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NEW EXAMPLES OF GENERALIZED SASAKIAN-SPACE-FORMS

Abstract. In this paper we study when a non-anti-invariant slant submanifold of a generalized Sasakian-space-form inherits such a structure, on the assumption that it is totally geodesic, totally umbilical, totally contact geodesic or totally contact umbilical. We obtain some general results (including some obstructions) and we also offer some explicit examples.

1. Introduction.

The study of generalized Sasakian-space-forms has been quickly developed since the first two named authors (jointly with David E. Blair) defined such a manifold in [1] as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor is given by

$$R \;=\; f_1R_1+f_2R_2+f_3R_3, \tag{1}$$

where f_1,f_2,f_3 are differential functions on M and

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X-g(X,Z)Y,\\
R_2(X,Y)Z &= g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z,\\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi,
\end{aligned}$$

for any vector fields X,Y,Z. We denote it by $M(f_1,f_2,f_3)$. Actually, we can refer to the recent papers [2], [3], [4], [5], [6], [7], [18], [20] and [25].

But, as in any new subject, one of the most important things is the search for new examples. In this sense, a natural question arises now: if M is a submanifold isometrically immersed in a generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, when does it inherit a generalized Sasakian-space-form structure, with functions f_1,f_2,f_3?

This is a non-trivial question, because two things have to be inherited from the ambient space. Firstly, the almost contact metric structure. Hence, we must work with some special classes of submanifolds, tangent to the structure vector field ξ. A natural election seems to be that of non-anti-invariant slant submanifolds (for some general background on the theory of slant submanifolds in almost contact metric manifolds, we recommend the survey paper [12]). But, secondly, in order to be a generalized Sasakian-space-form, the curvature tensor R of the submanifold M has to be written in a very special way. By using Gauss equation we have:

$$\begin{aligned}
&F_1\widetilde{R}_1(X,Y,Z,W)+F_2\widetilde{R}_2(X,Y,Z,W)+F_3\widetilde{R}_3(X,Y,Z,W)\\
&=R(X,Y,Z,W)+g(\sigma(X,Z),\sigma(Y,W))-g(\sigma(X,W),\sigma(Y,Z)),
\end{aligned} \tag{2}$$

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for any vector fields X,Y,Z,W tangent to M. Therefore, we must somehow control the second fundamental form σ of the immersion in order to obtain the correct writing for R.

Thus, after a preliminaries section containing some definitions and formulae for later use, in Section 3 we study the raised question for totally geodesic, totally umbilical, totally contact geodesic and totally contact umbilical non-anti-invariant slant submanifolds of a generalized Sasakian-space-form. We obtain some general results (Theorems 1, 2 and 3) as well as some obstructions (Theorems 4 and 5), and we also construct some explicit examples.

2. Preliminaries.

In this section, we recall some general definitions and basic formulas which we will use later. For more background on almost contact metric manifolds, we recommend the reference [8]. Anyway, we will recall some more specific notions and results in the following sections, when needed.

An odd-dimensional Riemannian manifold $(\widetilde{M},g)$ is said to be an *almost contact metric manifold* if there exist on $\widetilde{M}$ a $(1,1)$ tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi)=1$, $\phi^2(X)=-X+\eta(X)\xi$ and $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$, for any vector fields X,Y on $\widetilde{M}$. In particular, in an almost contact metric manifold we also have $\phi\xi=0$ and $\eta\circ\phi=0$.

Such a manifold is said to be a *contact metric* manifold if $d\eta=\Phi$, where $\Phi(X,Y)=g(X,\phi Y)$ is called the *fundamental 2-form* of $\widetilde{M}$. If, in addition, ξ is a Killing vector field, then $\widetilde{M}$ is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a K-contact manifold if and only if

$$\widetilde{\nabla}_X\xi=-\phi X, \tag{3}$$

for any vector field X on $\widetilde{M}$. On the other hand, the almost contact metric structure of $\widetilde{M}$ is said to be *normal* if $[\phi,\phi](X,Y)=-2d\eta(X,Y)\xi$, for any X,Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by $[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\widetilde{\nabla}_X\phi)Y=g(X,Y)\xi-\eta(Y)X,$$

for any X,Y.

