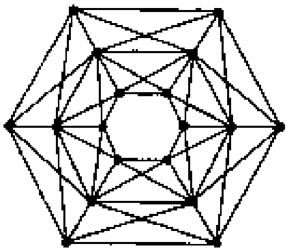

Figure 1: The strong product $C_6 \boxtimes P_3 = P_3 \boxtimes C_6$.

Introduction

ughout this paper, all the graphs are simple. Notation and terminology xplicitly given here can be found in the book by Chartrand and Lesniak

et G be a connected graph with vertex set $V = V(G)$, edge set $E =$, order $|V(G)| = n$ and size $|E(G)| = m$. The *girth* of a graph is rder of a shortest cycle. The distance between two vertices u and G is denoted by $d_G(u,v)$. The *eccentricity* in G of a vertex u is $u) = \max\{d_G(u,v)|v \in V\}$. Let $N_G^i(u) = \{v|d_G(u,v) = i\}, i \geq 1$. *diameter* of G is $D(G) = \max\{d_G(u,v)|u,v \in V\}$. The *transmission* is $\sigma(G) = \sum_{(u,v)\in V^2(G)} d_G(u,v)$ where the sum is taken over all the ed pairs (u,v) of vertices of G. The *average distance* $\mu(G)$ is defined

$$\mu(G) = \frac{\sum_{(u,v)\in V^2(G)} d_G(u,v)}{n(n-1)} = \frac{\sigma(G)}{n(n-1)}.$$

ne *strong product* $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is defined on artesian product of the vertex sets of the generators, such that two ct vertices (x_1,x_2) and (y_1,y_2) of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ $_2y_2 \in E(G_2)$, or $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1y_1 \in E(G_1)$ $_2y_2 \in E(G_2)$. From the definition, it clearly follows that the strong ct of two graphs is commutative and that it is connected if and only n graphs are connected. A picture of the strong product of a cycle of 6 and a path of order 3 is shown in Figure 1.

It is well-known (see [6]) that for any pair of vertices $x =$ $y = (y_1, y_2)$ of $G_1 \boxtimes G_2$,

$$\begin{aligned} d_{G_1\boxtimes G_2}(x,y) &= d_{G_1\boxtimes G_2}((x_1,x_2),(y_1,y_2)) \\ &= \max\{d_{G_1}(x_1,y_1), d_{G_2}(x_2,y_2)\}. \end{aligned}$$

Therefore $D(G_1 \boxtimes G_2) = \max\{D(G_1), D(G_2)\}$.

The product of graphs has been extensively investigated i a wide range of subjects, including connectivity [10], geodet [1], bandwidth [7] and independency [11], among others. A principle for network design is extendability. That is to sa bility of building larger versions of a network preserving cert properties. For designing large-scale interconnection network product is a useful method to obtain large graphs from smalle invariants can be easily calculated.

Many results exist on the average distance in different kind ite graphs (see for instance [5, 8, 9, 12]) but very few of them strong product. In this work we are interested in the averag the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 an der to avoid fractions, we work on the transmission. To get t $\mu(G_1 \boxtimes G_2)$, it is sufficient to divide $\sigma(G_1 \boxtimes G_2)$ by $n_1n_2(n_1n_2$ by $n_1(n_1-1)$ and $\sigma(G_2)$ by $n_2(n_2-1)$ where $n_1 = n(G_1)$ and

2 General expression of $\sigma(G_1 \boxtimes G_2)$ an bounds

For $k = 1, 2$, let G_k be a connected graph of vertex set V_k, o m_k, diameter D_k and transmission σ_k.

We define $R_i = \{(x_1,y_1) \in V_1^2 | d_{G_1}(x_1,y_1) = i\}$ for $i \in \{$ and $r_i = |R_i|$. Note that R_i is a set of ordered pairs (x_1,y_1) a unordered pair of vertices at distance $i > 0$ is counted twice i

Similarly, $S_j = \{(x_2,y_2) \in V_2^2 | d_{G_2}(x_2,y_2) = j\}$ for $j \in \{$ and $s_j = |S_j|$. With this notation, we have

$$r_0 = n_1,\ r_1 = 2m_1 \quad \text{and} \quad s_0 = n_2,\ s_1 = 2m_2$$

$$\sum_{i=0}^{D_1} r_i = n_1^2 \quad \text{and} \quad \sum_{j=0}^{D_2} s_j = n_2^2$$

$$\sum_{i=1}^{D_1} ir_i = \sigma_1 \quad \text{and} \quad \sum_{j=1}^{D_2} js_j = \sigma_2. \tag{4}$$

$K_1 \boxtimes G = G$ for all G, we suppose $n_k \geq 2$ for $k = 1, 2$.

st result provides a general expression of $\sigma(G_1 \boxtimes G_2)$ in terms of n_k, and s_j.

em 2.1 *For $k = 1, 2$, let G_k be a connected graph with order n_k, r D_k and transmission σ_k. Let r_i and s_j be defined as above. Then*

$$\sigma(G_1 \boxtimes G_2) = n_2\sigma_1 + n_1^2\sigma_2 + \sum_{j=1}^{D_2}\left(\sum_{i=j+1}^{D_1}(i-j)r_i\right)s_j.$$

more, if $D_1 \leq D_2$,

$$\sigma(G_1 \boxtimes G_2) = n_2\sigma_1 + n_1^2\sigma_2 + \sum_{j=1}^{D_1-1}\left(\sum_{i=j+1}^{D_1}(i-j)r_i\right)s_j. \tag{5}$$

