Generating partial Hadamard matrices as solutions to a Constraint Satisfaction Problem characterizing cliques[S]

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Abstract. A procedure is described looking for partial Hadamard matrices, as cliques of a particular subgraph G_t of Ito's Hadamard Graph $\Delta(4t)$ [9]. The key idea is translating the problem of extending a given clique C_m to a larger clique of size $m+1$ in G_t, into a constraint satisfaction problem, and look for a solution to this problem by means of Minion [6]. Iteration of this process usually ends with a large partial Hadamard matrix.

Keywords. *Hadamard matrix, Hadamard Graph, clique, Constraint Satisfaction Problem.*

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1 Introduction

Hadamard matrices consist in $\{1,-1\}$-square matrices whose rows are pairwise orthogonal. This nice property makes Hadamard matrices being objects for multiple applications (see [8] and the references therein, for instance).

It may be straightforwardly checked that such a matrix must be of size 1, 2 or a multiple of 4. The Hadamard Conjecture claims that a matrix of this type exists for every size multiple of 4. Many attempts have been devoted to prove this conjecture (both from a constructive way and also from a theoretical point of view in terms of asymptotic results of existence),but it remains unsolved so far.

From the practical point of view, taking into account possible applications, sometimes there is no need to consider a full Hadamard matrix. In fact, it suffices to meet a large amount of pairwise orthogonal rows. This has originated the interest in constructing *partial Hadamard matrices* PH, that is, $m \times 4t$ $(1,-1)$-matrices PH satisfying $PH \cdot PH^T = 4tI_m$, for $m \leq 4t$. We call m the *depth* of PH.

Although partial Hadamard matrices are as useful as Hadamard matrices themselves with regards to practical purposes, unfortunately it seems that their explicit construction is equally hard as well.

De Launey proved in [3] that partial Hadamard matrices of size about a third of a $4t \times 4t$ Hadamard matrix exist for large t. The proof gives a polynomial time algorithm

[S] This is a substantial progress on a previous and unpublished work of some of the authors [1], partially supported by the research project FMQ-016 from Junta de Andalucía.

Proceedings of X Encuentro Andaluz de Matemática Discreta, pp. 103-106. ISBN 978-84-697-4743-8.
(La Línea, Cádiz, July 10-11, 2017).

in t for constructing such a matrix. Furthermore, De Launey and Gordon proved in [4] that about a half of a Hadamard matrix $4t \times 4t$ exists for large t, assuming that the Riemann hypothesis is true. The idea is decomposing $2t - i$ as the sum of i odd prime numbers p_i, $2 \leq i \leq 3$, so that the juxtaposition of the corresponding Paley conference matrices provides a partial Hadamard matrix of depth $2 \min\{p_i\} + 2$. Unfortunately, none of these methods can provide a partial Hadamard matrix of depth greater than half of a full Hadamard matrix.

The aim of this work is to describe a procedure for constructing partial Hadamard matrices in terms of cliques of a certain subgraph G_t of Ito's Hadamard Graph $\Delta(4t)$ [9], induced by the $(1,-1)$-vectors simultaneously orthogonal to the three first rows of a normalized Hadamard matrix,

$$\left(\begin{array}{ccc|ccc|ccc|ccc} 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & \dots & 1 & -1 & \dots & -1 & -1 & \dots & -1 \\ 1 & \dots & 1 & -1 & \dots & -1 & 1 & \dots & 1 & -1 & \dots & -1 \\ & \dots & & & \dots & & & \dots & & & \dots & \end{array}\right).$$

It may be straightforwardly checked that the vertices of G_t consist in $(1,-1)$-vectors of length $4t$ where the $2t$ negative entries are distributed so that exactly k, $t-k$, $t-k$ and k negative entries occur respectively among the ranges $[1, \dots, t]$, $[t+1, \dots, 2t]$, $[2t+1, \dots, 3t]$ and $[3t+1, \dots 4t]$, for some $0 \leq k \leq t$. In turn, the set of vertices in G_t may be classified attending to the number k of negative entries which appear in positions 1 through t, from which the notion of k-vertex naturally follows.

