On average connectivity of the strong product of graphs

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A B S T R A C T

The average connectivity $\overline{\kappa}(G)$ of a graph G is the average, over all pairs of vertices, of the maximum number of internally disjoint paths connecting these vertices. The connectivity $\kappa(G)$ can be seen as the minimum, over all pairs of vertices, of the maximum number of internally disjoint paths connecting these vertices. The connectivity and the average connectivity are upper bounded by the minimum degree $\delta(G)$ and the average degree $\overline{d}(G)$ of G, respectively. In this paper the average connectivity of the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 is studied. A sharp lower bound for this parameter is obtained. As a consequence, we prove that $\overline{\kappa}(G_1 \boxtimes G_2) = \overline{d}(G_1 \boxtimes G_2)$ if $\overline{\kappa}(G_i) = \overline{d}(G_i)$, $i = 1, 2$. Also we deduce that $\kappa(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2)$ if $\kappa(G_i) = \delta(G_i)$, $i = 1, 2$.

Keywords:
Average connectivity Strong product of graphs Maximally connected graphs Average degree

1. Introduction

Throughout this paper, all the graphs are simple, that is, with neither loops nor multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [4].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The cardinalities of these sets are denoted by $|V(G)| = n$ and $|E(G)| = e$. Let u and v be two distinct vertices of G. A path from u to v, also called an uv-path in G, is a subgraph P with vertex set $V(P) = \{u = x_0, x_1, \ldots, x_r = v\}$ and edge set $E(P) = \{x_0x_1, \ldots, x_{r-1}x_r\}$. This path is usually denoted by $P : x_0x_1 \cdots x_r$ and r is the length of P, denoted by $l(P)$. Two uv-paths P and Q are said to be internally disjoint if $V(P) \cap V(Q) = \{u, v\}$. A cycle in G of length r is a path $C : x_0x_1 \cdots x_r$ such that $x_0 = x_r$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G, and if G contains no cycles, then $g(G) = \infty$. The set of adjacent vertices to $v \in V(G)$ is denoted by $N_G(v)$. The *degree* of v is $d_G(v) = |N_G(v)|$, whereas $\delta(G) = \min_{v\in V(G)} d_G(v)$ and $\overline{d}(G) = \frac{1}{n}\sum_{v\in V(G)} d_G(v) = 2e/n$ are the *minimum degree* and the *average degree* of G, respectively. The connectivity of a graph G, is the smallest number of vertices whose deletion from G produces a disconnected or a trivial graph. Clearly, a complete graph cannot be disconnected by deleting vertices, so that $\kappa(K_n) = n - 1$ is adopted. The connectivity between two distinct vertices u and v in a graph G, denoted by $\kappa_G(u, v)$, is the minimum number of vertices whose deletion separates u and v in G. Whitney [15] proved in 1932 that a graph G is r-connected, that is, $\kappa(G) \geq r$, if and only if every pair of vertices is connected by r internally disjoint paths. From this result, we know that the connectivity $\kappa_G(u, v)$ between two distinct vertices u and v in G is the maximum number of pairwise internally disjoint uv-paths in G. In this way, the connectivity of a graph can be seen as $\kappa(G) = \min_{u,v\in V(G)} \kappa_G(u, v)$. In [15] the author also showed that $\kappa(G) \leq \delta(G)$. The graph G is *maximally connected* if the previous bound is attained, that is, if $\kappa(G) = \delta(G)$.

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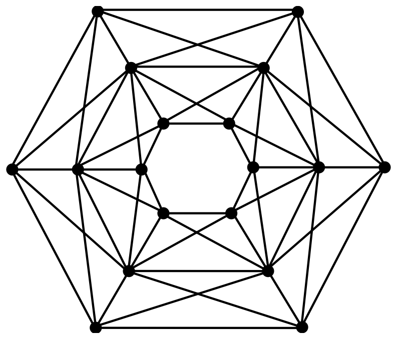

Fig. 1. The strong product of a cycle of length 6 and a path of order 3.

For a graph G of order n, the *average connectivity* $\overline{\kappa}(G)$ is defined as the average of the connectivities between all pairs of vertices of G, that is,

$$\overline{\kappa}(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} \kappa_G(u, v).$$

In order to avoid fractions, we also consider the *total connectivity* $K(G)$ of G, defined as $K(G) = \sum_{u,v \in V(G)} \kappa_G(u, v)$. While the connectivity is the minimum number of vertices whose removal separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices.