From the definition of the transmission and by (1),

$$\begin{aligned}\sigma(G_1 \boxtimes G_2) &= \sum_{(V_1\times V_2)^2} d_{G_1\boxtimes G_2}((x_1,x_2),(y_1,y_2)) \\ &= \sum_{V_1^2\times V_2^2} \max\{d_{G_1}(x_1,y_1), d_{G_2}(x_2,y_2)\}.\end{aligned} \tag{6}$$

nsider the partition $(R_i \times S_j)_{0\leq i\leq D_1, 0\leq j\leq D_2}$ of $V_1^2 \times V_2^2$. $, j)$ fixed, if $(x_1, y_1) \in R_i$, $(x_2, y_2) \in S_j$, then

$$d_{G_1\boxtimes G_2}((x_1,x_2),(y_1,y_2)) = \max\{i,j\}.$$

$R_i \times S_j$ contains $r_i s_j$ elements.

, for (i, j) fixed,

$$\sum_{(x_1,y_1)\in R_i \wedge (x_2,y_2)\in S_j} d_{G_1\boxtimes G_2}((x_1,x_2),(y_1,y_2)) = \max\{i,j\}s_j r_i.$$

It follows that $\sigma_{ij} = js_jr_i$ for $i < j$, and $\sigma_{ij} = js_jr_i + (i-j)s_jr_i$
Hence

$$\begin{aligned}\sigma(G_1 \boxtimes G_2) &= \sum_{j=0}^{D_2}\sum_{i=0}^{D_1}\sigma_{ij} \\ &= \sum_{j=0}^{D_2}\left(\sum_{i=0}^{j} js_jr_i + \sum_{i=j+1}^{D_1}(js_jr_i + (i-j)s_jr_i)\right) \\ &= \sum_{j=0}^{D_2} js_j \sum_{i=0}^{D_1} r_i + \sum_{j=0}^{D_2}\left(s_j \sum_{i=j+1}^{D_1}(i-j)r_i\right) \\ &= s_0\sum_{i=0}^{D_1} ir_i + \sum_{j=1}^{D_2} js_j \sum_{i=0}^{D_1} r_i + \sum_{j=1}^{D_2} s_j\left(\sum_{i=j+1}^{D_1}(i-\right.\end{aligned}$$

Some terms may be null. If $D_1 \leq D_2$, then $r_i = 0$ if $i > j+1 \geq$

$$\sigma(G) = s_0\sum_{i=0}^{D_1} ir_i + \sum_{j=1}^{D_2} js_j \sum_{i=0}^{D_1} r_i + \sum_{j=1}^{D_1-1} s_j\left(\sum_{i=j+1}^{D_1}(i-j)r\right.$$

We get the theorem by (1), (2) and (3). ■

The formula (5) of Theorem 2.1 easily give the exact exp $\sigma(G_1 \boxtimes G_2)$ when the graphs G_1 and G_2 are sufficiently regul the r_i's and s_j's easy to determine. For instance, in the next determine $\sigma(G_1 \boxtimes G_2)$ for paths and cycles. For general graphs, to find lower bounds on $\sigma(G_1 \boxtimes G_2)$. The following theorem prov of them in terms of the order, the size and the transmission of t G_1 and G_2.

Theorem 2.2 *For $k = 1, 2$, let G_k be a connected graph with size m_k, diameter D_k, transmission σ_k, minimum degree δ_k and $D_1 \leq D_2$.*

(i) $\sigma(G_1 \boxtimes G_2) \geq n_2\sigma_1 + n_1^2\sigma_2$.
Equality is attained if and only if $D_1 = 1$.

(ii) If $D_1 \geq 2$, then $\sigma(G_1 \boxtimes G_2) \geq n_2\sigma_1 + n_1^2\sigma_2 + 2m_2(\sigma_1 - n_1^2 +$
Equality is attained if and only if $D_1 = 2$.

(iii) If $D_1 \geq 3$, then $\sigma(G_1 \boxtimes G_2) \geq n_2\sigma_1 + n_1^2\sigma_2 + 2m_2(\sigma_1 - n$ $s_2(\sigma_1 + 2m_1 - 2n_1^2 + 2n_1)$. Equality is attained if and only if D_1

(iv) If $D_1 \geq 3$ and G_2 has girth at least 5, then

$$\sigma(G_1\boxtimes G_2) \geq n_2\sigma_1 + n_1^2\sigma_2 + 2m_2(\sigma_1 - n_1^2 + n_1) + \delta_2(2m_2 - n_2)(\sigma_1 + 2m$$

is attained if and only if $D_1 = 3$ and G_2 is regular.

ll the integers r_i, $i \le D_1$, and s_j, $j \le D_2$, are positive. Hence we trict lower bound on $\sigma(G_1 \boxtimes G_2)$ by stopping the formula (5) at of j strictly less than $D_1 - 1$ and the equality if and only if all are taken until $j = D_1 - 1$.

sult is obtained by taking no term s_j.

pping at $j = 1$, we obtain a strict lower bound if $D_1 > 2$ and the $D_1 = 2$.