A *maximum clique* is a clique with the maximum cardinality (which is called the maximum clique number). This notion is different from that of maximal clique, which refers to a clique which is not a proper subset of any other clique. Thus maximal cliques need not be maximum ones, though the converse is always true. Concerning cliques in G_t, this comes to say that a partial Hadamard matrix does not need to be a submatrix of a proper Hadamard matrix.

Given a graph, the maximum clique problem (MCP) is to find a maximum clique, and it is NP-complete [2]. Unfortunately, there is no polynomial-time algorithm for approximating the maximum clique within a factor of $n^{1-\epsilon}$ unless P=NP [7], where n is the number of the vertices of the graph. Moreover, there is no polynomial-time algorithm approximating the clique number within a factor of $\dfrac{n}{(\log n)^{1-\epsilon}}$ unless NP=ZPP [10].

Anyway, our purpose here is to design an algorithm for constructing sufficiently large cliques in G_t.

To this end, we firstly translate the problem of extending a given clique C_m to a larger clique of size $m+1$ in G_t, into a Constraint Satisfaction Problem [5] (CSP in brief, hereafter), and look for a solution to this problem by means of Minion [6], one of the fastest and most scalable constraint solvers using the "model and run" methodology. Starting from any single vertex (no matter which it is), iteration of this CSP ends in a large clique in G_t, providing a large partial Hadamard matrix in turn.

The explicit formulation of this CSP relays on a deeper knowledge of the properties of G_t. In particular, attending to the definition of G_t, it is straightforward to derive that two given vectors are orthogonal if and only if they share precisely $2t$ entries. Furthermore, fixed a k-vector $\mathbf{v}$, there exist s-vectors orthogonal to $\mathbf{v}$ if and only if $s \in [\lceil \frac{t}{2} \rceil - k, \lfloor \frac{t}{2} \rfloor]$. Surprisingly, there is experimental evidence that most of normalized full Hadamard matrices give row vectors for which s is equal either to $\frac{t-1}{2}$ or $\frac{t-3}{2}$ in all cases. We impose this restriction in order to state our CSP for extending a given clique C_m in G_t.

2 The Constraint Satisfaction Problem on Hadamard cliques

We have stated a CSP specifically designed for solving the problem of extending a given clique C_m to a larger clique of size $m+1$ in G_t.

The procedure consists in search for a new k-vertex $\mathbf{x} = (x_1, \ldots, x_{4t})$, so that the following constraints are simultaneously satisfied:

(C1) $k \in \{\frac{t-3}{2}, \frac{t-1}{2}\}$.

(C2) The number of -1s in the ranges $(x_1, \ldots, x_t)$, $(x_{t+1}, \ldots, x_{2t})$, $(x_{2t+1}, \ldots, x_{3t})$ and $(x_{3t+1}, \ldots, x_{4t})$ are k, $t-k$, $t-k$ and k, respectively.

(C3) The number of coincidences of $\mathbf{x}$ with each of the vectors already in C_m is $2t$.

In terms of the language and tools provided by Minion, (C1) consists in declaring a discrete variable `neg` admitting two possible values; the constraints (C2) translate as constraints of the type `occurrence([C[i,_]],1,neg)`; and the constraints (C3) translate as constraints of the type `occurrence([orto[i,_]],1,2t)` and `reify(element(C[i,_],k,` x_k`),orto[i,k])`, for a boolean matrix `orto`. Iteration of this process usually ends with a large partial Hadamard matrix, incomparably faster than any other method known so far by the authors. Some examples will be provided in the talk, since they cannot be included in this note for obvious space limitation reasons.

It is remarkable that the method described here may be straightforwardly adapted to fit to analogous CSPs involving similar graph structures, and opens a door to many other interesting and potential applications.

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