In [23], J. A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\widetilde{M}$ is a *trans-Sasakian manifold* if there exist two functions α and β on M such that

$$(\widetilde{\nabla}_X\phi)Y \;=\; \alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X), \tag{4}$$

for any X,Y on $\widetilde{M}$. If $\beta=0$, $\widetilde{M}$ is said to be an α-*Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha=1$. If $\alpha=0$, $\widetilde{M}$ is said to be a β-*Kenmotsu manifold*. Kenmotsu manifolds are particular examples with $\beta = 1$. If both α and β vanish, then M is a *cosymplectic manifold*.

In particular, from (4) it is easy to see that the following equations hold for a trans-Sasakian manifold:

$$\widetilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{5}$$

$$d\eta = \alpha\Phi.$$

Let now M be a submanifold of an almost contact metric manifold $(\widetilde{M},\phi,\xi,\eta,g)$, tangent to the structure vector field ξ. We will denote also by g the induced metric on M and, if F is a differentiable function on $\widetilde{M}$, we will denote also by F the composition $F\circ x$, where $x : M \to \widetilde{M}$ is the corresponding immersion. We will write the Gauss and Weingarten formulas for this immersion as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{6}$$

$$\widetilde{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{7}$$

for any X,Y (resp. U) tangent (resp. normal) to M. It is well-known that

$$g(A_U X,Y) = g(\sigma(X,Y),U). \tag{8}$$

If we define

$$(\overline{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{9}$$

for any X,Y,Z tangent to M, then Codazzi's equation is given by

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X \sigma)(Y,Z) - (\overline{\nabla}_Y \sigma)(X,Z), \tag{10}$$

where $(\widetilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\widetilde{R}(X,Y)Z$.

For any vector field X tangent to M, we write

$$\phi X = TX + NX,$$

where TX (resp. NX) is the tangential (resp. normal) component of ϕX. Similarly, for any vector field U normal to M, we denote by tU the tangential component of ϕU, and it is well-known that

$$g(NX,U) = -g(X,tU). \tag{11}$$

If $\widetilde{M}$ is K-contact, from (3) and (6) we have

$$\nabla_X \xi = -TX, \tag{12}$$

$$\sigma(X,\xi) = -NX, \tag{13}$$

for any X on M. Similarly, if $\widetilde{M}$ is a trans-Sasakian manifold, if follows from (5) and (6) that:

$$\nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi), \tag{14}$$

$$\sigma(X,\xi) = -\alpha NX. \tag{15}$$

The submanifold M is said to be *invariant* (resp. *anti-invariant*) if ϕX is tangent (resp. normal) to M, for any tangent vector field X, i.e., $N \equiv 0$ (resp. $T \equiv 0$). On the other hand, M is said to be *slant* if for any $x \in M$ and any $X \in T_xM$, linearly independent on ξ, the angle between ϕX and T_xM is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of M in $\widetilde{M}$ [21]. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper* slant immersion. In [9], it was proved that a submanifold M, tangent to the structure vector field ξ of an almost contact metric manifold, is θ-slant if and only if $T^2 = -\cos^2\theta(I - \eta \otimes \xi)$. Moreover, for any vector fields X, Y tangent to such a submanifold, we have:

$$g(TX, TY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{16}$$

$$g(NX, NY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{17}$$

If M is a non-anti-invariant θ-slant submanifold (i.e., $\theta \in [0, \pi/2)$), then it was proved in [10] that $(\overline{\phi}, \xi, \eta, g)$ is an almost contact metric structure on M, where $\overline{\phi} = (\sec\theta)T$. If, in addition, the equation

$$(\nabla_X T)Y = \cos^2\theta(g(X,Y)\xi - \eta(Y)X) \tag{18}$$

is satisfied for any X, Y tangent to M, it was pointed out also in [10] that

$$(\nabla_X \overline{\phi})Y = \cos\theta(g(X,Y)\xi - \eta(Y)X),$$

which means that M is a $\overline{\alpha}$-Sasakian manifold, with $\overline{\alpha} = \cos\theta$. Actually, it was shown in [9] that slant submanifolds satisfying equation (18) play a very important role in that theory. Slant submanifolds in trans-Sasakian manifolds have been investigated in [15, 16, 17].