It is well known that most networks can be modeled by a graph $G = (V, E)$, where V is the set of mainly elements and E is the set of communication links between them in the network. The best known measure of reliability of a graph is its connectivity, defined above. As the connectivity is a worst-case measure, it does not always reflect what happens throughout the graph. For example, a tree and the graph obtained by appending an end-vertex to a complete graph both have connectivity 1. Nevertheless, for large order the latter graph is far more reliable than the former. Interest in the vulnerability and reliability of networks such as transportation and communication networks, has given rise to a host of other measures of reliability, see for example [1]. In this paper we pay attention to a measure for the reliability of a graph, the *average connectivity*, introduced by Beineke, Oellermann and Pippert [3].

There is a lot of research on the connectivity of a graph (see [10]). Many works provide sufficient conditions for a graph to be maximally connected or super connected [5,8,14]. Others study the maximal connectivity in networks that are constructed from graph generators, as Cartesian product graphs [6,12,16], line graphs [11,13], permutation graphs [2,9]. There are two excellent papers where the average connectivity has been investigated. In the first one, Beineke, Oellermann and Pippert [3] find upper and lower bounds on the average connectivity of a graph G in terms of its order n and its average degree $\overline{d}(G)$. In the second one, Dankelmann and Oellermann [7] obtain sharp upper bounds for some families of graphs, such as planar and outerplanar graphs and Cartesian product of graphs. In this paper, we study the average connectivity of one kind of product graphs, the so called *strong product of graphs*.

For a large system, configuration processing is one of the most tedious and time-consuming parts of the analysis. Different methods have been proposed for configuration processing and data generation. Some of them are structural models which can be seen as the product graph of two given graphs, known as generators. Many properties of structural models can be obtained by considering the properties of their generators. In this sense, a usual objective in network design is the extension of a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough. To achieve this goal, many works in Graph Theory have studied fault-tolerant properties of some products of graphs, such as the Cartesian product, the direct product or the strong product of graphs, among others.

We focus on this last one. The *strong product* $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is defined on the Cartesian product of the vertex sets of the generators, so that two distinct vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. From the definition, it clearly follows that the strong product of two graphs is commutative. A picture of the strong product of a cycle of length 6 and a path of order 3 is shown in Fig. 1.

In this work we provide, by a constructive method, a lower bound on the average connectivity of the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 with at least three vertices and girth at least 5. As a consequence, we prove that the strong product of two maximally connected graphs of girth at least 5 is maximally connected, and also, that $\overline{\kappa}(G_1 \boxtimes G_2) = \overline{d}(G_1 \boxtimes G_2)$ if $\overline{\kappa}(G_i) = \overline{d}(G_i)$, $i = 1, 2$.

2. Main results

To estimate the average connectivity of the strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2, we must find a lower bound on the number of internally disjoint paths that join any two arbitrary vertices in $V(G_1 \boxtimes G_2)$. The following two lemmas provide these estimations.

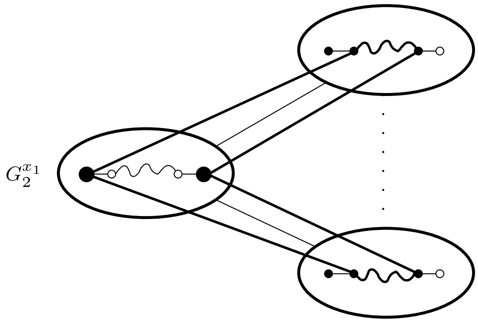

Fig. 2. Construction of paths $R_{u,j}$ in Lemma 2.1.

Given two vertices $x_1, y_1 \in V(G_1)$, we denote by $k = \kappa_{G_1}(x_1, y_1)$ and let $P_1, \ldots, P_k$ be k internally disjoint x_1y_1-paths in G_1. Similarly, for vertices $x_2, y_2 \in V(G_2)$, we denote by $\ell = \kappa_{G_2}(x_2, y_2)$ and let $Q_1, \ldots, Q_\ell$ be ℓ internally disjoint x_2y_2-paths in G_2. Without loss of generality we assume that $l(P_1) = \min\{l(P_1), \ldots, l(P_k)\}$ and that $l(Q_1) = \min\{l(Q_1), \ldots, l(Q_\ell)\}$. Observe that for every $u \in V(G_1)$, the subgraph of $G_1 \boxtimes G_2$ induced by the set $\{(u, x_2) : x_2 \in V(G_2)\}$ is isomorphic to G_2, and so, this subgraph will be denoted by G_2^u. Thus, for each x_2y_2-path Q_j in G_2, there exists an $(u, x_2)(u, y_2)$-path in G_2^u, which will be denoted by Q_j^u.

