(Pseudo)-cocyclic (structured) Hadamard matrices over (quasi)groups

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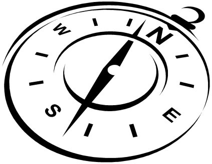
5th Workshop on Real and Complex Hadamard Matrices and Applications

Outline

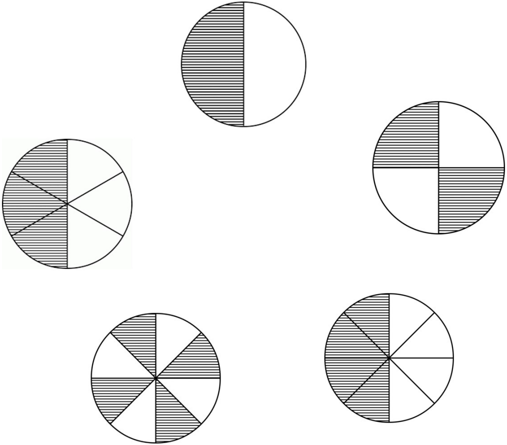
1. Cocyclic constructions for Hadamard matrices
2. (Pseudo)cocyclic Hadamard matrices over quasigroups
3. The Goethals-Seidel arrays are pseudo-cocyclic
4. Searching for large cocyclic Hadamard matrices
5. Future work



The cocyclic framework

$H = \left(\psi(g_i, g_j)\right)$ is a G-cocyclic Hadamard matrix, $|G| = 4t$.

$$\psi(g_i, g_j)\,\psi(g_i g_j, g_k)\psi(g_i, g_j g_k)\,\psi(g_j, g_k) = 1, \quad g_i, g_j, g_k \in G.$$



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	Cocyclic	Non cocyclic
Sylvester	$\mathbb{Z}_2^{\log_2 4t}$	
Williamson	$\mathbb{Z}_2^2 \times \mathbb{Z}_t$	
Paley I	D_{4t}	
Paley II	$\mathbb{Z}_2^2 \times \mathbb{Z}_t$	
Ito	D_{4t}	
1-circulant core	$\mathbb{Z}_{4t-1}$-cocyclic structured	
2-circulant core	D_{4t-2}-cocyclic structured	
Goethals-Seidel		always?
Twin prime power		always?

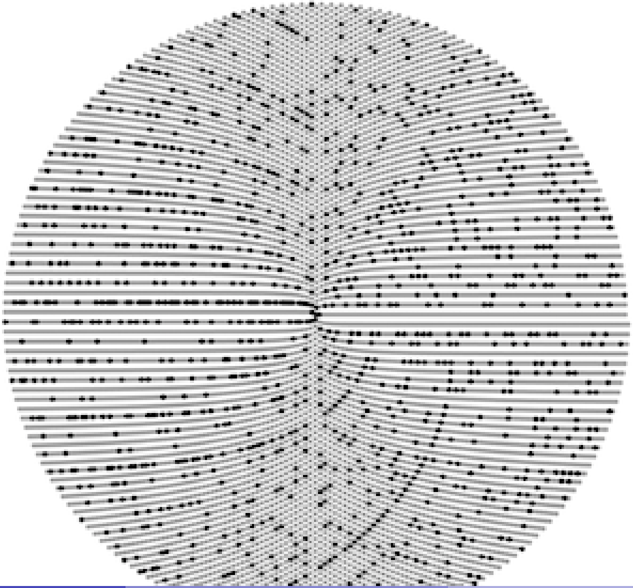
(Dis)advantages

- Faster Hadamard test

$$\sum_{j=1}^{4t} \psi(g_i, g_j) = 0, \text{ for } 2 \leq i \leq 4t.$$

(Dis)advantages

- Search space is reduced 

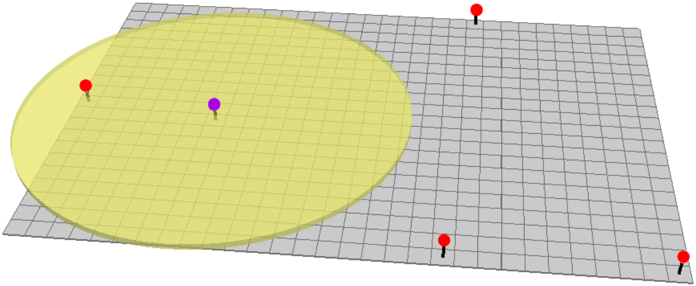


(Dis)advantages

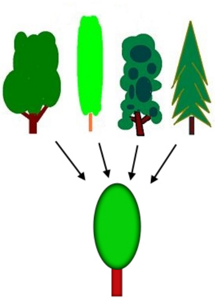
- The proportion of Hadamard matrices is reduced in turn 😳

order	2	4	8	12	16	20	24	28	32	36
$\sim_H$	1	1	1	1	5	3	60	487	13710027	$\geq 3 \cdot 10^6$
$\sim_{H+c}$	1	1	1	1	5	3	**16**	**6**	**100**	**35**