$$\begin{aligned} G_2) &\geq n_2\sigma_1 + n_1^2\sigma_2 + s_1 \sum_{i=2}^{D_1}(i-1)r_i \\ &= n_2\sigma_1 + n_1^2\sigma_2 + s_1\left(\sum_{i=2}^{D_1} ir_i - \sum_{i=2}^{D_1} r_i\right) \\ &= n_2\sigma_1 + n_1^2\sigma_2 + 2m_2\left((\sigma_1 - 2m_1) - (n_1^2 - n_1 - 2m_1)\right) \\ &\quad \text{by (2), (3), (4).} \\ G_2) &= n_2\sigma_1 + n_1^2\sigma_2 + 2m_2\left(\sigma_1 - n_1^2 + n_1\right). \end{aligned}$$

pping at $j = 2$, we obtain a strict lower bound if $D_1 > 3$ and y if $D_1 = 3$. Using again (2), (3) and (4), we get

$$\begin{aligned} \boxtimes G_2) &\geq n_2\sigma_1 + n_1^2\sigma_2 + s_1\sum_{i=2}^{D_1}(i-1)r_i + s_2\sum_{i=3}^{D_1}(i-2)r_i \\ &= n_2\sigma_1 + n_1^2\sigma_2 + s_1\left(\sum_{i=2}^{D_1} ir_i - \sum_{i=2}^{D_1} r_i\right) \\ &\quad + s_2\left(\sum_{i=3}^{D_1} ir_i - 2\sum_{i=3}^{D_1} r_i\right) \\ &= n_2\sigma_1 + n_1^2\sigma_2 + 2m_2\left(\sigma_1 - n_1^2 + n_1\right) \\ &\quad + s_2\left(\sigma_1 - 2n_1^2 + 2n_1 + 2m_1\right). \end{aligned}$$

aphs G_2, s_2 is difficult to determine. We consider in the fourth y of graphs G_2 in which a lower bound on s_2 is known.

raph G_2 has girth $g_2 \geq 5$, the neighborhood of each vertex v is and contains $d(v)(d(v)-1)$ ordered pairs of vertices, all of them nce 2. Moreover any two vertices x and y at distance 2 have neighbor v in common. Therefore $s_2 = \sum_{v\in V_2} d(v)(d(v)-1) \geq$

$\delta_2 \sum_{v\in V_2}(d(v) - 1) = \delta_2(2m_2 - n_2)$ with equality if and only if d for every vertex v, i. e., if G_2 is regular. Since the coefficient (5) is positive, we get a lower bound on $\sigma(G_1 \boxtimes G_2)$ by replaci $\delta_2(2m_2 - n_2)$ in the bound of (iii).

Note that $\delta_2(2m_2 - n_2) \geq 2m_2(\delta_2 - 1)$ with equality if and G_2 is regular. Therefore we can get weaker but simpler lower on $\sigma(G_1 \boxtimes G_2)$, still attained if and only if G_2 is regular, by $\delta_2(2m_2 - n_2)$ by $2m_2(\delta_2 - 1)$ in the expression above. This gives

$$\begin{aligned} \sigma(G_1 \boxtimes G_2) &\geq n_2\sigma_1 + n_1^2\sigma_2 + 2m_2\delta_2(\sigma_1 + 2m_1 - 2n_1(n_1-1)) \\ &\quad + 2m_2(n_1^2 - n_1 - 2m_1) \\ &\geq n_2\sigma_1 + n_1^2\sigma_2 + 2m_2\delta_2(\sigma_1 + 2m_1 - 2n_1(n_1-1)). \end{aligned}$$

■

Corollary 2.1 *For $k = 1, 2$, let G_k be a connected graph with o size m_k, diameter D_k and transmission σ_k.*
(i) (Pattabiraman and Paulraja [8, Collorary 6]) If $D_1 = 1$, $G_1 = K_{n_1}$, then

$$\sigma(K_{n_1} \boxtimes G_2) = n_1(n_1 - 1)n_2 + n_1^2\sigma_2.$$

(ii) If $D_1 = 2$ then

$$\sigma(G_1 \boxtimes G_2) = 2n_2(n_1^2 - n_1 - m_1) + n_1^2\sigma_2 + 2m_2(n_1^2 - n_1 - 2n$$

Proof. (i) This is the equality case of Theorem 2.2 (i) since $\sigma($ $n_1(n_1 - 1)$.

(ii) By (2), (3), (4), $D_1 = 2$ implies $r_1 + r_2 = n_1(n_1 - 1)$ an $2(r_1 + r_2) - r_1 = 2n_1^2 - 2n_1 - 2m_1$. Hence the result is the equality Theorem 2.2 (ii) when $D_2 \geq 2$ and comes from (i) when $D_2 = 1$.

We will see at the end of the last section that Corollary 2.1 (i ralizes a result in [8].