In the first result we estimate the connectivity between two vertices $(x_1, x_2), (y_1, y_2)$ in $V(G_1 \boxtimes G_2)$ such that either $x_1 = y_1$ or $x_2 = y_2$. In the former case, it means that both vertices belong to a subgraph isomorphic to G_2, namely the copy $G_2^{x_1}$ corresponding to the vertex $x_1 \in V(G_1)$.

Lemma 2.1. *Let G_1 and G_2 be two connected graphs with at least three vertices and girth at least* 5. *Let $x_i, y_i \in V(G_i)$ be two distinct vertices, $i = 1, 2$. Then the following assertions hold:*

(i) *There exist $d_{G_1}(x_1)\kappa_{G_2}(x_2, y_2) + d_{G_1}(x_1) + \kappa_{G_2}(x_2, y_2)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$.*
(ii) *There exist $\kappa_{G_1}(x_1, y_1)d_{G_2}(x_2) + \kappa_{G_1}(x_1, y_1) + d_{G_2}(x_2)$ internally disjoint $(x_1, x_2)(y_1, x_2)$-paths in $G_1 \boxtimes G_2$.*

Proof. By the commutativity of the strong product of two graphs, it suffices to prove (i). Given vertices $x_1 \in V(G_1)$ and x_2, y_2 in $V(G_2)$, let us denote by $\ell = \kappa_{G_2}(x_2, y_2)$. For any (x_2, y_2)-path Q_j in G_2, we will denote by Q_j' the corresponding path obtained from Q_j by removing its end-vertices.

Now, we introduce some general constructions of $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$. Let $u \in N_{G_1}(x_1)$ and $j \in \{1, \ldots, \ell\}$. If $l(Q_j) \geq 2$, then vertices (x_1, x_2) and (x_1, y_2) are adjacent to the first and to the last internal vertex of Q_j^u, respectively. Hence, it makes sense to consider the path $R_{u,j} : (x_1, x_2)(Q_j^u)'(x_1, y_2)$ in $G_1 \boxtimes G_2$ constructed as above (see Fig. 2). Also, when there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$, we can consider the $(x_1, x_2)(x_1, y_2)$-path $R_{w_u} : (x_1, x_2)(u, x_2)(Q_1^{w_u})'(u, y_2)(x_1, y_2)$.

Observe that vertices (x_1, x_2) and (x_1, y_2) belong to the same copy $G_2^{x_1}$ of $G_1 \boxtimes G_2$, therefore, $Q_1^{x_1}, \ldots, Q_\ell^{x_1}$ are ℓ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$. To construct the $(\ell+1)d_{G_1}(x_1)$ remaining paths we distinguish whether x_2y_2 belongs to $E(G_2)$ or not.

First, assume that $x_2y_2 \in E(G_2)$, that is, $l(Q_1) = 1$. Let $u \in N_{G_1}(x_1)$. The paths $\widetilde{R}_u : (x_1, x_2)(u, x_2)(x_1, y_2)$ and $\widehat{R}_u : (x_1, x_2)(u, y_2)(x_1, y_2)$ are contained in $G_1 \boxtimes G_2$. Moreover, since G_2 is a simple graph, for every $j \in \{2, \ldots, \ell\}$, the path Q_j have length at least 2 and there exists the path $R_{u,j}$. Hence, $Q_1^{x_1}, \ldots, Q_\ell^{x_1}, \widetilde{R}_u, \widehat{R}_u, R_{u,2}, \ldots, R_{u,\ell}$, for every $u \in N_{G_1}(x_1)$ are at least $\ell + 2\delta(G_1) + \delta(G_1)(\ell - 1) = (\delta(G_1) + 1)\ell + \delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$.