Ó Catháin, Röder 2011



What next? Cocycles over quasigroups...



Cocycles ψ over a *quasigroup* Q (i.e. associativity fails)

$$\forall a, b \in Q,\ \exists! x, y \in Q /\ ax = b, ya = b.$$

$$\psi(g_i, g_j)\ \psi(g_ig_j, g_k)\psi(g_i, g_jg_k)\ \psi(g_j, g_k) = 1,\ \ g_i, g_j, g_k \in Q.$$

What next? Cocycles over quasigroups...

Although the usual Hadamard cocyclic test is available ...

$$\sum_{k=1}^{4t} \psi(g_h, g_k)\psi(g_j, g_k) = 0 \Leftrightarrow \sum_{k=1}^{4t} \psi(g_i, g_k) = 0 \qquad (1)$$

Proposition

...A necessary condition for a Q-cocyclic matrix M_ψ being Hadamard is that Q is actually endowed with a loop structure.

Example: a Q-cocyclic Hadamard matrix of order 8

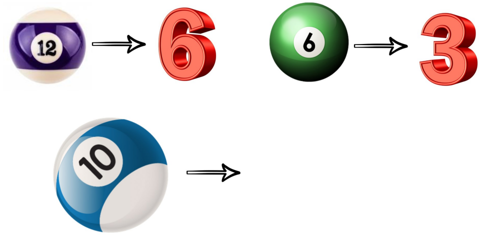
Consider the quasigroup Q of given law $((5 \cdot 6) \cdot 7 = 6 \neq 5 = 5 \cdot (6 \cdot 7))$:

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	8	7	3	4	2	1
6	5	7	8	4	3	1	2
7	8	6	5	1	2	4	3
8	7	5	6	2	1	3	4

$$BN_2 \otimes \mathbf{1}_4, \overset{\partial_2, \partial_3, \partial_4,}{\begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & + & - \\ + & - & - & - \end{pmatrix}} \otimes \mathbf{1}_2$$

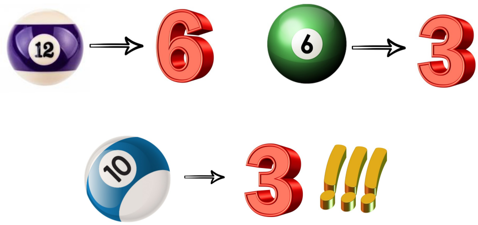
4 out of 32 are Hadamard: $\partial_2\partial_3$, $\partial_2\partial_3\partial_4$, ∂_3, ∂_4.

... Or even pseudo-cocycles over quasigroups!



... Or even pseudo-cocycles over quasigroups!

Formal *coboundaries* **might not** be cocyclic!



Lemma

The elementary map $\partial\delta_h$ actually constitutes a genuine cocycle if and only if

$$g_i(g_jg_k) = g_h \Leftrightarrow (g_ig_j)g_k = g_h, \quad g_i, g_j, g_k \in Q.$$

... Or even pseudo-cocycles over quasigroups!

Those maps which are formally coboundaries but not truly cocycles are called *pseudo-coboundaries*. It is of interest considering *pseudo-cocyclic* matrices $M_{\psi\cdot\phi}$ resulting from the product of a genuine cocycle ψ and a pseudocoboundary ϕ for which the Hadamard test (1) still applies, no matter they are not truly cocyclic.

$\{H: H = M_\psi\} \subset \{H: H = M_{\psi \cdot \phi}\}$

order	2	4	8	12	16	20	24	28	32	36
$\sim_H$	1	1	1	1	5	3	60	487	13710027	$\geq 3 \cdot 10^6$
$\sim_{H+sc}$	1	1	1	1	5	3	??	??	??	??
$\sim_{H+c}$	1	1	1	1	5	3	**16**	**6**	**100**	**35**

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Goethals-Seidel	GS_{4t}-pseudo-cocyclic	
Twin prime power		always?