With respect to the behavior of its second fundamental form, a submanifold is said to be *totally geodesic* if σ vanishes identically, and it is called *totally umbilical* if

$$\sigma(X,Y) = g(X,Y)H \tag{19}$$

for any tangent vector fields X, Y, where H denotes the mean curvature vector. Any totally geodesic submanifold is totally umbilical, and the converse is true if and only if $H = 0$, i.e., if and only if the submanifold is minimal. But there are some other kinds of submanifolds more interesting in almost contact Riemannian geometry. Hence, a submanifold M of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ is said to be *totally contact geodesic* if

$$\sigma(X,Y) = \eta(X)\sigma(Y,\xi) + \eta(Y)\sigma(X,\xi), \tag{20}$$

for any X,Y tangent to M, and it is called *totally contact umbilical* if there exists a normal vector field V such that

$$\sigma(X,Y) = (g(X,Y)-\eta(X)\eta(Y))V+\eta(X)\sigma(Y,\xi)+\eta(Y)\sigma(X,\xi), \tag{21}$$

for any X,Y on M. Once again, any totally contact geodesic submanifold is totally contact umbilical (with $V=0$). If $\widetilde{M}$ is K-contact or trans-Sasakian, it is easy to see that $V=((m+1)/m)H$, where $m+1$ is the dimension of M. Therefore, in such two cases, if M is totally contact umbilical, then it is totally contact geodesic if and only if it is minimal.

3. Slant submanifolds of a generalized Sasakian-space-form.

In this section we obtain some new examples of generalized Sasakian-space-forms, by working with a non-anti-invariant θ-slant submanifold M of a generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, under certain conditions. We always consider on the submanifold the induced almost contact metric structure $(\overline{\phi},\xi,\eta,g)$ described in the previous section. Of course, the key tool to relate curvature tensors on both the submanifold and the ambient manifold is Gauss' equation (2). Actually, it is clear that

$$\widetilde{R}_i(X,Y)Z = R_i(X,Y)Z, \quad i=1,3,$$

for any tangent vector fields X,Y,Z. On the other hand, with respect to $\widetilde{R}_2$, we have

$$\widetilde{R}_2(X,Y,Z,W) = g(X,TZ)g(TY,W)-g(Y,TZ)g(TX,W)+2g(X,TY)g(TZ,W),$$

for any X,Y,Z,W tangent to M. But, since $\overline{\phi}=(\sec\theta)T$, the above equation can be written as

$$\widetilde{R}_2(X,Y,Z,W) = \cos^2\theta R_2(X,Y,Z,W),$$

and so Gauss' equation turns into:

$$\begin{aligned} &F_1R_1(X,Y,Z,W)+\cos^2\theta F_2R_2(X,Y,Z,W)+F_3R_3(X,Y,Z,W)\\ &=R(X,Y,Z,W)+g(\sigma(X,Z),\sigma(Y,W))-g(\sigma(X,W),\sigma(Y,Z)).\end{aligned} \tag{22}$$

Therefore, we can obtain the following result:

THEOREM 1. *Let M be a θ-slant submanifold of a generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta\in[0,\pi/2)$.*

i) If M is totally geodesic, then it is a generalized Sasakian-space-form, with functions $f_1=F_1$, $f_2=\cos^2\theta F_2$, $f_3=F_3$.

ii) If M is totally umbilical, then it is a generalized Sasakian-space-form, with functions $f_1=F_1+||H||^2$, $f_2=\cos^2\theta F_2$, $f_3=F_3$.

