|f^*| > \lambda\} \cap I^-) \leq k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(I^-), \tag{4.21}$$

using just that $Nu(e^{-i\delta}) \leq \alpha$.

Then, summing up (4.20) and (4.21), Proposition 4.25 will follow.

Let Q be the right half-disc

$$Q = \{z \in \mathbb{D};\ \operatorname{Re} z > 0\},$$

and denote by ψ the (unique) homeomorphism from $\overline{\mathbb{D}}$ to $\overline{Q}$, which is a conformal mapping from $\mathbb{D}$ onto Q and sends 1 to 1, i to i, and $-i$ to $-i$.

We can construct ψ as the composition of a Moebius transformation T, with the square root function and then with T^{-1}. Namely, let:

$$Tz = -i\frac{z+i}{z-i};$$

T maps $\mathbb{D}$ onto the upper-half plane, sending $-i$ into 0, -1 into -1, 1 into 1, and also 0 into i, and i into ∞. The square root function maps the upper-half plane into the first quadrant and T^{-1}:

$$T^{-1}z = \frac{z-i}{1-iz}$$

maps this quadrant onto the half-disk Q.

It is not difficult to see that $\psi(-1) = 0$, $\psi(0) = \sqrt{2} - 1$, and that there exist $\rho \in (0, \pi/2)$ such that $\psi(e^{i\rho}) = e^{\pi i/4}$ and $\psi(e^{-i\rho}) = e^{-\pi i/4}$ (we must have $e^{i\rho} = 1/3 + (\sqrt{8})i/3$; hence $\rho = \arctan(\sqrt{8})$).

Let J be the arc:

$$J = \{e^{it}\,;\ -\rho < t < \rho\}.$$

The map ψ is regular on J, and so there exist two constants γ_1 and $\gamma_2 > 0$ such that for every Borel subset E of J, one has:

$$\gamma_1\, m(E) \le m\big(\psi(E)\big) \le \gamma_2\, m(E).$$

If now $\beta \in (0,1)$, we put:

$$\psi_\beta(z) = (\psi(z))^\beta.$$

Then, it is easy to see that for every Borel subset E of J, one has:

$$m\big(\psi_\beta(E)\big) = \beta\, m\big(\psi(E)\big),$$

and so

$$\gamma_1 \beta\, m(E) \le m\big(\psi_\beta(E)\big) \le \gamma_2 \beta\, m(E).$$

In order to prove (4.20), consider the function $F\colon \mathbb{D} \to \mathscr{C}$ defined by:

$$F(z) = f\left(e^{i\delta}\psi_\beta(z)\right), \quad \text{where } \beta = 4\delta/\pi.$$

Then:

$$\operatorname{Re}\big(F(0)\big) = u\big((1-\sqrt{2})^\beta e^{i\delta}\big) \le Nu(e^{i\delta}) \le \alpha.$$

Let us call χ the map:

$$\chi(z) = e^{i\delta}\psi_\beta(z).$$

It is clear that I^+ is contained in $\chi(J)$. If $A = \{|f^*| > \lambda\} \cap I^+$, then $E = \chi^{-1}(A)$ is a Borel subset of J, and for every $\xi \in E$, one has $|F^*(\xi)| > \lambda$. Then:

$$\begin{aligned} m(A) &= m\big(\chi(E)\big) = m\big(\psi_\beta(E)\big) \le \gamma_2 \beta\, m(E) \\ &\le 8\gamma_2 \frac{\delta}{2\pi} m(\{|F^*| > \lambda\}) \\ &= 8\gamma_2\, m(I^+)\, m(\{|F^*| > \lambda\}) \end{aligned}$$

and using Lemma 4.24 for F,

$$\begin{aligned} &\le 8\gamma_2\, m(I^+) \times 8c_3 \left(\frac{\operatorname{Re} F(0)}{\lambda}\right)^3 \\ &\le 64\gamma_2 c_3 \left(\frac{\alpha}{\lambda}\right)^3 m(I^+) = k_2 \left(\frac{\alpha}{\lambda}\right)^3 m(I^+)\,. \end{aligned}$$

The proof of (4.20) is finished, and Proposition 4.25 follows. □

4.4.3 Proof of Theorem 4.19

Using the fact $h \mapsto \mu_\phi\big(S(\xi,h)\big)$ is nondecreasing, it is enough to prove that there exist $h_0 > 0$, and $\varepsilon_0 > 0$, such that (4.15) is true for $0 < h < h_0(1-|\phi(0)|)$, and $0 < \varepsilon < \varepsilon_0$, because changing the constant k_1, if necessary, the theorem will follow.

We can also suppose that $\xi = 1$.