3 Strong products of paths and cycles

The exact expressions of the transmission of strong products of pa cycles are given by the formula (5) of Theorem 2.1. We used M rm the calculations.

path P_n of order n has diameter $n-1$, transmission $\dfrac{n^3-n}{3}$ and admits 2i ordered pairs of vertices at distance i for $1 \le i < n-1$.

ycle C_n of order n has diameter $D = \lfloor \frac{n}{2} \rfloor$ and transmission $\lfloor \frac{n^2}{4} \rfloor n$. s odd, C_n admits $2n$ ordered pairs of vertices at distance i for $1 \le i \le$ n is even, C_n admits $2n$ ordered pairs of vertices at distance i for $\le D-1$ and n ordered pairs of vertices at distance D.

llary 3.1 *Let n_1, n_2 be two integers such that $2 \le n_1 \le n_2$. Then*

$$\sigma(P_{n_1} \boxtimes P_{n_2}) = n_1^2 n_2 \left(\frac{n_1^2}{6} + \frac{n_2^2}{3} - \frac{1}{2}\right) - \frac{n_1}{3}\left(\frac{n_1^4}{10} - \frac{n_1^2}{2} + \frac{2}{5}\right).$$

f. Since $n_1 \le n_2$, $D(P_{n_1}) \le D(P_{n_2})$ and we apply Theorem 2.1 with P_{n_1} and $G_2 = P_{n_2}$. Then $r_i = 2n_1 - 2i$ for $i \in \{1, \dots, n_1 - 1\}$ and $2n_2 - 2j$ for $j \in \{1, \dots, n_2 - 1\}$.

$$\begin{aligned} {}_{n_1} \boxtimes P_{n_2}) &= n_2\sigma_1 + n_1^2\sigma_2 + \sum_{j=1}^{n_1-2}\left((2n_2-2j)\sum_{i=j+1}^{n_1-1}(i-j)(2n_1-2i)\right) \\ &= \frac{n_1^3-n_1}{3}n_2 + n_1^2\frac{n_2^3-n_2}{3} \\ &\quad + \frac{n_1 n_2}{3}\left(\frac{n_1^3}{2} - n_1^2 - \frac{n_1}{2} + 1\right) - \frac{n_1^5}{30} + \frac{n_1^3}{6} - \frac{2n_1}{15} \\ &= n_1^2 n_2\left(\frac{n_1^2}{6} + \frac{n_2^2}{3} - \frac{1}{2}\right) - \frac{n_1}{3}\left(\frac{n_1^4}{10} - \frac{n_1^2}{2} + \frac{2}{5}\right). \end{aligned}$$

llary 3.2 *Let n_1, n_2 be two integers such that $3 \le n_1 \le n_2$. Then*

$$\sigma(C_{n_1} \boxtimes C_{n_2}) = n_1^2 n_2\left(\frac{n_1^2}{12} + \lfloor\frac{n_2^2}{4}\rfloor + c\right),$$

ere $c = \frac{1}{6}$ if n_1 is even and $c = \frac{-1}{12}$ if n_1 is odd.

f. Since $n_1 \le n_2$, $D(C_{n_1}) \le D(C_{n_2})$ and we apply Theorem 2.1 with C_{n_1} and $G_2 = C_{n_2}$.

In G_1, $r_i = 2n_1$, for $i \in \{1, \dots, D_1 - 1\}$ and $r_{D_1} = \begin{cases} n_1, & \text{if} \\ 2n_1, & \text{if} \end{cases}$

Similarly, in G_2, $s_j = 2n_2$, for $j \in \{1, \dots, D_2 - 1\}$ and

$$s_{D_2} = \begin{cases} n_2, & \text{if } n_2 \text{ is even} \\ 2n_2, & \text{if } n_2 \text{ is odd.} \end{cases}$$

First, suppose that n_1 is even. Then $D_1 = \frac{n_1}{2}$ and $\sigma_1 = \frac{n_1^3}{4}$. H

$$\begin{aligned} \sigma(C_{n_1} \boxtimes C_{n_2}) &= \frac{n_1^3}{4}n_2 + n_1^2\lfloor\frac{n_2^2}{4}\rfloor n_2 \\ &\quad + \sum_{j=1}^{\frac{n_1-2}{2}} 2n_2\left((\frac{n_1}{2} - j)n_1 + \sum_{i=j+1}^{\frac{n_1-2}{2}}(i-j \right. \\ &= \frac{n_1^3}{4}n_2 + n_1^2\lfloor\frac{n_2^2}{4}\rfloor n_2 + \frac{n_1^4 n_2}{12} - \frac{n_1^3 n_2}{4} + \\ &= n_1^2 n_2\left(\frac{n_1^2}{12} + \lfloor\frac{n_2^2}{4}\rfloor + \frac{1}{6}\right). \end{aligned}$$

Second, suppose that n_1 is odd. Then $D_1 = \frac{n_1-1}{2}$ and $\sigma_1 =$ Hence,

$$\begin{aligned} \sigma(C_{n_1} \boxtimes C_{n_2}) &= \frac{n_1^3-n_1}{4}n_2 + n_1^2\lfloor\frac{n_2^2}{4}\rfloor n_2 + \sum_{j=1}^{\frac{n_1-3}{2}} 2n_2\left(\sum_{i=j+1}^{\frac{n_1-1}{2}} (\right. \\ &= \frac{n_1^3-n_1}{4}n_2 + n_1^2\lfloor\frac{n_2^2}{4}\rfloor n_2 + \frac{n_1^4 n_2}{12} - \frac{n_1^3 n_2}{4} - \\ &= n_1^2 n_2\left(\frac{n_1^2}{12} + \lfloor\frac{n_2^2}{4}\rfloor - \frac{1}{12}\right). \end{aligned}$$

■

Corollary 3.3 *Let $n \ge 2$ and $m \ge 3$ be two integers.*

$< \lfloor \frac{m}{2} \rfloor + 1$, *then* $\sigma(P_n \boxtimes C_m) = n^2 m \left(\frac{n^2}{6} + \lfloor \frac{m^2}{4} \rfloor - \frac{1}{6} \right)$.