Proof. Statement $i)$ follows directly from (22), because in this case $\sigma \equiv 0$. To prove statement $ii)$, we just have to take into account that

$$\begin{aligned} g(\sigma(X,Z),\sigma(Y,W)) &= g(X,Z)g(Y,W)||H||^2, \\ g(\sigma(X,W),\sigma(Y,Z)) &= g(X,W)g(Y,Z)||H||^2, \end{aligned}$$

for any X,Y,Z,W tangent to M, and so

$$g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)) = -||H||^2 R_1(X,Y,Z,W). \tag{23}$$

Therefore, the aimed writing (1) for R is obtained from (22) and (23). □

Of course, a particular case in Theorem 1 is that of invariant submanifolds ($\theta = 0$). In such a case, $\overline{\phi} = \phi$ so statement $i)$ is more than expected. Moreover, if $\widetilde{M}$ is K-contact or (α,β) trans-Sasakian with $\alpha \neq 0$ at any point of M, then it follows from either (13) or (15) and the totally geodesic condition that $N \equiv 0$, which means that the invariant case is the only one under such two conditions. Nevertheless, if $\alpha = 0$, we can find nice examples of proper slant submanifolds satisfying statement $i)$ of Theorem 1. To show them, let us consider a function $f > 0$ on $\mathbb{R}$ and a θ-slant submanifold M_2 of a complex-space-form $\widetilde{M}_2(c)$, where $\theta \in [0,\pi/2)$. It is clear that $\mathbb{R} \times_f M_2$ is a submanifold isometrically immersed in $\mathbb{R} \times_f \widetilde{M}_2$, and it was proved in [1] that this manifold can be endowed with a natural structure of β-Kenmotsu generalized Sasakian-space-form, with $\beta = f'/f$ and functions

$$F_1 = \frac{c - 4f'^2}{4f^2}, \quad F_2 = \frac{c}{4f^2}, \quad F_3 = \frac{c - 4f'^2}{4f^2} + \frac{f''}{f}. \tag{24}$$

Moreover, it is easy to see that $\mathbb{R} \times_f M_2$ is also a θ-slant submanifold, and it follows from [14, Theorem 1] that it is totally geodesic in $\mathbb{R} \times_f \widetilde{M}_2$ if M_2 is a totally geodesic submanifold of $\widetilde{M}_2$. Therefore, by using [13, Example 2.1] we have:

EXAMPLE 1. For any differentiable function $f > 0$ on $\mathbb{R}$ and any $\theta \in [0,\pi/2)$,

$$x(t,u,v) = (t, u\cos\theta, u\sin\theta, v, 0)$$

defines a 3-dimensional, totally geodesic, θ-slant submanifold M in $\mathbb{R} \times_f \mathbb{C}^2$. Thus, by virtue of Theorem 1 and (24), we obtain that M is a generalized Sasakian-space-form, with functions:

$$f_1 = -\frac{f'^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{f'^2}{f^2} + \frac{f''}{f}.$$

The above example can be easily extended to some others with higher dimensions.

Concerning statement $ii)$ of Theorem 1, let us first point out that, if $\widetilde{M}$ is K-contact or trans-Sasakian, then it follows from either (13) or (15) and (19) that M

should be minimal, and so totally geodesic. On the other hand, it was proved in [3] that, if M is a connected and totally umbilical submanifold, and $F_2 \neq 0$ at any point of M, then M is also an invariant manifold. In such a case, statement $ii)$ of Theorem 1 was already obtained in [3, Theorem 6.2].

With respect to totally contact geodesic submanifolds, we have to impose the additional condition of being trans-Sasakian to the ambient manifold in order to obtain a suitable writing for R:

THEOREM 2. *Let M be a θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. If M is totally contact geodesic, then it is a generalized Sasakian-space-form, with functions*

$$f_1 = F_1, \quad f_2 = \cos^2\theta F_2, \quad f_3 = F_3 + \alpha^2 \sin^2\theta.$$

Proof. As before, we start at (22). Now, if M is totally contact geodesic, then it follows from (15) and (20) that

$$\sigma(X,Y) \;=\; -\alpha\eta(X)NY - \alpha\eta(Y)NX, \tag{25}$$

for any X,Y tangent to M. Therefore, a direct calculation from (17) and (25) gives:

$$g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)) = -\alpha^2\sin^2\theta R_3(X,Y,Z,W), \tag{26}$$

for any X,Y,Z,W tangent to M. The result is then obtained by putting (26) in (22). □

In particular, if the ambient manifold $\widetilde{M}$ is a Sasakian-space-form with constant ϕ–sectional curvature c, the functions in Theorem 2 would be given by

$$f_1 = \frac{c+3}{4}, \quad f_2 = \cos^2\theta\frac{c-1}{4}, \quad f_3 = \frac{c-1}{4} + \sin^2\theta, \tag{27}$$

taking into account that such a manifold is Sasakian (i.e., $\alpha = 1$ and $\beta = 0$).