Second, assume that $x_2y_2 \notin E(G_2)$. For $j \in \{1, \ldots, \ell\}$ and $u \in N_{G_1}(x_1)$, we consider the path $R_{u,j}$. Thus, we have $(d_{G_1}(x_1) + 1)\ell$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths. If there exists a vertex $u \in N_{G_1}(x_1)$ such that $d_{G_1}(u) = 1$, notice that $d_{G_1}(x_1) \geq 2$ and then $(d_{G_1}(x_1) + 1)\ell \geq 3\ell \geq 2\ell + 1 = (\delta(G_1) + 1)\ell + \delta(G_1)$. Otherwise, there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$ for every $u \in N_{G_1}(x_1)$. Since $g(G_1) \geq 5$, then $w_u \neq w_v$ for all $u, v \in N_{G_1}(x_1)$ with $u \neq v$. Hence, the paths R_{w_u}, $u \in N_{G_1}(x_1)$, are at least $\delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$. □

Now we study in the following lemma the number of internally disjoint paths between two vertices in $G_1 \boxtimes G_2$ which come from two different vertices in G_1 and from another two different ones in G_2.

Lemma 2.2. *Let G_1 and G_2 be two connected graphs with at least three vertices and girth at least* 5. *Then for every two distinct vertices $x_1, y_1 \in V(G_1)$ and every two distinct vertices $x_2, y_2 \in V(G_2)$, there exist $\kappa_{G_1}(x_1, y_1)\kappa_{G_2}(x_2, y_2) + \kappa_{G_1}(x_1, y_1) + \kappa_{G_2}(x_2, y_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$.*

Proof. Let us denote by $k = \kappa_{G_1}(x_1, y_1)$ and $\ell = \kappa_{G_2}(x_2, y_2)$. Let $P_1, \ldots, P_k$ be k internally disjoint x_1y_1-paths in G_1, and $Q_1, \ldots, Q_\ell$ be ℓ internally disjoint x_2y_2-paths in G_2. Let us denote by $P_i : u_0^i u_1^i \ldots u_{r_i}^i$ and $Q_j : v_0^j v_1^j \ldots v_{s_j}^j$, so that $(x_1, x_2) = (u_0^i, v_0^j)$ and $(y_1, y_2) = (u_{r_i}^i, v_{s_j}^j)$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$. The proof is constructive, that is, we provide next $k\ell + k + \ell$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$.

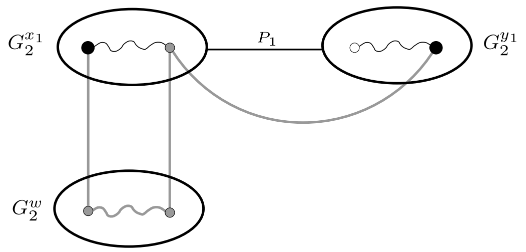

Fig. 3. Construction of path R^* if $k = 1$.

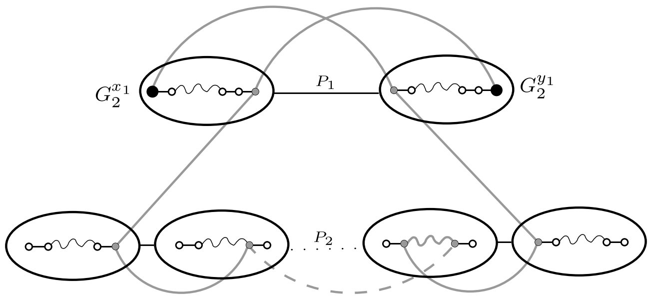

Fig. 4. Construction of path R^* if $k \geq 2$.

(I) Associated to the x_1y_1-path P_1 in G_1 and the x_2y_2-path Q_1 in G_2, we construct 3 internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$, denoted by $R'_{1,1}$, $\widetilde{R}_{1,1}$ and R^*, depending on the lengths of P_1 and Q_1.

(a) If $l(P_1) = 1$ and $l(Q_1) = 1$, that is, if $P_1 : x_1y_1$ and $Q_1 : x_2y_2$, then

$R'_{1,1} : (x_1, x_2)(x_1, y_2)(y_1, y_2)$.

$\widetilde{R}_{1,1} : (x_1, x_2)(y_1, x_2)(y_1, y_2)$.

$R^* : (x_1, x_2)(y_1, y_2)$.

(b) If $l(P_1) = 1$ and $l(Q_1) \geq 2$ (the case $l(P_1) \geq 2$ and $l(Q_1) = 1$ is analogous by the commutativity of the strong product of graphs), then

$R'_{1,1} : (u_0^1, v_0^1) \dots (u_0^1, v_{s_1-1}^1)(u_1^1, v_{s_1}^1)$.