$> \lfloor \frac{m}{2} \rfloor + 1$, *then*

$P_n) = m^2 n \left(\frac{n^2}{3} + \frac{m^2}{12} - \frac{1}{6} \right) - \frac{m^3}{24} \left(\frac{m^2}{4} - 1 \right)$, *m even.*

$P_n) = m^2 n \left(\frac{n^2}{3} + \frac{m^2}{12} - \frac{5}{12} \right) - \frac{m}{16} \left(\frac{m^4}{6} - \frac{5m^2}{3} + \frac{3}{2} \right)$, *m odd.*

ince $D(P_n) = n - 1$ and $D(C_m) = \lfloor \frac{m}{2} \rfloor$, $D(P_n) \leq D(C_m)$ if and $\leq \lfloor \frac{m}{2} \rfloor + 1$.

se that $n \leq \lfloor \frac{m}{2} \rfloor + 1$. Then $D_1 \leq D_2$ with $G_1 = P_n$ and $G_2 = C_m$. $= 2n - 2i$ for $i \in \{1, \ldots, n-1\}$ and in C_m, $s_j = 2m$ for $j \in$ $(C_m) - 1\}$.

$$\begin{aligned} C_m) &= \frac{n^3 - n}{3} m + n^2 \lfloor \frac{m^2}{4} \rfloor m + \sum_{j=1}^{n-2} 2m \left(\sum_{i=j+1}^{n-1} (i-j)(2n-2i) \right) \\ &= \frac{n^3 - n}{3} m + n^2 \lfloor \frac{m^2}{4} \rfloor m + \frac{n^4 m}{6} - \frac{n^3 m}{3} - \frac{n^2 m}{6} + \frac{nm}{3} \\ &= n^2 m \left(\frac{n^2}{6} + \lfloor \frac{m^2}{4} \rfloor - \frac{1}{6} \right). \end{aligned}$$

e that $n > \lfloor \frac{m}{2} \rfloor + 1$. Then $D_1 < D_2$ with $G_1 = C_m$ and $G_2 = P_n$. $= 2n - 2j$ for $j \in \{1, \ldots, n-1\}$ and in C_m, $r_i = 2m$ for $D(C_m) - 1\}$ and $r_{D(C_m)} = \begin{cases} m, & \text{if } m \text{ even,} \\ 2m, & \text{if } m \text{ odd.} \end{cases}$

uish two cases depending on the parity of the order of the cycle.

n,

$$\begin{aligned} \sigma(C_m \boxtimes P_n) &= n \frac{m^3}{4} + m^2 \frac{n^3 - n}{3} \\ &\quad + \sum_{j=1}^{\frac{m-2}{2}} (2n - 2j) \left((\frac{m}{2} - j)m + \sum_{i=j+1}^{\frac{m-2}{2}} (i-j)2n \right. \\ \sigma(C_m \boxtimes P_n) &= n \frac{m^3}{4} + m^2 \frac{n^3 - n}{3} + \frac{nm^4}{12} - \frac{nm^3}{4} + \frac{nm^2}{6} - \\ &= m^2 n \left(\frac{n^2}{3} + \frac{m^2}{12} - \frac{1}{6} \right) - \frac{m^3}{24} \left(\frac{m^2}{4} - 1 \right). \end{aligned}$$

If m is odd,

$$\begin{aligned} \sigma(C_m \boxtimes P_n) &= n \frac{m^3 - m}{4} + m^2 \frac{n^3 - n}{3} + \sum_{j=1}^{\frac{m-3}{2}} (2n - 2j) \left(\sum_{i=j+1}^{\frac{m}{2}} \right. \\ &= n \frac{m^3 - m}{4} + m^2 \frac{n^3 - n}{3} + \frac{nm^4}{12} - \frac{nm^3}{4} - \frac{nm^2}{12} \\ &\quad + \frac{nm}{4} - \frac{m^5}{96} + \frac{5m^3}{48} - \frac{3m}{32} \\ &= m^2 n \left(\frac{n^2}{3} + \frac{m^2}{12} - \frac{5}{12} \right) - \frac{m}{16} \left(\frac{m^4}{6} - \frac{5m^2}{3} + \right. \end{aligned}$$

■

4 Upper bound

If G is a connected graph of order n, it is known [3] that $\sigma(G)$ with equality if and only if G is the path P_n. We generalize this the strong product of two graphs.

Theorem 4.1 *For $k \in \{1, 2\}$, let G_k be connected graphs with Then*

$$\sigma(G_1 \boxtimes G_2) \leq \sigma(P_{n_1} \boxtimes G_2)$$

with equality if and only if G_1 is the path P_{n_1}.