Now again, there are nice examples of proper slant submanifolds satisfying Theorem 2. To show some of them, let $(\mathbb{R}^{2n+1},\phi,\xi,\eta,g)$ denote the manifold $\mathbb{R}^{2n+1}$ with its usual Sasakian structure given by

$$\eta \;=\; 1/2(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi \;=\; 2\frac{\partial}{\partial z},$$

$$g \;=\; \eta\otimes\eta + 1/4\left(\sum_{i=1}^{n}(dx^i\otimes dx^i + dy^i\otimes dy^i)\right),$$

$$\phi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) \;=\; \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}) + \sum_{i=1}^{n} Y_i y^i\frac{\partial}{\partial z},$$

where (x^i, y^i, z), $i = 1 \dots n$ are the cartesian coordinates. It is well-known that, with this structure, $\mathbb{R}^{2n+1}$ is a Sasakian-space-form with constant ϕ–sectional curvature -3 (see for example [8]). Then, by virtue of Theorem 2 and (27), any totally contact geodesic θ-slant ($\theta \in [0, \pi/2)$) submanifold of $\mathbb{R}^{2n+1}(-3)$ is a generalized Sasakian-space-form with functions:

$$f_1 = 0, \quad f_2 = f_3 = -\cos^2\theta. \tag{28}$$

Actually, we do have examples of such a submanifold:

EXAMPLE 2. For any $\theta \in [0, \pi/2)$,

$$x(u,v,t) \;=\; 2(u\cos\theta, u\sin\theta, v, 0, t)$$

defines a 3-dimensional minimal θ-slant submanifold in $\mathbb{R}^5(-3)$ [9]. Moreover, it was proved in [11] that all these submanifolds are totally contact geodesic.

EXAMPLE 3. For any $\theta \in [0, \pi/2)$,

$$x(u,v,w,s,t) \;=\; 2(u, 0, w, 0, v\cos\theta, v\sin\theta, s\cos\theta, s\sin\theta, t)$$

defines a 5-dimensional minimal θ-slant submanifold in $\mathbb{R}^9(-3)$ [9]. As in Example 2, it can be checked that all these submanifolds are also totally contact geodesic.

We can now ask what is the structure of new generalized Sasakian-space-forms given by above examples. In fact, it was proved in [9] that all of them satisfy equation (18) and so, as we pointed out in the preliminaries section, they are $\overline{\alpha}$-Sasakian manifolds, with $\overline{\alpha} = \cos\theta$. In this way, we can obtain examples of $\overline{\alpha}$-Sasakian generalized Sasakian-space-forms for any constant value $0 < \overline{\alpha} \leq 1$. Let us mention how functions f_1, f_2, f_3 given by (28) satisfy Theorem 4.2 of [2], saying that if $M(f_1, f_2, f_3)$ is a connected $\overline{\alpha}$-Sasakian generalized Sasakian-space-form, with dimension greater than or equal to 5 (which is the case of submanifolds given in Example 3) then, f_1, f_2, f_3 are constant functions such that $f_1 - \overline{\alpha}^2 = f_2 = f_3$.

On the other hand, submanifolds given in Example 1 also satisfy Theorem 2, because it follows from (15) and (20) that a submanifold of a β-Kenmotsu manifold is totally contact geodesic if and only if it is totally geodesic.

Moreover, with the techniques we have been using in this paper, we can also obtain a new obstruction result for totally contact geodesic slant submanifolds:

COROLLARY 1. *Let M be a θ-slant submanifold of a Sasakian-space-form $\widetilde{M}(c)$, with $\theta \in [0, \pi/2)$. If M is of dimension greater than or equal to 5, it satisfies (18) and it is totally contact geodesic, then M is invariant or $c = -3$.*

Proof. Under these conditions, it follows from Theorem 2 and the above remarks that M is a $\overline{\alpha}$-Sasakian generalized Sasakian-space-form with $\overline{\alpha} = \cos\theta$ and functions f_1, f_2, f_3 satisfying (27). But, by applying Theorem 4.2 of [2], we have $f_2 = f_3$, which implies that $\sin^2\theta(c+3)/4 = 0$, and so $\theta = 0$ or $c = -3$. $\square$