$\widetilde{R}_{1,1} : (u_0^1, v_0^1)(u_1^1, v_1^1) \dots (u_1^1, v_{s_1}^1)$.

Observe that it is impossible to construct in $G_1 \boxtimes G_2$ one more path induced only by P_1 and Q_1. We solve this problem in two different ways depending on the value k.

If $k = 1$, since $x_1y_1 \in E(G_1)$ and G_1 has at least three vertices, there exists a vertex $w \in V(G_1)$ such that either $wx_1 \in E(G_1)$ or $wy_1 \in E(G_1)$. Without loss of generality, we consider that $wx_1 \in E(G_1)$ and hence the end-vertices of the path Q_1^w are adjacent in $G_1 \boxtimes G_2$ to (x_1, x_2) and (x_1, y_2), respectively. Thus, we obtain the $(x_1, x_2)(y_1, y_2)$-path (see Fig. 3)

$R^* : (x_1, x_2)(Q_1^w)(x_1, y_2)(y_1, y_2)$.

If $k \geq 2$, since $g(G_1) \geq 5$ and $l(P_1) = 1$, the path P_2 exists and $l(P_2) \geq 4$. Also, by the hypothesis, $l(Q_1) \geq 2$. Notice that $u_0^1 = u_0^2 = x_1$, $u_1^1 = u_{r_2}^2 = y_1$, $v_0^1 = v_0^2 = x_2$ and $v_{s_1}^1 = v_{s_2}^2 = y_2$. Hence, (see Fig. 4)

$R^* : (u_0^1, v_0^1)(u_1^1, v_0^1)(u_{r_2-1}^2, v_0^1)(u_{r_2-2}^2, v_1^1) \dots (u_{r_2-2}^2, v_{s_1-1}^1) \dots (u_2^2, v_{s_1-1}^1)(u_1^2, v_{s_1}^1)(u_0^2, v_{s_1}^1)(u_1^1, v_{s_1}^1)$

(c) If $l(P_1) \geq 2$ and $l(Q_1) \geq 2$, then

$R'_{1,1} : (u_0^1, v_0^1) \dots (u_0^1, v_{s_1}^1) \dots (u_{r_1}^1, v_{s_1}^1)$.

$\widetilde{R}_{1,1} : (u_0^1, v_0^1) \dots (u_{r_1}^1, v_0^1) \dots (u_{r_1}^1, v_{s_1}^1)$.

$R^* : (u_0^1, v_0^1)(u_1^1, v_1^1) \dots (u_1^1, v_{s_1-1}^1) \dots (u_{r_1-1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1)$.

Notice that these three paths prove constructively the desired result when $k = 1$ and $\ell = 1$.

(II) If $\ell \geq 2$ then associated to the x_1y_1-path P_1 in G_1 and the x_2y_2-paths $Q_2, \dots, Q_\ell$ in G_2, we construct the following $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$:

$$\left.\begin{array}{l} R'_{1,j} : (u_0^1, v_0^j) \dots (u_0^1, v_{s_j-1}^j) \dots (u_{r_1-1}^1, v_{s_j-1}^j)(u_{r_1}^1, v_{s_j}^j) \\ \widetilde{R}_{1,j} : (u_0^1, v_0^j)(u_1^1, v_1^j) \dots (u_{r_1}^1, v_1^j) \dots (u_{r_1}^1, v_{s_j}^j) \end{array}\right\}, \quad \text{for } j \in \{2, \dots, \ell\}.$$

As $g(G_2) \geq 5$, we have $l(Q_j) \geq 3$ for every $j \in \{2, \dots, \ell\}$. This fact has made possible the construction of the previous $2(\ell - 1)$ pairwise internally disjoint paths.

If $k = 1$ then (I) and (II) provide $3 + 2(\ell - 1) = 2\ell + 1$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ and the proof is finished.

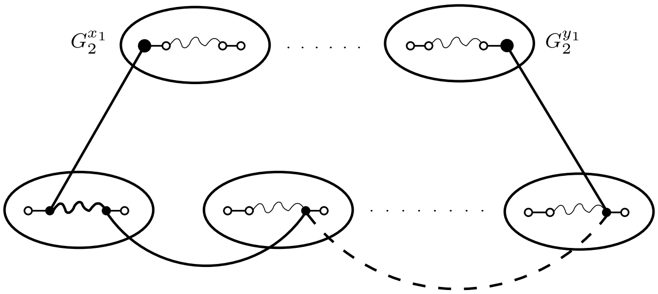

Fig. 5. Construction of path $R_{i,j}$.