The Goethals-Seidel arrays

$$\begin{pmatrix} A & BR & CR & DR \\ BR & -A & RD & -RC \\ CR & -RD & -A & RB \\ DR & RC & -RB & -A \end{pmatrix} \qquad \begin{array}{l} A, B, C, D \text{ circulants,} \\ R \leftarrow^{b} [0, \ldots, 0, 1] \end{array}$$

It is Hadamard if $AA^T + BB^T + CC^T + DD^T = 4tI_t$.

The Goethals-Seidel arrays

$$GS_{4t} = \langle a, b, c, d : \; a^t = b^2 = c^2 = d^2 = 1, (a^i x)a^j = a^{i+j}x,$$
$$a^i(a^j y) = a^{j-i}y, (a^i y)(a^j y) = a^{j-i}, (a^i y_1)(a^j y_2) = a^{t-2-j-i}y_3\rangle$$

for $x \in \{1, b, c, d\}$, $y \in \{b, c, d\}$, $\{y_1, y_2, y_3\} = \{b, c, d\}$.

$$1, a, \ldots, a^{t-1}, b, ab, \ldots a^{t-1}b, c, ac, \ldots, a^{t-1}c, d, ad, \ldots, a^{t-1}d.$$

The Goethals-Seidel arrays are GS_{4t}-pseudo cocyclic

$$\begin{pmatrix} A & BR & CR & DR \\ BR & -A & RD & -RC \\ CR & -RD & -A & RB \\ DR & RC & -RB & -A \end{pmatrix} \quad \begin{pmatrix} {}^bA & B^c & C^c & D^c \\ {}^bB & A^c & {}^b\bar{D} & {}^b\bar{C} \\ {}^bC & {}^b\bar{D} & A^c & {}^b\bar{B} \\ {}^bD & {}^b\bar{C} & {}^b\bar{B} & A^c \end{pmatrix}$$

Theorem

The Goethals-Seidel array is pseudo-cocyclic over the loop GS_{4t}.

Range	(i,j,k)	$i(jk)$
$2 \le h \le t$	$(t+1, 2t+1, 4t+2-h)$	$1 + (h - 3 \bmod t)$
$t+1 \le h \le 2t$	$(2, 2t+1, 5t+1-h)$	$t + 1 + (h - 3 \bmod t)$
$2t+1 \le h \le 3t$	$(2, t+1, 6t+1-h)$	$2t + 1 + (h - 3 \bmod t)$
$3t+1 \le h \le 4t$	$(2, 2t+1, 7t+1-h)$	$3t + 1 + (h - 3 \bmod t)$

$(ij)k = h \neq i(jk)$ and **none** of the formal coboundaries are cocyclic!

The Goethals-Seidel arrays are GS_{4t}-pseudo cocyclic

$$\begin{pmatrix} A & BR & CR & DR \\ BR & -A & RD & -RC \\ CR & -RD & -A & RB \\ DR & RC & -RB & -A \end{pmatrix} \quad \begin{pmatrix} {}^bA & B^c & C^c & D^c \\ {}^bB & A^c & {}^b\bar{D} & {}^b\bar{C} \\ {}^bC & {}^b\bar{D} & A^c & {}^b\bar{B} \\ {}^bD & {}^b\bar{C} & {}^b\bar{B} & A^c \end{pmatrix}$$

Theorem

The Goethals-Seidel array is pseudo-cocyclic over the loop GS_{4t}.

$$M_\psi = (\prod_{h\in H} M_{\partial_h})R, R = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{pmatrix} \otimes \mathbf{1}_t$$

Permute the pairs of rows $(i, t+2-i)$, for $2 \leq i \leq \frac{t+1}{2}$.

The Goethals-Seidel arrays are GS_{4t}-pseudo cocyclic

Theorem

The Goethals-Seidel array is Hadamard if and only if the related GS_{4t}-pseudococyclic matrix satisfies the usual cocyclic test.

$$\langle Row_{ij}, Row_j \rangle = \sum_{k=1}^{4t} \left(\prod_{h \in H} \delta_{h,i(jk)} \delta_{h,(ij)k} \right) \psi(i,j)\psi(i,jk) =$$

$$\psi(i,j) \sum_{k=1}^{4t} \sigma_k \psi(i,jk) = \psi(i,j) \sum_{k=1}^{4t} \psi(i,k).$$

Furthermore, it suffices to check rows $2 \leq i \leq \frac{t+1}{2}$.