The real part of $1/\big(1-\phi(z)\big)$ is positive, in fact greater than $1/2$, for every $z \in \mathbb{D}$. Take $0 < h < h_0$, and consider the analytic function f defined by:

$$f(z) = \left(\frac{h}{1-\phi(z)}\right)^{1/3},$$

where the cubic root is taken in order that, for every $z \in \mathbb{D}$, $f(z)$ belongs to the cone

$$\mathscr{C} = \{z \in \mathbb{C} : -\pi/6 < \operatorname{Arg}(z) < \pi/6\}.$$

Clearly $\mu_\phi\big(S(1,h)\big) = m(\{|f^*| > 1\})$ and $\mu_\phi\big(S(1,\varepsilon h)\big) = m(\{|f^*| > 1/\sqrt[3]{\varepsilon}\})$. We also have:

$$|f(0)| \leq \left(\frac{h}{1-|\phi(0)|}\right)^{1/3} < \sqrt[3]{h_0}.$$

We shall write $f = u + iv$ where u and v are real-valued harmonic functions. Observe that:

$$|v(z)| < \frac{1}{\sqrt{3}}\, u(z), \quad \text{for every } z \in \mathbb{D}.$$

It is known that $f \in H^p$, for every $p < 3$ (see [12], Theorem 3.2), and so u and v are the Poisson integrals of u^* and v^* (in particular $u, v \in h^1$).

We are looking for a control of $m(\{|f^*| > 1/\sqrt[3]{\varepsilon}\})$ by ε times $m(\{|f^*| > 1\})$. Proposition 4.26 provides this control replacing $m(\{|f^*| > 1\})$ by $m(\{Nu > 2\})$:

$$m(\{|f^*| > 1/\sqrt[3]{\varepsilon}\}) \leq 8k_2\varepsilon m(\{Nu > 2\}),$$

when $m(\{Nu > 2\}) < 1/4$.

As f is valued in $\mathscr{C}$, $|f^*|$ is controlled by u^*. Then what we need in fact is a control of the measure of level sets of Nu by the measure of level sets of u^*. This will be done by using Proposition 4.27.

Indeed, by Proposition 4.27, we have:

$$m(\{Nu > 2\}) \leq m(\{Mu^* > 2\}) \leq k_4\, m(\{u^* > 1\}).$$

We know that $||u^*||_1 = u(0) \leq |f(0)| \leq h_0^{1/3}$. Then, choosing h_0 small enough, we can have $k_4 h_0^{1/3} < 1/4$, and so $m(\{Nu > 2\}) < 1/4$; therefore, we can use Proposition 4.26. Moreover we also have $\{u^* > 1\} \subseteq \{|f^*| > 1\}$. Taking $\varepsilon_0 < 3\sqrt{3}/64$, we have, for $0 < \varepsilon < \varepsilon_0$, by Proposition 4.26:

$$\begin{aligned} m(\{|f^*| > 1/\sqrt[3]{\varepsilon}\}) &\leq 8k_2\,\varepsilon\, m(\{Nu > 2\}) \leq 8k_4k_2\,\varepsilon\, m(\{u^* > 1\}) \\ &\leq k_1\,\varepsilon\, m(\{|f^*| > 1\})\,, \end{aligned}$$

taking $k_1 = 8k_4k_2$. □

5 Bergman spaces

5.1 Bergman-Orlicz spaces

Definition 5.1 *Let* $d\mathscr{A}(z) = \frac{dx\,dy}{\pi}$ *(*$z = x + iy$*) be the normalized Lebesgue measure on* $\mathbb{D}$*. The* Bergman-Orlicz space $\mathscr{B}^\Psi$ *denotes the space of analytic functions* $f\colon \mathbb{D} \to \mathbb{C}$ *which are in the Orlicz space* $L^\Psi(\mathbb{D}, d\mathscr{A})$*. The* Bergman-Morse-Transue space *is the subspace* $\mathscr{B}M^\Psi = \mathscr{B}^\Psi \cap M^\Psi(\mathbb{D}, d\mathscr{A})$*.*

$\mathscr{B}^\Psi$, equipped with the induced norm of $L^\Psi(\mathbb{D}, d\mathscr{A})$, is a Banach space, as an obvious consequence of the following lemma, analogous to Lemma 3.11.

Lemma 5.2 *For every* $a \in \mathbb{D}$*, the norm of the evaluation functional* δ_a*, which maps* $f \in \mathscr{B}^\Psi$ *to* $f(a)$*, is:*

$$\|\delta_a\| \approx \Psi^{-1}\Big(\frac{1}{(1-|a|)^2}\Big)\,.$$

Proof. For every analytic function $g\colon \mathbb{D} \to \mathbb{C}$, the mean-value property gives:

$$g(0) = \int_{\mathbb{D}} g(z)\, d\mathscr{A}(z).$$

Hence if $\phi_a\colon \mathbb{D} \to \mathbb{D}$ denotes the analytic automorphism

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}\,,$$

whose inverse is $\phi_a^{-1} = \phi_{-a}$, one has, for every $f \in \mathscr{B}^\Psi$, using the change of variable formula:

$$\begin{aligned} f(a) = f\circ\phi_{-a}(0) &= \int_{\mathbb{D}} f\circ\phi_{-a}(z)\, d\mathscr{A}(z) = \int_{\mathbb{D}} f(w)\,|\phi_a'(w)|^2\, d\mathscr{A}(w) \\ &= \int_{\mathbb{D}} f(w) H_a(w)\, d\mathscr{A}(w)\,, \end{aligned}$$

where:

$$H_a(w) = |\phi_a'(w)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}w|^4}\,\cdot$$