(III) If $k \geq 2$ then associated to the x_1y_1-paths $P_2, \dots, P_k$ in G_1 and the x_2y_2-path Q_1 in G_2, we find the following $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$:

$$R_{i,1} : \begin{cases} (u_0^i, v_0^1)(u_1^i, v_1^1)\dots(u_{r_i}^i, v_1^1), & \text{if } l(Q_1) = 1 \\ (u_0^i, v_0^1)(u_1^i, v_1^1)\dots(u_1^i, v_{s_1-1}^1)(u_2^i, v_{s_1}^1)\dots(u_{r_i}^i, v_{s_1}^1), & \text{if } l(Q_1) \geq 2 \end{cases}$$

$$\widehat{R}_{i,1} : \begin{cases} (u_0^i, v_0^1)\dots(u_{r_i-1}^i, v_0^1)(u_{r_i}^i, v_1^1), & \text{if } l(Q_1) = 1 \\ (u_0^i, v_0^1)\dots(u_{r_i-2}^i, v_0^1)(u_{r_i-1}^i, v_1^1)\dots(u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } l(Q_1) \geq 2 \end{cases}$$

for $i \in \{2, \dots, k\}$. As $g(G_1) \geq 5$, we have $l(P_i) \geq 3$ for every $i \in \{2, \dots, k\}$, yielding that the previous $2(k-1)$ paths are pairwise internally disjoint.

If $\ell = 1$, then (I) and (III) provide $3 + 2(k-1)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$, which finishes the proof.

(IV) If $k \geq 2$ and $\ell \geq 2$, then associated to the x_1y_1-paths $P_2, \dots, P_k$ in G_1 and the x_2y_2-paths $Q_2, \dots, Q_\ell$ in G_2, we obtain the remaining $(k-1)(\ell-1)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths $R_{i,j}$, for $i \in \{2, \dots, k\}, j \in \{2, \dots, \ell\}$, given as (see Fig. 5)

$R_{i,j} : (u_0^i, v_0^j)(u_1^i, v_1^j)\dots(u_1^i, v_{s_j-1}^j)\dots(u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j)$.

Hence, (I)–(IV) provide $3 + 2(\ell-1) + 2(k-1) + (k-1)(\ell-1) = k\ell + k + \ell$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$. □

The previous lemmas together with the fact that the minimum degree of $G_1 \boxtimes G_2$ is $\delta(G_1 \boxtimes G_2) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2)$, give a sufficient condition to guarantee maximal connectivity of $G_1 \boxtimes G_2$.

Theorem 2.1. *Let G_1 and G_2 be two connected graphs with at least three vertices and girth at least 5. If both G_1 and G_2 are maximally connected, then $G_1 \boxtimes G_2$ is maximally connected.*

Proof. Denote by $G = G_1 \boxtimes G_2$ and let $(x_1, x_2), (y_1, y_2)$ be two vertices of $V(G)$. If $x_1 = y_1$ then by Lemma 2.1 we have

$$\begin{aligned} \kappa_G((x_1, x_2), (y_1, y_2)) &\geq d_{G_1}(x_1)\kappa_{G_2}(x_2, y_2) + d_{G_1}(x_1) + \kappa_{G_2}(x_2, y_2) \\ &\geq \delta(G_1)\kappa(G_2) + \delta(G_1) + \kappa(G_2) \\ &= \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2), \end{aligned}$$

the last equality due to the maximal connectivity of G_2. The reasoning is analogous if $x_2 = y_2$. Finally, if $x_1 \neq y_1$ and $x_2 \neq y_2$ then, from Lemma 2.2 and the fact that both G_1 and G_2 are maximally connected, it follows that

$$\begin{aligned} \kappa_G((x_1, x_2), (y_1, y_2)) &\geq \kappa_{G_1}(x_1, y_1)\kappa_{G_2}(x_2, y_2) + \kappa_{G_1}(x_1, y_1) + \kappa_{G_2}(x_2, y_2) \\ &\geq \kappa(G_1)\kappa(G_2) + \kappa(G_1) + \kappa(G_2) \\ &= \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2). \end{aligned}$$

Hence, $\delta(G) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) \leq \kappa_G((x_1, x_2), (y_1, y_2))$.