Counting -1s

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$$M_\psi = (\prod_{h \in H} M_{\partial_h})R, \qquad \partial_h(i,j) = \delta_{h,i}\delta_{h,j}\delta_{h,ij}$$

- Every M_{∂_h} contributes two -1s at row k at positions (k,h) (head,) and $(k, k^{-1}h)$ (tail,).

Counting -1s

$$M_\psi = (\prod_{h\in H} M_{\partial_h})R, \qquad \partial_h(i,j) = \delta_{h,i}\delta_{h,j}\delta_{h,ij}$$

- Whenever two different $M_{\partial_{h_1}}$ and $M_{\partial_{h_1}}$ share a tail and a head at row k, they constitute a path at row k.

- Consequently $\prod_{h\in H} M_{\partial_h}$ contributes twice as many -1s as maximal paths there exist at row k.

Counting -1s

$$M_\psi = (\prod_{h \in H} M_{\partial_h})R, \qquad \partial_h(i,j) = \delta_{h,i}\delta_{h,j}\delta_{h,ij}$$

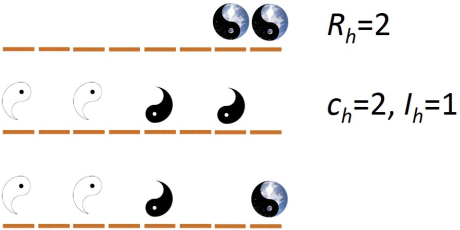
- Following the same principle, whenever a head or a tail of a path is shared by R, an intersection occurs and this tentative -1 is lost.

Counting -1s

$$M_\psi = (\prod_{h \in H} M_{\partial_h})R, \qquad \partial_h(i,j) = \delta_{h,i}\delta_{h,j}\delta_{h,ij}$$

- Consequently, the -1s of M_ψ at row h come from heads, tails and those of R which do not contribute any intersections at all,

$$2c_h + R_h - 2I_h = 2t$$

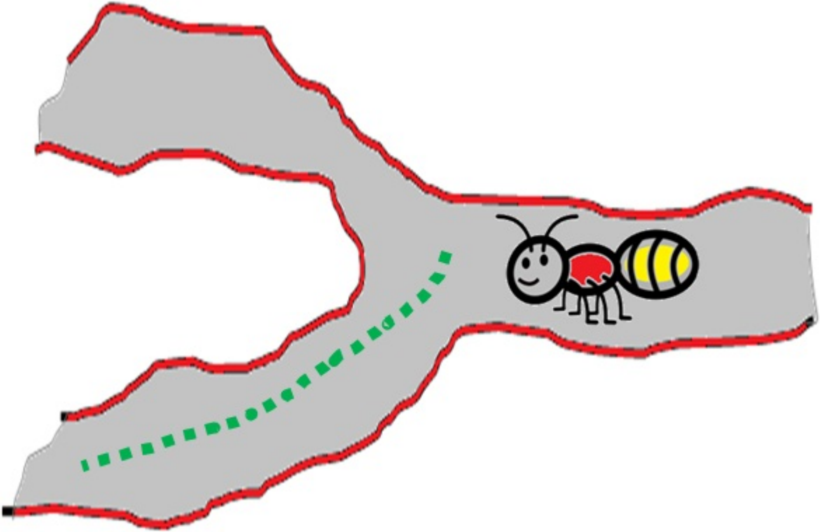


Counting -1s in practise

- Exhaustive search: $t \leq 7$ (2003).

Counting -1s in practise

- Heuristic search: Fitness = number of Hadamard rows (GA 2006, ACS 2009), $t \leq 13$.



Counting -1s in practise

- Exhaustive search via ingredients and recipes (2011), $t \leq 11$, $t \leq 23$.

Counting -1s in practise

- Cocyclic Hadamard ideals (2016), $t \leq 39$.

Counting -1s in practise

- Heuristic search (GA 2017) + local search (CSP), $t = 47$??