The kernel H_a plays for $\mathscr{B}^\Psi$ the role that the Poisson kernel P_a plays for H^Ψ: the *analytic reproducing kernel* K_a for $\mathscr{B}^2$ being $K_a(z) = \frac{1}{(1-\bar{a}z)^2}\,\cdot$

We therefore have (using [30], Proposition 4, page 61, or [5], Theorem 8.14):

$$|f(a)| \leq 2\|f\|_\Psi \|H_a\|_\Phi\,,$$

which proves the continuity of δ_a. To estimate its norm, we are going to majorize $\|H_a\|_\Phi$, with the help of Lemma 3.9. Let us notice that, on the one hand, $\|H_a\|_1 = 1$ (take $f = \mathbb{1}$ in the above identity); and on the other hand:

$$\|H_a\|_\infty = \frac{(1-|a|^2)^2}{(1-|a|)^4} = \frac{(1+|a|)^2}{(1-|a|)^2}\,\cdot$$

We get, setting $b = \|H_a\|_\infty$, and using Lemma 3.9 for $\| \ \|_\Phi$:

$$\|H_a\|_\Phi \leq \frac{b}{\Phi^{-1}(b)} \cdot$$

But $b \leq \Phi^{-1}(b)\Psi^{-1}(b)$ (see [30], Proposition 1 (ii), page 14). Hence $\|H_a\|_\Phi \leq \Psi^{-1}(b)$. Now:

$$b \leq \frac{4}{(1-|a|)^2} \cdot$$

We have $\Psi^{-1}(4t) \leq 4\Psi^{-1}(t)$ for all $t > 0$. It follows that

$$\|H_a\|_\Phi \leq C\Psi^{-1}\Big(\frac{1}{(1-|a|)^2}\Big) \cdot$$

Since $\|\delta_a\| \leq 2\|H_a\|_\Phi$, we get the upper bound in Lemma 5.2.

For the lower bound, we simply observe that $H_a = |G_a|$, where $G_a(z) = \dfrac{(1-|a|^2)^2}{(1-\overline{a}z)^4}$, and by Lemma 3.9:

$$\begin{aligned} \|\delta_a\| &\geq \frac{|G_a(a)|}{\|G_a\|_\Psi} = \frac{|H_a(a)|}{\|H_a\|_\Psi} \geq \frac{\dfrac{1}{(1-|a|^2)^2}}{b/\Psi^{-1}(b)} = \frac{\Psi^{-1}(b)}{b(1-|a|^2)^2} \\ &= \frac{\Psi^{-1}(b)}{(1+|a|)^4} \geq \frac{1}{16}\Psi^{-1}(b) \geq \frac{1}{16}\Psi^{-1}\Big(\frac{1}{(1-|a|)^2}\Big), \end{aligned}$$

since $b \geq 1/(1-|a|)^2$. □

Proposition 5.3 *We have the following properties*

i) $\mathscr{B}M^\Psi$ *is the closure of* $H^\infty(\mathbb{D})$ *in* $L^\Psi(\mathbb{D}, \mathscr{A})$ *and actually the algebraic polynomials are dense in* $\mathscr{B}M^\Psi$.

ii) On the unit ball of $\mathscr{B}^\Psi$, *the weak-star topology* $\sigma\big(L^\Psi(\mathbb{D},\mathscr{A}), M^\Phi(\mathbb{D},\mathscr{A})\big)$ *coincides with the topology of convergence on compact subsets of* $\mathbb{D}$.

iii) $\mathscr{B}^\Psi$ *is closed in* $L^\Psi(\mathbb{D},\mathscr{A})$ *for the weak-star topology.*

iv) If $\Psi \in \nabla_2$, $\mathscr{B}^\Psi$ *is (isometric to) the bidual of* $\mathscr{B}M^\Psi$.

Proof. For the first point, let us fix $f \in \mathscr{B}M^\Psi$. Setting $f_r(z) = f(rz)$ for $z \in \mathbb{D}$ and $0 \leq r < 1$, it suffices to show that $\|f_r - f\|_\Psi \underset{r\to 1}{\longrightarrow} 0$, since, being analytic in the disk $r\overline{\mathbb{D}} \subset \mathbb{D}$, f_r can be uniformly approximated on $\overline{\mathbb{D}}$ by its Taylor series. But the norm of M^Ψ is absolutely continuous (see [30], Theorem 14, page 84) and therefore, for every $\varepsilon > 0$, there is some $R > 0$, with $1/3 \leq R < 1$, such that $\|f\mathbb{1}_{\mathbb{D}\setminus R\mathbb{D}}\|_\Psi \leq \varepsilon$; hence:

$$\int_{\mathbb{D}\setminus R\mathbb{D}} \Psi\Big(\frac{|f|}{4\varepsilon}\Big)\, d\mathscr{A} \leq \int_{\mathbb{D}\setminus R\mathbb{D}} \frac{1}{4}\Psi\Big(\frac{|f|}{\varepsilon}\Big)\, d\mathscr{A} \leq \frac{1}{4}\cdot$$