Therefore, $\delta(G) \leq \min\{\kappa_G((x_1, x_2), (y_1, y_2)) : (x_1, x_2), (y_1, y_2) \in V(G_1 \boxtimes G_2)\} = \kappa(G) \leq \delta(G)$, it follows that $\kappa(G) = \delta(G)$, that is, $G = G_1 \boxtimes G_2$ is maximally connected. □

Let G_1 and G_2 be two connected graphs of order n_1 and n_2, size e_1 and e_2, average connectivity $\overline{\kappa}(G_1)$ and $\overline{\kappa}(G_2)$, and average degree $\overline{d}(G_1)$ and $\overline{d}(G_2)$, respectively. From the previous lemmas, we obtain a lower bound on the average connectivity of $G_1 \boxtimes G_2$ in terms of the aforementioned parameters of G_1 and G_2. To do that, let us denote by $\mathcal{P}$ the set of non-ordered pairs of vertices of $V(G_1 \boxtimes G_2)$. Then the following sets

$$A = \bigcup_{x_2, y_2 \in V(G_2)} \{\{(u, x_2), (u, y_2)\} : u \in V(G_1)\}$$

$$B = \bigcup_{x_1, y_1 \in V(G_1)} \{\{(x_1, v), (y_1, v)\} : v \in V(G_2)\}$$

$$C = \bigcup_{(x_1,x_2),(y_1,y_2) \in V(G_1 \boxtimes G_2)} \{\{(x_1, x_2), (y_1, y_2)\} : x_1 \neq y_1 \text{ and } x_2 \neq y_2\}$$

form a partition of $\mathcal{P}$. Indeed, $|V(G_1 \boxtimes G_2)| = n_1 n_2$, $|A| = n_1 \binom{n_2}{2}$, $|B| = n_2 \binom{n_1}{2}$ and $|C| = 2\binom{n_1}{2}\binom{n_2}{2}$.

Theorem 2.2. *Let G_1 and G_2 be two connected graphs with orders $n_1, n_2 \geq 3$, respectively, and girth at least 5. Then*

$$\begin{aligned} \overline{\kappa}(G_1 \boxtimes G_2) \geq\ & \frac{1}{n_1 n_2 - 1}\Big[(n_1 - 1)(n_2 + \overline{d}(G_2))\overline{\kappa}(G_1) + (n_2 - 1)(n_1 + \overline{d}(G_1))\overline{\kappa}(G_2) \\ & + (n_1 - 1)(n_2 - 1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2 - 1)\overline{d}(G_1) + (n_1 - 1)\overline{d}(G_2)\Big]. \end{aligned}$$

Proof. Let $G = G_1 \boxtimes G_2$. Since the elements of $A \cup B$ satisfy the hypothesis of Lemma 2.1, it follows that

$$\begin{aligned} \sum_A \kappa_G((u, x_2), (u, y_2)) &\geq \sum_A \left[(1 + d(u))\kappa_{G_2}(x_2, y_2) + d(u)\right] \\ &= \sum_{x_2, y_2 \in V(G_2)} \kappa_{G_2}(x_2, y_2) \sum_{u \in V(G_1)} (1 + d(u)) + \binom{n_2}{2} \sum_{u \in V(G_1)} d(u) \\ &= \sum_{x_2, y_2 \in V(G_2)} \kappa_{G_2}(x_2, y_2)(n_1 + 2e_1) + 2e_1 \binom{n_2}{2} \\ &= (n_1 + 2e_1)K(G_2) + 2e_1 \binom{n_2}{2}. \end{aligned}$$

By applying Lemma 2.1 and the commutativity of the strong product of graphs, we also deduce that

$$\sum_B \kappa_G((x_1, v), (y_1, v)) \geq (n_2 + 2e_2)K(G_1) + 2e_2 \binom{n_1}{2}.$$