Let us point out that the condition of M satisfying (18) is not strange at all. Actually, it was proved in [9] that any 3-dimensional proper slant submanifold of a K-contact manifold does. On the other hand, the result of $\widetilde{M}(c)$ being a Sasakian-space-form with $c = -3$ means that, if it is complete and simply connected, it must be $\mathbb{R}^{2n+1}$ with its usual Sasakian structure (by virtue of the well-known classic result of S. Tanno given in [24]).

Now, in order to study what happens with totally contact umbilical θ-slant submanifolds, we state the following lemma:

LEMMA 1. *Let M be an $(m+1)$-dimensional θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. If M is totally contact umbilical, then*

$$
\begin{aligned}
R(X,Y)Z &= (F_1+||V||^2)R_1(X,Y)Z \\
&\quad +\cos^2\theta F_2 R_2(X,Y)Z \\
&\quad +(F_3+\alpha^2\sin^2\theta+||V||^2)R_3(X,Y)Z \\
&\quad +\alpha\{\eta(X)g(Z,tV)Y-\eta(Y)g(Z,tV)X \\
&\quad +g(X,Z)\eta(Y)tV-g(Y,Z)\eta(X)tV \\
&\quad +g(X,tV)\eta(Z)Y-g(Y,tV)\eta(Z)X \\
&\quad +g(X,Z)g(Y,tV)\xi-g(Y,Z)g(X,tV)\xi\},
\end{aligned}
\tag{29}
$$

for any vector fields X,Y,Z tangent to M, where $V=((m+1)/m)H$.

Proof. It follows from (15) and (21) that

$$
\sigma(X,Y) = (g(X,Y)-\eta(X)\eta(Y))V-\alpha\eta(X)NY-\alpha\eta(Y)NX, \tag{30}
$$

for any X,Y tangent to M. Thus, equation (29) follows from (11), (17), (22) and (30), through a quite long computation. □

Lemma 1 shows that, in general, a totally contact umbilical slant submanifold does not inherit the aimed structure from the ambient manifold. Nevertheless, with some additional conditions, it does:

THEOREM 3. *Let M be an $(m+1)$-dimensional θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. Let us suppose that M is totally contact umbilical and at least one of the following conditions holds:*

i) $\alpha = 0$, ii) M is minimal, iii) M is invariant.

Then, M is a generalized Sasakian-space-form, with functions

$$f_1=F_1+((m+1)^2/m^2)\,||H||^2,\ f_2=\cos^2\theta F_2,\ f_3=F_3+\alpha^2\sin^2\theta+((m+1)^2/m^2)\,||H||^2.$$

Proof. It is clear that, in both cases $i)$ and $ii)$, the last terms of (29) vanish. It also happens in case $iii)$, because M being invariant implies $t \equiv 0$. □

Let us now see what can we say about conditions $i) - iii)$ of the above theorem. Firstly, if condition $i)$ holds, then we have the following direct corollary:

COROLLARY 2. *Let M be an $(m+1)$-dimensional totally contact umbilical θ-slant submanifold of a β-Kenmotsu generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. Then, M is a generalized Sasakian-space-form, with functions*

$$f_1 = F_1 + ((m+1)^2/m^2)\,||H||^2, \quad f_2 = \cos^2\theta F_2, \quad f_3 = F_3 + ((m+1)^2/m^2)\,||H||^2.$$

Secondly, if condition $ii)$ of Theorem 3 holds, then M is a totally contact geodesic submanifold, and so the corresponding result was already obtained in Theorem 2. Thirdly, with some additional conditions, we can see that condition $iii)$ holds. We obtain this result as a particular case of the following theorem:

THEOREM 4. *Let M be a connected, totally contact umbilical submanifold, tangent to the structure vector field of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. If $\dim M > 3$ and*

$$F_2 \neq -\alpha^2 \tag{31}$$

at any point of M, then M is either invariant or anti-invariant.