Alternative fitness

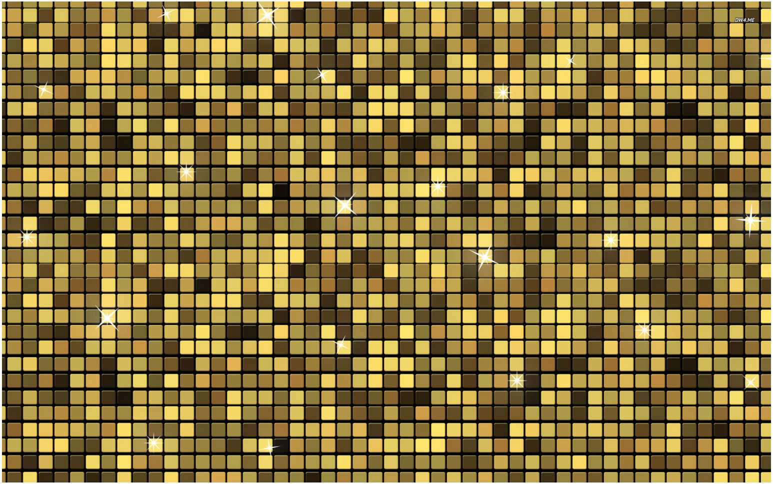
$$F(paths, intersections) = constant \tag{2}$$

Group	$F(p, I)$	$\overrightarrow{\mathbf{k}}$	Rows
$\mathbb{Z}_2^2 \times \mathbb{Z}_t$	p	$(t, \ldots, t)$	$r \equiv 1 \bmod t$
D_{4t}	$p - I$	$(t-1, \ldots, 1)$	$2, \ldots, t$
GS_{4t}	$p_A + p_B + p_C + p_D$	t	$2, \ldots, \frac{t+1}{2}$

IDEA:
$\| \overrightarrow{F}(p, I) - \overrightarrow{\mathbf{k}} \|_\infty$ instead of hamming distance!

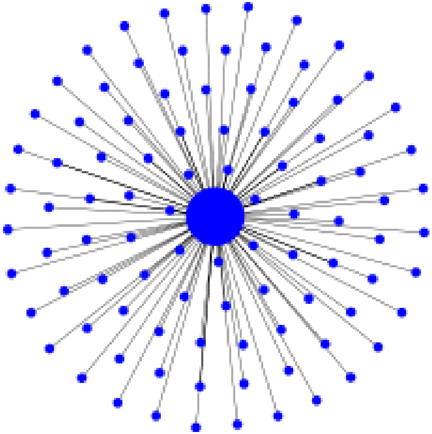
The case $D_{4\cdot 47}$

Fitness of 10000 random individuals runs on $[5, 15]$.



The case $D_{4 \cdot 47}$

Perform a heuristic such that you move to a neighbor as soon as fitness improves.



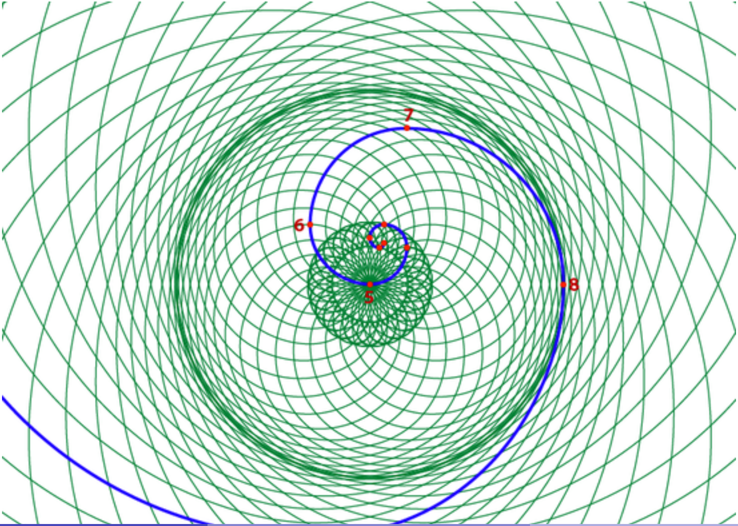
The case $D_{4\cdot 47}$

In case that none of the $4t$ neighbors works, jump to a random individual at a prefixed hamming distance (6 seems to work fine).

The case $D_{4\cdot 47}$

Reaches fitness 2 immediately!

Reaches **fitness 1** almost every run, after no more than 1000 iterations!



The case $D_{4\cdot 47}$

Unfortunately, there are many local minima 😳

The case $D_{4\cdot 47}$

Second step: local search.

Perform a radial search (radius 4 = **51.512.518** instances).

The case $D_{4\cdot 47}$

Faster by means of a Constraint Satisfaction Problem

What to come?

Thank you Mate and Ferenc!