Since the elements of C satisfy the hypothesis of Lemma 2.2, we have

$$\begin{aligned} \sum_C \kappa_G((x_1, x_2), (y_1, y_2)) &\geq \sum_C \left[\kappa_{G_1}(x_1, y_1)\kappa_{G_2}(x_2, y_2) + \kappa_{G_1}(x_1, y_1) + \kappa_{G_2}(x_2, y_2)\right] \\ &= \sum_C \left[\left(\kappa_{G_1}(x_1, y_1) + 1\right)\left(\kappa_{G_2}(x_2, y_2) + 1\right) - 1\right] \\ &= 2 \sum_{x_1, y_1 \in V(G_1)} (\kappa_{G_1}(x_1, y_1) + 1) \sum_{x_2, y_2 \in V(G_2)} (\kappa_{G_2}(x_2, y_2) + 1) - |C| \\ &= 2K(G_1)K(G_2) + 2\binom{n_2}{2}K(G_1) + 2\binom{n_1}{2}K(G_2). \end{aligned}$$

Thus, from the partition of $\mathcal{P}$ into the sets A, B, C, we deduce that

$$\begin{aligned} K(G_1 \boxtimes G_2) &= \sum_{\{(x_1,x_2),(y_1,y_2)\} \in \mathcal{P}} \kappa_G((x_1, x_2), (y_1, y_2)) \\ &\geq (n_1 + 2e_1)K(G_2) + 2e_1\binom{n_2}{2} + (n_2 + 2e_2)K(G_1) + 2e_2\binom{n_1}{2} \\ &\quad + 2K(G_1)K(G_2) + 2\binom{n_2}{2}K(G_1) + 2\binom{n_1}{2}K(G_2) \\ &= (n_2^2 + 2e_2)K(G_1) + (n_1^2 + 2e_1)K(G_2) + 2K(G_1)K(G_2) + 2e_1\binom{n_2}{2} + 2e_2\binom{n_1}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} \overline{\kappa}(G_1 \boxtimes G_2) &= \frac{2}{n_1 n_2(n_1 n_2 - 1)} K(G_1 \boxtimes G_2) \\ &\geq \frac{1}{n_1 n_2 - 1}\Big[(n_1 - 1)(n_2 + \overline{d}(G_2))\overline{\kappa}(G_1) + (n_2 - 1)(n_1 + \overline{d}(G_1))\overline{\kappa}(G_2) \\ &\quad + (n_1 - 1)(n_2 - 1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2 - 1)\overline{d}(G_1) + (n_1 - 1)\overline{d}(G_2)\Big]. \quad \square \end{aligned}$$

Theorem 2.2 is best possible in the sense that the hypothesis of girth at least 5 cannot be relaxed. Indeed, let G_1 be the graph formed by two cycles of length 5 which share a common vertex z, and let G_2 be a cycle of length 4. Clearly G_1 is 1-connected, since z is a cut vertex of G_1, and G_2 is 2-connected. Let us consider two distinct vertices $x_1, y_1 \in V(G_1) \setminus \{z\}$ such that any x_1y_1-path in G_1 pass through z. For any two vertices $x_2, y_2 \in V(G_2)$, it is impossible to find five internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$, because each of these paths must contain a vertex of the subgraph G_2^z. But this graph has only four vertices because it is isomorphic to G_2, that is, to the cycle of length 4.

In [3] the following result is proved.

Theorem 2.3 (*See [3]*)**.** *Let G be a graph on n vertices and e edges with $e \geq n$, and let $r = 2e - n\lfloor 2e/n \rfloor$. Then*

$$\overline{\kappa}(G) \leq \overline{d}(G) - \frac{r(n-r)}{n(n-1)}.$$

From Theorem 2.3 it directly follows that

$$\overline{d}(G) \geq \overline{\kappa}(G), \tag{1}$$

for any connected graph G of minimum degree at least 2. Hence, applying (1) to the inequality of Theorem 2.2, we have

$$\overline{\kappa}\,(G_1 \boxtimes G_2) \geq \overline{\kappa}(G_1)\overline{\kappa}(G_2) + \overline{\kappa}(G_1) + \overline{\kappa}(G_2).$$

Since the average degree of $G_1 \boxtimes G_2$ is $\overline{d}\,(G_1 \boxtimes G_2) = \overline{d}(G_1)\overline{d}(G_2) + \overline{d}(G_1) + \overline{d}(G_2)$, we obtain the following corollary whose proof is immediate.

Corollary 2.1. *Let G_1 and G_2 be two connected graphs with at least three vertices and girth at least 5. If $\overline{\kappa}(G_i) = \overline{d}(G_i)$, for $i = 1, 2$, then*

$$\overline{\kappa}\,(G_1 \boxtimes G_2) = \overline{d}\,(G_1 \boxtimes G_2)\,.$$

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