Proof. A direct computation from (9) and (21) gives

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,Z) \;&= (g(Y,Z)-\eta(Y)\eta(Z))\nabla^{\perp}_X V\\ &+g(Y,\nabla_X\xi)\sigma(Z,\xi)+g(Z,\nabla_X\xi)\sigma(Y,\xi)\\ &-g(Y,\nabla_X\xi)\eta(Z)V-g(Z,\nabla_X\xi)\eta(Y)V\\ &+\eta(Y)\sigma(Z,\nabla_X\xi)+\eta(Z)\sigma(Y,\nabla_X\xi)\\ &+\eta(Y)(\overline{\nabla}_X\sigma)(Z,\xi)+\eta(Z)(\overline{\nabla}_X\sigma)(Y,\xi),\end{aligned}\tag{32}$$

for any vector fields X,Y,Z tangent to M. Thus, by using (14), (15) and (32), Codazzi's equation can be written as

$$\begin{aligned}(\widetilde{R}(X,Y)Z)^{\perp} \;&= g(Y,Z)\nabla^{\perp}_X V - g(X,Z)\nabla^{\perp}_Y V\\ &+\alpha^2 g(Z,TX)NY - \alpha^2 g(Z,TY)NX\\ &-\alpha\beta g(X,Z)NY + \alpha\beta g(Y,Z)NX\\ &-2\alpha^2 g(X,TY)NZ,\end{aligned}\tag{33}$$

for any tangent X,Y,Z, orthogonal to ξ. Since $\dim M > 3$, given a tangent vector field X, orthogonal to ξ, we can choose an unit tangent vector field Y such that it is orthogonal to X, ϕX and ξ. Then, for such a choice, equation (33) reduces to

$$(\widetilde{R}(X,Y)Y)^{\perp} \;=\; \nabla^{\perp}_X V + \alpha\beta NX. \tag{34}$$

But, as $\widetilde{M}$ is a generalized Sasakian-space-form,

$$(\widetilde{R}(X,Y)Y)^{\perp} = F_2(\widetilde{R}_2(X,Y)Y)^{\perp} = 0. \tag{35}$$

Therefore, from (34) and (35) we deduce that

$$\nabla^{\perp}_X V = -\alpha\beta NX, \tag{36}$$

for any X orthogonal to ξ, and from (33) and (36) it follows that

$$\begin{aligned}(\widetilde{R}(X,Y)Z)^{\perp} &= +\alpha^2 g(Z,TX)NY - \alpha^2 g(Z,TY)NX \\ &\quad -2\alpha^2 g(X,TY)NZ \\ &= -\alpha^2(\widetilde{R}_2(X,Y)Z)^{\perp},\end{aligned} \tag{37}$$

for any tangent X,Y,Z, orthogonal to ξ. But, once again, as $\widetilde{M}$ is a generalized Sasakian-space-form,

$$(\widetilde{R}(X,Y)Z)^{\perp} = F_2(\widetilde{R}_2(X,Y)Z)^{\perp}. \tag{38}$$

Thus, from (37) and (38) we obtain that

$$(F_2+\alpha^2)(\widetilde{R}_2(X,Y)Z)^{\perp} = 0,$$

which gives that $(\widetilde{R}_2(X,Y)Z)^{\perp} = 0$, for any tangent X,Y,Z orthogonal to ξ, since we are working under the assumption of $F_2 \neq -\alpha^2$ at any point of M. In particular, for any X,Y orthogonal to ξ, we have:

$$(\widetilde{R}_2(X,Y)X)^{\perp} = 3g(X,TY)NX = 0. \tag{39}$$

Since M is connected, (39) implies that either $T \equiv 0$ (i.e., M is anti-invariant) or $N \equiv 0$ (i.e., M is invariant) and the proof concludes. □