When $r \geq \frac{2R}{R+1} \geq 1/2$, we therefore have:

$$\int_{\mathbb{D}\setminus\frac{1+R}{2}\mathbb{D}} \Psi\Big(\frac{|f_r|}{4\varepsilon}\Big)\, d\mathscr{A} \leq 1,$$

and, by convexity of Ψ:

$$\begin{aligned}\int_{\mathbb{D}} \Psi\Big(\frac{|f_r - f|}{8\varepsilon}\Big)\, d\mathscr{A} &\leq \int_{\frac{1+R}{2}\mathbb{D}} \Psi\Big(\frac{|f_r - f|}{8\varepsilon}\Big)\, d\mathscr{A} \\ &\qquad + \int_{\mathbb{D}\setminus\frac{1+R}{2}\mathbb{D}} \frac{1}{2}\Big[\Psi\Big(\frac{|f_r|}{4\varepsilon}\Big) + \Psi\Big(\frac{|f|}{4\varepsilon}\Big)\Big]\, d\mathscr{A} \\ &\leq \int_{\frac{1+R}{2}\mathbb{D}} \Psi\Big(\frac{|f_r - f|}{8\varepsilon}\Big)\, d\mathscr{A} \\ &\qquad + \frac{1}{2}\int_{\mathbb{D}\setminus\frac{1+R}{2}\mathbb{D}} \Psi\Big(\frac{|f_r|}{4\varepsilon}\Big)\, d\mathscr{A} + \frac{1}{2}\int_{\mathbb{D}\setminus R\mathbb{D}} \Psi\Big(\frac{|f|}{4\varepsilon}\Big)\, d\mathscr{A} \\ &\leq 1,\end{aligned}$$

for r close enough to 1 since $f_r - f$ tends to 0 uniformly on $\frac{1+R}{2}\mathbb{D}$. Hence, for some $r_0 < 1$, one has $\|f_r - f\|_\Psi \leq 8\varepsilon$ for $r_0 \leq r < 1$. This was the claim.

ii) It suffices to use a sequential argument since the topologies are metrizable on balls (the space $M^\Phi(\mathbb{D}, \mathscr{A})$ is separable). Assume that $f \in \mathscr{B}^\Psi$ (with $\|f\|_\Psi \leq 1$) is the weak-star limit of a sequence of analytic functions $f_n \in \mathscr{B}^\Psi$ (with $\|f_n\|_\Psi \leq 1$). Testing this with the function $h_k(z) = (k+1)\overline{z}^k$, we obtain that the k^{th} Taylor coefficient $a_k(n)$ of f_n converges to the k^{th} Taylor coefficient a_k of f:

$$a_k(n) = \int_{\mathbb{D}} f_n h_k\, d\mathscr{A} \longrightarrow \int_{\mathbb{D}} f h_k\, d\mathscr{A} = a_k\ , \text{ for every } k \geq 0.$$

Fix a compact $K \subset \mathbb{D}$, there exists an $r \in (0,1)$ such that $K \subset r\mathbb{D}$. We have:

$$\sup_{z\in K} |f_n(z) - f(z)| \leq \sum_{k\geq 0} |a_k(n) - a_k| r^k \longrightarrow 0$$

by the dominated convergence Theorem (observe that $|a_k - a_k(n)| \leq 2(k+1)$ for every $k \geq 0$ and every $n \geq 0$).

For the converse, suppose now that $f_n \in \mathscr{B}^\Psi$ (with $\|f_n\|_\Psi \leq 1$) converges uniformly on every compact subsets of $\mathbb{D}$ to $f \in \mathscr{B}^\Psi$ (with $\|f\|_\Psi \leq 1$). Fixing $g \in M^\Phi(\mathbb{D}, \mathscr{A})$ and $\varepsilon > 0$, there exists an $r \in (0,1)$ such that $\|g\mathbb{1}_{\mathbb{D}\setminus r\mathbb{D}}\|_\Phi \leq \varepsilon$. We have:

$$\Big|\int_{\mathbb{D}} (f_n - f) g\, d\mathscr{A}\Big| \leq \Big|\int_{r\mathbb{D}} (f_n - f) g\, d\mathscr{A}\Big| + 2\varepsilon \leq \sup_{z\in r\mathbb{D}} |f_n(z) - f(z)|.\|g\|_1 + 2\varepsilon.$$

By hypothesis $\sup_{z\in r\mathbb{D}} |f_n(z) - f(z)| \longrightarrow 0$. The conclusion follows.

iii) By the classical Theorem of Banach-Dieudonné, it is sufficient to prove that the balls are weak-star closed (equivalently weak-star compact) and by separability of $M^\Phi(\mathbb{D}, \mathscr{A})$, the weak-star topology is metrizable on balls. The previous fact ii) shows that it is equivalent to prove that the unit ball of $\mathscr{B}^\Psi$ is compact for the topology of convergence on compact subsets. But this is easy: indeed, if f_n in the unit ball of $\mathscr{B}^\Psi$. This is a normal family thanks to Lemma 5.2. A subsequence converges to an analytic function f on compact subsets of $\mathbb{D}$ and the Fatou Lemma implies that f actually lies in the unit ball.