Proof. We keep on with the notation of Section 2. Without los erality, we may suppose that $n_1 \geq 3$. Let T be a spanning tree of also suppose $G_1 = T$, as $\sigma(G_1 \boxtimes G_2) \leq \sigma(T \boxtimes G_2)$ with equality if and if $G_1 = T$.

he proof is by induction on n_1. For $n_1 = 3$, we have $T = P_3$ and $\boxtimes G_2) = \sigma(P_3 \boxtimes G_2)$. From now on, $n_1 \geq 4$. Suppose that the result s for $n_1 - 1$, we prove it for n_1.

enote by V_1 and V_2 the vertex sets of T and G_2, respectively, with $= n_1$ and $|V_2| = n_2$. By (6),

$$\sigma(T \boxtimes G_2) = \sum_{V_1^2 \times V_2^2} \max\{d_T(u_1, v_1), d_{G_2}(u_2, v_2)\}.$$

e denote by p the first projection in $V(T\boxtimes G_2)$, that is $p : V_1 \times V_2 \to V_1$. $u_1, u_2) = u_1$.

t x_1 be a pendant vertex of T. Consider A_1 and A_2 defined as follows,

$$A_1 = \sum_{(p(u)\neq x_1)\wedge(p(v)\neq x_1)} d_{T\boxtimes G_2}(u, v)$$

$$A_2 = \sum_{(p(u)=x_1)\vee(p(v)=x_1)} d_{T\boxtimes G_2}(u, v).$$

lear that $\sigma(T \boxtimes G_2) = A_1 + A_2$.

rst, we prove that $A_1 = \sigma((T - \{x_1\}) \boxtimes G_2)$. Let $u = (u_1, u_2), v =$) be two vertices of $(V_1 - \{x_1\}) \times V_2$. As x_1 is a pendant vertex of $(u_1, v_1) = d_{T-\{x_1\}}(u_1, v_1)$. Hence

$$_{G_2}(u, v) = \max\{d_{T-\{x_1\}}(u_1, v_1), d_{G_2}(u_2, v_2)\} = d_{(T-\{x_1\})\boxtimes G_2}(u, v).$$

erefore, it follows that

$$A_1 = \sum_{(p(u)\neq x_1)\wedge(p(v)\neq x_1)} d_{T\boxtimes G_2}(u, v) = \sigma((T - \{x_1\}) \boxtimes G_2).$$

induction, we conclude that $A_1 \leq \sigma(P_{n_1-1} \boxtimes G_2)$ with equality if ly if $T - \{x_1\} = P_{n_1-1}$.

cond, we compute A_2. Notice that there exists a bijection between lowing two sets

$\times V_2) \times (V_1 \times V_2)) \cup ((V_1 \times V_2) \times (\{x_1\} \times V_2))$, subset of $(V_1 \times V_2)^2$,

and $((\{x_1\} \times V_1) \cup (V_1 \times \{x_1\})) \times V_2^2$, subset of V_1^2

We consider the partition $(R_i(x_1) \times S_j)_{i,j}$ of the last subs $i \leq D_1$,

$$\begin{aligned} R_i(x_1) &= R_i \cap ((\{x_1\} \times V_1) \cup (V_1 \times \{x_1\})) \\ &= \{(u_1, v_1) \in R_i | u_1 = x_1 \text{ or } v_1 = x_1\}. \end{aligned}$$

Let $r_i(x_1) = |R_i(x_1)|$. Note that $r_0(x_1) = 1$ and tha $2|N^i_{G_1}(x_1)|$ is even for any $i \geq 1$ and is positive for $i \leq ecc_T(x$

Furthermore, $\sum_{i\geq 1} r_i(x_1) = 2(n_1 - 1)$.

For a fixed $i \in \{1, \ldots, n_1\}$, $|R_i(x_1) \times S_j| = r_i(x_1)s_j$ fo $\{0, \ldots, n_2\}$. Then, by (1),

$$\begin{aligned} \sum_{(u_1,v_1)\in R_i(x_1)} d_{T\boxtimes G_2}(u, v) &= r_i(x_1) \left(\sum_j \max\{i, j\} s_j\right) \\ &= r_i(x_1) \left(\sum_{j\leq i} i s_j + \sum_{j\geq i+1} j s \right. \end{aligned}$$

Hence, we get

$$\begin{aligned} A_2 &= \sum_{i=0}^{ecc_T(x_1)} \sum_{(u_1,v_1)\in R_i(x_1)} d_{T\boxtimes G_2}(u, v) \\ &= \sum_{i=0}^{ecc_T(x_1)} r_i(x_1) \left(\sum_{j\leq i} i s_j + \sum_{j\geq i+1} j s_j\right) \\ &= \sum_{i=0}^{ecc_T(x_1)} r_i(x_1) \left(\sum_{j<i} (i - j) s_j + \sum_{j\geq 0} j s_j\right). \end{aligned}$$

Let us remark that, since $s_j > 0$ for $0 \leq j \leq D_2$,

(R) the coefficient $f_i := \sum_{j\leq i} (i - j) s_j + \sum_{j\geq 0} j s_j$ is a stri sing function of i.