If $\widetilde{M}$ has dimension greater than or equal to 5 (which it what happens if M is a slant submanifold with dimension greater than 3), then it was proved by J. C. Marrero in [22] that it is either an α-Sasakian or a β-Kenmotsu manifold. In the first case, it was proved in [2] that α, F_1, F_2, F_3 are constant functions such that $F_1 - \alpha^2 = F_2 = F_3$. Therefore, condition (31) is equivalent to $F_1 \neq 0$. Thus, Theorem 4 is a generalization to trans-Sasakian manifolds of a classical result of I. Ishihara and M. Kon, given in [19] for a Sasakian-space-form with constant ϕ-sectional curvature $c \neq -3$, because in such a space $F_1 = (c+3)/4$. In the second case, i.e., if $\widetilde{M}$ is a β-Kenmotsu manifold, condition (31) just means that F_2 does not vanish on M. Hence, Theorem 4 also implies that, if we want to look for non-invariant slant submanifolds satisfying Corollary 2, then we should ask F_2 to vanish. We can obtain such a β-Kenmotsu generalized Sasakian-space-form by considering a warped product $\mathbb{R} \times_f \mathbb{C}^n$.

For 3-dimensional slant submanifolds in a 5-dimensional ambient manifold, we have that at least one of conditions *ii*) and *iii*) of Theorem 3 holds, with no assumptions about functions F_1, F_2, F_3:

THEOREM 5. *Let M be a connected 3-dimensional totally contact umbilical θ-slant submanifold of a 5-dimensional (α,β) trans-Sasakian manifold $\widetilde{M}$, with $\theta \in [0,\pi/2)$. Then M is minimal or invariant.*

Proof. Let M be a 3-dimensional θ-slant submanifold of an (α,β) trans-Sasakian manifold. Then, it was proved in [17] that

$$A_{NX}Y = A_{NY}X + \alpha \sin^2\theta(\eta(X)Y - \eta(Y)X), \tag{40}$$

for any vector fields X,Y tangent to M, which means that

$$A_{NX}Y = A_{NY}X \tag{41}$$

for any tangent X,Y, orthogonal to ξ. If, in addition, M is totally contact umbilical, a direct computation by using (8), (15) and (21) gives

$$A_{NX}Y = g(NX,V)Y - \alpha g(NX,NY)\xi. \tag{42}$$

Thus, from (41) and (42), we obtain that

$$g(NX,V)Y = g(NY,V)X,$$

which means that

$$g(NX,V) = 0, \tag{43}$$

for any tangent vector field X, orthogonal to ξ. But, as $\dim\widetilde{M} = 5$, we know that, if M is not invariant, then $T_p^{\perp}(M)$ is spanned at any point $p \in M$ by

$$\{(NX)_p |\ X \text{ is orthogonal to } \xi\}.$$

Therefore, equation (43) implies that, if M is not invariant, then $V \equiv 0$ and so it is minimal. □

Theorem 5 can be extended to an $(m+1)$-dimensional totally contact umbilical θ-slant submanifold M of an $(2m+1)$-dimensional (α,β) trans-Sasakian generalized Sasakian-space-form, such that

$$(\nabla_X T)Y = \alpha\cos^2\theta(g(X,Y)\xi - \eta(Y)X) + \beta(g(TX,Y)\xi - \eta(Y)TX), \tag{44}$$

for any vector fields X,Y tangent to M, because it was proved in [17] that equation (44) is equivalent to (40), and so the above proof works. Actually, slant submanifolds satisfying (44) play a similar role in trans-Sasakian manifolds to those satisfying (18) in Sasakian ones.

Finally, we can give more information for a totally contact umbilical anti-invariant submanifold M, tangent to the structure vector field of an α-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. From (14) and (32) we have

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,Z) &= (g(Y,Z) - \eta(Y)\eta(Z))\nabla^{\perp}_X V \\ &+ \eta(Y)(\overline{\nabla}_X\sigma)(Z,\xi) + \eta(Z)(\overline{\nabla}_X\sigma)(Y,\xi),\end{aligned} \tag{45}$$

for any vector fields X,Y,Z tangent to M. Assume that M has parallel second fundamental form. Then from (45) we have

$$(g(Y,Z)-\eta(Y)\eta(Z))\nabla^{\perp}_{X}V=0,$$

which gives us $\nabla^{\perp}_{X}V=0$. Hence M has parallel mean curvature vector and we can state the following result:

THEOREM 6. *Let M be a totally contact umbilical anti-invariant submanifold, tangent to the structure vector field of an α-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. If M has parallel second fundamental form, then the mean curvature vector of M is parallel.*

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