iv) Assume now that Ψ satisfies ∇_2. Since $(M^\Psi)^{**} = L^\Psi(\mathbb{D}, \mathscr{A})$, we have $(\mathscr{B}M^\Psi)^{**} = \overline{\mathscr{B}M^\Psi}^{w^*}$, in the space $L^\Psi(\mathbb{D}, \mathscr{A})$. Hence it suffices to show that $\overline{\mathscr{B}M^\Psi}^{w^*} = \mathscr{B}^\Psi$. We already know that $\mathscr{B}^\Psi$ is weak-star closed. Now, let $f \in \mathscr{B}^\Psi$. Obviously, $f_r \in \mathscr{B}M^\Psi$ for every $r \in (0,1)$, where $f_r(z) = f(rz)$. Moreover $\|f_r\|_\Psi \leq \|f\|_\Psi$. But this is clear that f_r is uniformly convergent to f on compact subsets of $\mathbb{D}$, when $r \to 1^-$. By ii), the conclusion follows. □

In the previous proof, we can see the points ii) and iii) in a slighlty different way: the unit ball B of $\mathscr{B}^\Psi$ is compact for the topology τ of uniform convergence on compact subsets. Then observe that the identity from B, equipped with the topology τ, to B, equipped with the weak-star topology, is continuous (the sequential argument is sufficient by metrizability). This implies, since the weak-star topology is separated, that B is weak-star compact (hence closed) and that the topologies coincide. Now, by Banach-Dieudonné, the space $\mathscr{B}^\Psi$ is weak-star closed.

5.2 Compact composition operators on Bergman-Orlicz spaces

We shall begin with showing that, as in the Hardy-Orlicz case, every symbol ϕ defines a bounded composition operator.

Proposition 5.4 *Every analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator $C_\phi\colon \mathscr{B}^\Psi \to \mathscr{B}^\Psi$. Moreover, $C_\phi f \in \mathscr{B}M^\Psi$ for every $f \in \mathscr{B}M^\Psi$; hence, when $\Psi \in \nabla_2$, the former operator is the bi-transposed of $C_\phi\colon \mathscr{B}M^\Psi \to \mathscr{B}M^\Psi$.*

Proof. It suffices to follow the lines of Proposition 3.12, and to integrate the integrals written there between 0 and 1, with respect to the measure $2rdr$. □

Before stating and proving the main theorem, we are going to prove the following auxiliary result, interesting in itself, and which reinforces an example of J. H. Shapiro ([36], Example, page 185).

Proposition 5.5 *There exists a Blaschke product B having angular derivative at no point of $\mathbb{T} = \partial\mathbb{D}$, in the following sense:*

$$(5.1) \qquad (\forall \varepsilon > 0)\ (\exists c_\varepsilon > 0) \quad 1 - |B(z)| \geq c_\varepsilon (1 - |z|)^\varepsilon\,, \ \ \forall z \in \mathbb{D}.$$

Proof. We shall take:

$$B(z)=\prod_{n=1}^{+\infty}\frac{|z_n|}{z_n}\,\frac{z_n-z}{1-\bar z_n z}\,,$$

where:

$$\{z_n\,;\ n\geq 1\}=\bigcup_{n\geq 1}A_n,$$

with:

$$A_n=\{r_n\omega_n^j\,;\ \omega_n=\mathrm{e}^{2\pi i/p_n}\,,\ 0\leq j\leq p_n-1\}\,,$$

where $(r_n)_{n\geq 1}$ is a (strictly) increasing sequence with $0<r_n<1$, and the integers p_n will have to be adjusted, satisfying the Blaschke condition:

$$\sum_{n=1}^{+\infty}(1-|z_n|)=\sum_{n=1}^{+\infty}p_n(1-r_n)<+\infty.$$

One has:

$$|B(z)|^2=\prod_{n=1}^{+\infty}\Big|\frac{z_n-z}{1-\bar z_n z}\Big|^2=\prod_{n=1}^{+\infty}\bigg[1-\frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar z_n z|^2}\bigg]\leq\exp\big(-S(z)\big)\,,$$

where:

$$S(z)=\sum_{n=1}^{+\infty}\frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar z_n z|^2}\,\cdot$$

We now proceed to minorize $S(z)$. For this purpose, we shall need the following simple lemma, whose proof will be temporarily postponed.

Lemma 5.6 *For every positive integer p and every $a\in\mathbb{D}$, one has, setting $\omega=\mathrm{e}^{2\pi i/p}$:*

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{1}{|1-a\omega^k|^2}=\frac{1-|a|^{2p}}{1-|a|^2}\,\frac{1}{|1-a^p|^2}\geq\frac{1}{4}p\,|a|^p.$$

Then, setting $r=|z|$, we have:

$$\begin{aligned}S(z)&\geq(1-r)\sum_{n=1}^{+\infty}(1-r_n)\sum_{k=0}^{p_n-1}\frac{1}{|1-r_n\omega_n^{-k}z|^2}\\&\geq\frac{1-r}{4}\sum_{n=1}^{+\infty}p_n^2(1-r_n)(r_nr)^{p_n}.\end{aligned}$$