If $ecc_T(x_1) = n_1 - 1$, then $T = P_{n_1}$. If $ecc_T(x_1) \leq n_1 - 2$, $i_0 \in \{1, \ldots, n_1 - 1\}$ such that $r_{i_0}(x_1) > 2$. We note that if we

$r = (\rho_1, \ldots, \rho_{i_1}, \ldots, \rho_p, 0, \ldots, 0, 0)$ with $\rho_i \geq 2$ for each i and $\rho_{i_1} > 2$, sequence $r' = (\rho_1, \ldots, \rho_{i_1} - 2, \ldots, \rho_p, 2, 0, \ldots, 0, 0)$ of the same length, sequence of the same sum. By the previous remark (R), we have

$$\sum_{i \leq p} \rho_i f_i < \sum_{i \leq p+1} \rho_i' f_i.$$

ratively the previous operation, from the sequence

$$(1, r_1(x_1), \ldots, r_{ecc(x_1)}(x_1), 0, \ldots, 0)$$

n_1, we obtain the sequence $(1, 2, \ldots, 2, 2, \ldots, 2)$ of the same length same sum. This last sequence is $r_0(x) = 1$, $r_i(x) = 2|N^i_{P_n}(x)| = 2$ where x is a pendant vertex of the graph P_{n_1}.

, we get

$$A_2 < f_0 + 2 \sum_{i=1}^{n_1 - 1} f_i = \sum_{(p(u)=x_1) \vee (p(v)=x_1)} d_{P_{n_1} \boxtimes G_2}(u, v).$$

y, by summing A_1 and A_2 we obtain the desired result $\sigma(T \boxtimes G_2) \leq$ $G_2)$.

$\boxtimes G_2) = \sigma(P_{n_1} \boxtimes G_2)$ then both A_1 and A_2 attain their upper his imply $T - \{x_1\} = P_{n_1 - 1}$ for A_1 and $ecc_T(x_1) = n_1 - 1$ for A_2. $= P_{n_1}$. ■

llowing corrolary results from Theorem 4.1 by the commutativity ong product.

y 4.1 *For $k \in \{1, 2\}$, let G_k be connected graphs with order n_k.*

$$\sigma(G_1 \boxtimes G_2) \leq \sigma(P_{n_1} \boxtimes P_{n_2})$$

ity if and only if G_1 is the path P_{n_1} and G_2 the path P_{n_2}.

ener and hyper-Wiener indices

al graph theory (see [12]), the transmission $\sigma(G)$ is most often y the *Wiener index*

$$W(G) = \frac{1}{2} \sum_{(u,v) \in V^2(G)} d_G(u, v) = \frac{\sigma(G)}{2}.$$

Another well studied parameter is the *hyper-Wiener index*

$$\begin{aligned} WW(G) &= \frac{1}{4} \sum_{(u,v) \in V^2(G)} \left(d_G(u, v) + d_G^2(u, v)\right) \\ &= \frac{1}{4} \left(\sigma(G) + \sum_{(u,v) \in V^2(G)} d_G^2(u, v) \right). \end{aligned}$$

We can determine the general expression of $WW((G_1 \boxtimes G_2)$ by same method as in Section 1. We denote $WW(G_1)$ by WW_1, W WW_2, and we observe that

$$4WW_1 = \sum_{i=1}^{D_1} (i + i^2) r_i \quad \text{and} \quad 4WW_2 = \sum_{j=1}^{D_2} (j + j^2) s_j.$$

Theorem 5.1 *For $k = 1, 2$, let G_k be a connected graph with diameter D_k and hyper-Wiener index WW_k. Then*

$$WW(G_1 \boxtimes G_2) = n_2 WW_1 + n_1^2 WW_2 + \frac{1}{4} \sum_{j=1}^{D_1 - 1} \left(\sum_{i=j+1}^{D_1} (i - j)(i + j + \right.$$

Proof. From the definition of the hyper-Wiener index,

$$\begin{aligned} 4WW(G_1 \boxtimes G_2) &= \sum_{(V_1 \times V_2)^2} d_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) \\ &\quad + \sum_{(V_1 \times V_2)^2} d^2_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)). \end{aligned}$$

To calculate

$$\sum_{(V_1 \times V_2)^2} d^2_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = \sum_{V_1^2 \times V_2^2} \max\{d^2_{G_1}(x_1, y_1), d^2_{G_2}$$

by (1), it is sufficient to replace everywhere i by i^2 and j by j^2 in t lation of $\sum_{V_1^2 \times V_2^2} \max\{d_{G_1}(x_1, y_1), d_{G_2}(x_2, y_2)\}$ done in the proof of 2.1. This gives by (7)

$$\begin{aligned} 4WW(G_1 \boxtimes G_2) &= s_0 \sum_{i=0}^{D_1} (i + i^2) r_i + \sum_{j=1}^{D_2} (j + j^2) s_j \times \sum_{i=0}^{D_1} \\ &\quad + \sum_{j=1}^{D_1 - 1} s_j \left(\sum_{i=j+1}^{D_1} ((i + i^2) - (j + j^2)) r_i \right) \end{aligned}$$

ult follows from (2), (3), (8). ■

n Theorem 5.1, it is not difficult to obtain bounds on $WW(G_1 \boxtimes G_2)$ as those on $\sigma(G_1 \boxtimes G_2)$ in Theorem 2.2. We just present here the us of Corollary 2.1 for the comparison with [8].

ry 5.1 *For $k = 1, 2$, let G_k be a connected graph with order n_k, diameter D_k and hyper-Wiener index WW_k.*
tabiraman and Paulraja [8, Collorary 9]) If $D_1 = 1$, i. e., if $G_1 =$ en

$$WW(K_{n_1} \boxtimes G_2) = \frac{n_1(n_1-1)}{2} n_2 + n_1^2 WW_2.$$

$_1 = 2$ then

$$\boxtimes G_2) = \frac{n_2}{2}(3n_1^2 - 3n_1 - 4m_1) + n_1^2 WW_2 + 2m_2(n_1^2 - n_1 - 2m_1).$$

i) If $D_1 = 1$, (9) only contains the first two terms. Moreover, by 2), $WW_1 = \frac{r_1}{2} = m_1 = \frac{n_1(n_1-1)}{2}$.