We shall take:

$$p_n=\big[(1-r_n)^{\varepsilon_n-1}\big]+1\,,$$

with:

$$\varepsilon_n=\frac{1}{\sqrt{n}}\qquad\text{and}\qquad r_n=1-\frac{1}{2^n}$$

and where [ ] stands for the integer part. More explicitly:

$$p_n = \left[2^{n-\sqrt{n}}\right] + 1.$$

If $r \geq 1/2$, let $N \geq 1$ be such that $r_N < r \leq r_{N+1}$. One has:

$$S(z) \geq \frac{1-r}{4}\,(1-r_N)^{2\varepsilon_N - 1}\, r_N^{2p_N} \geq \frac{1}{8}(1-r_N)^{2\varepsilon_N}\, r_N^{2p_N}\,,$$

since $1 - r \geq 1 - r_{N+1} = (1-r_N)/2$. Moreover:

$$p_N \leq 2(1-r_N)^{\varepsilon_N - 1} = 2.2^{N(1-\varepsilon_N)} \leq 2.2^N\,,$$

so that:

$$S(z) \geq \frac{1}{8}(1-r)^{2\varepsilon_N}\,(1-2^{-N})^{2^{N+1}} \geq c(1-r)^{2\varepsilon_N}\,,$$

where c is a positive numerical constant.

Hence, setting:

$$\varepsilon(z) = 2\varepsilon_N \qquad \text{for } |z| \geq 1/2 \text{ and } r_N < r \leq r_{N+1}\,,$$

one has:

$$\varepsilon(z) \underset{|z| \overset{<}{\to} 1}{\longrightarrow} 0\,,$$

and we get:

$$1 - |B(z)|^2 \geq 1 - \mathrm{e}^{-S(z)} \geq 1 - \mathrm{e}^{-c(1-|z|)^{\varepsilon(z)}} \geq c'(1-|z|)^{\varepsilon(z)}\,,$$

where c' is another positive numerical constant. This gives condition (5.1), since $1 - |B(z)| \geq \dfrac{1-|B(z)|^2}{2}$.

Finally, the Blaschke condition is satisfied, since:

$$p_n(1-r_n) \leq 2(1-r_n)^{\varepsilon_n} = 2.2^{-\sqrt{n}}.$$

This ends the proof of Proposition 5.5. □

Proof of Lemma 5.6. Let G_p be the finite group of p^{th} roots of unity, equipped with its normalized Haar measure. For $u\colon G_p \to \mathbb{C}$ and $0 \leq k \leq p-1$, we denote by $\hat{u}(k)$ the k^{th} Fourier coefficient of u, *i.e.*:

$$\hat{u}(k) = \frac{1}{p}\sum_{z\in G_p} u(z) z^{-k}.$$

Then, the Plancherel-Parseval formula for G_p reads:

$$\sum_{k=0}^{p-1} |\hat{u}(k)|^2 = \frac{1}{p}\sum_{z\in G_p} |u(z)|^2.$$

Applying this to

$$u(z) = \frac{1}{1 - az} = \sum_{\substack{l \geq 0 \\ 0 \leq k \leq p-1}} a^{lp+k} z^k = \sum_{k=0}^{p-1} \hat{u}(k) z^k \,,$$

with

$$\hat{u}(k) = \sum_{l \geq 0} a^{lp+k} = \frac{a^k}{1 - a^p} \,,$$

we get:

$$\frac{1}{p} \sum_{k=0}^{p-1} \frac{1}{|1 - a\omega^k|^2} = \sum_{k=0}^{p-1} \frac{|a|^{2k}}{|1 - a^p|^2} = \frac{1 - |a|^{2p}}{(1 - |a|^2)|1 - a^p|^2} \,.$$

To finish, we note that $|1 - a^p| \leq 2$, and that, by the arithmetico-geometric inequality, we have, with $x = |a|^2$:

$$\begin{aligned} \frac{1 - |a|^{2p}}{1 - |a|^2} &= 1 + x + \cdots + x^{p-1} \\ &\geq p\big(x^{1+2+\cdots+(p-1)}\big)^{1/p} = p\, x^{\frac{p-1}{2}} \geq p\, x^{p/2} = p\, |a|^p. \end{aligned}$$

□

Theorem 5.7 *If the composition operator $C_\phi \colon \mathscr{B}^\Psi \to \mathscr{B}^\Psi$ is compact, then*

$$\frac{\Psi^{-1}\Big[\dfrac{1}{(1 - |\phi(a)|)^2}\Big]}{\Psi^{-1}\Big[\dfrac{1}{(1 - |a|)^2}\Big]} \underset{|a| \overset{<}{\to} 1}{\longrightarrow} 0. \tag{5.2}$$

This condition is sufficient if $\Psi \in \Delta^2$.

Before giving the proof of this theorem, let us note that in case where $\Psi = \Psi_2$, with $\Psi_2(x) = \mathrm{e}^{x^2} - 1$, it reads: *the composition operator $C_\phi \colon \mathscr{B}^{\Psi_2} \to \mathscr{B}^{\Psi_2}$ is compact if and only if:*

$$(\forall \varepsilon > 0)\ (\exists c_\varepsilon > 0) \qquad 1 - |\phi(z)| \geq c_\varepsilon (1 - |z|)^\varepsilon \,, \quad \forall z \in \mathbb{D}. \tag{5.3}$$

Indeed, $\Psi_2 \in \Delta^2$, and:

$$\frac{\Psi_2^{-1}\Big[\dfrac{1}{(1 - |\phi(a)|)^2}\Big]}{\Psi_2^{-1}\Big[\dfrac{1}{(1 - |a|)^2}\Big]} = \frac{\sqrt{\log \dfrac{1}{(1 - |\phi(a)|)^2}}}{\sqrt{\log \dfrac{1}{(1 - |a|)^2}}} \,;$$

hence C_ϕ is compact if and only if

$$\frac{\log \dfrac{1}{(1 - |\phi(a)|)^2}}{\log \dfrac{1}{(1 - |a|)^2}} \underset{|a| \to 1}{\longrightarrow} 0.$$

Then, for every $\varepsilon > 0$, we can find some $c_\varepsilon > 0$ such that, for all $a \in \mathbb{D}$:

$$\log \frac{1}{1 - |\phi(a)|} \leq \varepsilon \, \log \frac{1}{(1 - |a|)^2} + c_\varepsilon \, ;$$

which is equivalent to (5.3). □

That allows to have compact composition operators on $\mathscr{B}^{\Psi_2}$ which are not compact on H^{Ψ_2}. However it is very likely that this is the case for every $\Psi \in \Delta^2$, but we have not tried to see this full generality.

Theorem 5.8 *There exist symbols $\phi\colon \mathbb{D} \to \mathbb{D}$ such that the composition operators C_ϕ is compact from $\mathscr{B}^{\Psi_2}$ into itself, but not compact from H^{Ψ_2} into itself, and even C_ϕ is an isometry onto its image.*

A similar example is well-known for the Hilbert spaces $\mathscr{B}^2$ and H^2 (see [36], pages 180–186).

Proof. Let B be a Blaschke product verifying the condition of Proposition 5.5. We introduce $\phi(z) = zB(z)$. The function still verifies $1 - |\phi(z)| \geq C_\varepsilon (1 - |z|)^\varepsilon$, since $|\phi(z)| \leq |B(z)|$ on $\mathbb{D}$. From Theorem 5.7, it follows, since $\Psi_2 \in \Delta^2$, that $C_\phi \colon \mathscr{B}^{\Psi_2} \to \mathscr{B}^{\Psi_2}$ is compact.

We are now going to see that $C_\phi \colon H^{\Psi_2} \to H^{\Psi_2}$ is an isometry. Indeed, recall the following well-known fact, that we already used (see [29], Theorem 1): since ϕ is an inner function, the image $\phi(m)$ of the Haar measure m of $\mathbb{T}$ under ϕ is equal to $P_a.m$, where $a = \phi(0)$ and P_a is the Poisson kernel at a. Here $\phi(0) = 0$ so $\phi(m) = m$. It follows that for every $f \in H^{\Psi_2}$, one has, for $C > 0$:

$$\int_{\mathbb{T}} \Psi_2\Big(\frac{|f \circ \phi|}{C}\Big) \, dm = \int_{\mathbb{T}} \Psi_2\Big(\frac{|f|}{C}\Big) \, dm$$

so that $\|f\|_{\Psi_2} = \|f \circ \phi\|_{\Psi_2}$. □

We shall need also the following lemma, which completes Lemma 5.2.

Lemma 5.9 *For every $f \in \mathscr{B}M^\Psi$, one has:*

$$f(a) = o\left(\Psi^{-1}\Big(\frac{1}{(1 - |a|)^2}\Big)\right) \quad \textit{as} \quad |a| \xrightarrow{<} 1.$$

Proof. This is obvious for the monomials $e_n \colon z \mapsto z^n$ since $|e_n(a)| \leq 1$, whereas $\Psi^{-1}\big(1/(1 - |a|)^2\big) \underset{|a| \to 1}{\longrightarrow} +\infty$. Since the evaluation δ_a is bounded on $\mathscr{B}M^\Psi$ and $\big\|\delta_a / \big(\Psi^{-1}(1/1 - |a|^2)\big)\big\| = O\,(1)$, it suffices to use that the polynomials are dense in $\mathscr{B}M^\Psi$; but this was already proved in Proposition 5.3.

□

Proof of Theorem 5.7. If $C_\phi \colon \mathscr{B}^\Psi \to \mathscr{B}^\Psi$ is compact, then so is the restriction $C_\phi \colon \mathscr{B}M^\Psi \to \mathscr{B}M^\Psi$ and its adjoint $C_\phi^* = C_\phi \colon \big(\mathscr{B}M^\Psi\big)^* \to \big(\mathscr{B}M^\Psi\big)^*$. Since

$C_\phi^*(\delta_a) = \delta_{\phi(a)}$, Lemma 5.9 gives $\delta_a / \|\delta_a\| \xrightarrow[|a|\to 1]{w^*} 0$. Compactness of C_ϕ^* now leads us to

$$C_\phi^*\Big(\frac{\delta_a}{\|\delta_a\|}\Big) \xrightarrow[|a|\to 1]{\|\ \|} 0.$$

That gives (5.2), in view of Lemma 5.2.