$= 2$ then, by (2), (3), (8), $r_2 = n_1(n_1-1) - r_1 = n_1(n_1-1) - 2m_1$, $r_1 + 3r_2 = 3(r_1 + r_2) - 2r_1 = 3(n_1^2 - n_1) - 4m_1$ and $s_1 r_2 =$ $_1 - 1) - 2m_1)$. The result comes from (9) when $D_2 \geq 2$ and from $D_2 = 1$. ■

Pattabiraman and Paulraja determined the values of $W(G_1 \boxtimes G_2)$ $G_1 \boxtimes G_2)$ when G_1 is a complete multipartite graph and G_2 is ected graph. Replacing our notation by theirs in Corollaries 2.1 1 (ii) gives their results. Hence Corollaries 2.1 and 5.1 generalize 1 and 2 of [8] to all graphs G_1 of diameter 2.

ly, we can obtain an upper bound on $WW(G_1 \boxtimes G_2)$ by adapting of Theorem 4.1. This generalizes the known result $WW(G_n) \leq$ or every connected graph G_n of order n [4].

5.2 *For $k \in \{1, 2\}$, let G_k be a connected graph with order n_k.*

$_1 \boxtimes G_2) \leq WW(P_{n_1} \boxtimes G_2)$ with equality if and only if $G_1 = P_{n_1}$.
$G_1 \boxtimes G_2) \leq WW(P_{n_1} \boxtimes P_{n_2})$ with equality if and only if $G_1 =$ $_2 = P_{n_2}$.

Proof. It is sufficient to replace in the proof of Theorem 4.1 A_1 A_2 by A_2' defined as follows.

$$\begin{aligned} A_1' &= A_1 + \sum_{(p(u) \neq x_1) \wedge (p(v) \neq x_1)} d^2_{T \boxtimes G_2}(u, v) \\ &= WW((T - \{x_1\}) \boxtimes G_2) \leq WW((P_{n_1 - 1} - \{x_1\}) \boxtimes G \end{aligned}$$

A_2' is obtained by replacing i by $i + i^2$ and j by $j + j^2$ in the dete of A_2. In the case $ecc_T(x_1) \leq n_1 - 2$, the coefficient f_i is re $f_i' = f_i + \sum_{j \geq 0} j^2 s_j + \sum_{j \leq i} (i^2 - j^2) s_j$ which is still a strictly increasin of i. Hence

$$A_2' < f_0' + 2 \sum_{i=1}^{n_1 - 1} f_i' - \sum_{(p(u) = x_1) \vee (p(v) = x_1)} (d_{P_{n_1} \boxtimes G_2}(u, v) + d^2_{P_{n_1} \boxtimes G}$$

■

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A Graph Theoretic Division Algori[t...]

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Dedicated to Professor Teresa Haynes

Abstract

For two graphs H and G, a decomposition $\mathcal{D} = \{H_1, H_2$ [...] $H_k, R\}$ of G is called an H-maximal k-decomposition if H_i [...] for $1 \le i \le k$ and R contains no subgraph isomorphic to H [...] $\mathrm{Min}(G, H)$ and $\mathrm{Max}(G, H)$ be the minimum and maximu[m ...] respectively, for which G has an H-maximal k-decompos[ition ...] A graph H without isolated vertices is said to possess the i[nter]mediate decomposition property if for each connected gra[ph ...] and each integer k with $\mathrm{Min}(G, H) \le k \le \mathrm{Max}(G, H)$, ther[e ex]ists an H-maximal k-decomposition of G. For a set S of gr[aphs] and a graph G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ [of] G is called an S-maximal k-decomposition if $H_i \cong H$ for s[ome] $H \in S$ for each integer i with $1 \le i \le k$ and R contain[s no] subgraph isomorphic to any subgraph in S. Let $\mathrm{Min}(G, S)$ [and] $\mathrm{Max}(G, S)$ be the minimum and maximum k , respectively[, for] which G has an S-maximal k-decomposition. A set S of gr[aphs] without isolated vertices is said to possess the intermediat[e de]composition property if for every connected graph G and [...] integer k with $\mathrm{Min}(G, S) \le k \le \mathrm{Max}(G, S)$, there exists a[n S-]maximal k-decomposition of G. All graphs of size 3 or les[s are] determined that possess the intermediate decomposition p[rop]erty. Sets of graphs having size 3 that possess the intermed[iate] decomposition property are investigated. In particular, all s[uch] sets consisting of two graphs are determined.

Key Words: maximal decompositions, remainder subgraph, int[...] decomposition property.

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