Conversely, assume that (5.2) is verified. Observe first that, since $\Psi \in \Delta^2$, one has:

$$\Psi^{-1}(x^2) \leq \alpha \Psi^{-1}(x) \qquad \text{for } x \text{ large enough.} \tag{5.4}$$

Indeed, let $x_0 > 0$ be such that $\Psi(\alpha x) \geq \big(\Psi(x)\big)^2$ for $x \geq x_0$. For $x \geq y_0 = \sqrt{\Psi(\alpha x_0)}$, with $y = \Psi^{-1}(x^2)$, one has $x^2 = \Psi(y) \geq \big(\Psi(y/\alpha)\big)^2$, *i.e.* $x \geq \Psi(y/\alpha)$, and hence $\Psi^{-1}(x^2) = y \leq \alpha \Psi^{-1}(x)$.

Therefore condition (5.2), which reads:

$$\Psi^{-1}\Big(\frac{1}{(1-|\phi(z)|)^2}\Big) = o\left(\Psi^{-1}\Big(\frac{1}{(1-|z|)^2}\Big)\right),$$

reads as well, because of (5.4):

$$\Psi^{-1}\Big(\frac{1}{1-|\phi(z)|}\Big) = o\left(\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\right), \qquad \text{as } |z| \to 1. \tag{5.5}$$

We have to prove that (5.5) implies the compactness of $C_\phi \colon \mathscr{B}^\Psi \to \mathscr{B}^\Psi$. So, by Proposition 3.8, we have to prove that: for every sequence $(f_n)_n$ in the unit ball of $\mathscr{B}^\Psi$ which converges uniformly on compact sets of $\mathbb{D}$, one has $\|f_n \circ \phi\|_\Psi \xrightarrow[n\to\infty]{} 0$.

But (5.2) and (5.4) imply that, for some $C > 0$:

$$|f_n(z)| \leq C\, \Psi^{-1}\Big(\frac{1}{1-|z|}\Big), \qquad \forall z \in \mathbb{D}. \tag{5.6}$$

Let $\varepsilon > 0$ and set $\varepsilon_0 = \varepsilon/\alpha C$. Due to (5.5), we can find some r with $0 < r < 1$ such that:

$$\begin{cases} \sqrt{1-r} \leq \dfrac{1}{8}\,; \qquad \Psi^{-1}\Big(\dfrac{1}{1-r}\Big) \geq \alpha x_0\,; \\[2ex] \Psi^{-1}\Big(\dfrac{1}{1-|\phi(z)|}\Big) \leq \varepsilon_0 \Psi^{-1}\Big(\dfrac{1}{1-|z|}\Big) \text{ if } z \in \mathbb{D} \setminus r\mathbb{D}. \end{cases} \tag{5.7}$$

Then, since $(f_n)_n$ converges uniformly on $r\mathbb{D}$, we have, for n large enough ($n \geq n_0$):

$$\int_{r\mathbb{D}} \Psi\Big(\frac{|f_n \circ \phi|}{\varepsilon}\Big)\, d\mathscr{A}(z) \leq \frac{1}{2}\,.$$

On the other hand, by (5.7):

$$
\begin{aligned}
\int_{\mathbb{D}\setminus r\mathbb{D}} \Psi\Big(\frac{|f_n\circ\phi|}{\varepsilon}\Big)\,d\mathscr{A}(z) &\leq \int_{\mathbb{D}\setminus r\mathbb{D}} \Psi\bigg(\frac{C}{\varepsilon}\Psi^{-1}\Big(\frac{1}{1-|\phi(z)|}\Big)\bigg)\,d\mathscr{A}(z)\\
&\leq \int_{\mathbb{D}\setminus r\mathbb{D}} \Psi\bigg(\frac{\varepsilon_0 C}{\varepsilon}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\bigg)\,d\mathscr{A}(z)\\
&= \int_{\mathbb{D}\setminus r\mathbb{D}} \Psi\bigg(\frac{1}{\alpha}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\bigg)\,d\mathscr{A}(z)\\
&\leq \int_{\mathbb{D}\setminus r\mathbb{D}} \sqrt{\Psi\bigg(\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\bigg)}\,d\mathscr{A}(z)\\
&\qquad\qquad \text{since } \Psi^{-1}\Big(\frac{1}{1-r}\Big)\geq \alpha x_0\,,\\
&= \int_{\mathbb{D}\setminus r\mathbb{D}} \frac{1}{\sqrt{1-|z|}}\,d\mathscr{A}(z) = 2\int_r^1 \frac{\rho\,d\rho}{\sqrt{1-\rho}}\\
&\leq 2\int_r^1 \frac{d\rho}{\sqrt{1-\rho}} = 4\sqrt{1-r}\leq \frac{1}{2}\cdot
\end{aligned}
$$

Putting together these two inequalities, we get, for $n \geq n_0$:

$$\int_{\mathbb{D}} \Psi\Big(\frac{|f_n\circ\phi|}{\varepsilon}\Big)\,d\mathscr{A}(z) \leq \frac{1}{2}+\frac{1}{2} = 1,$$

and hence: $\|f_n\circ\phi\|_\Psi \leq \varepsilon$, which ends the proof of Theorem 5.7. □

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