The 3-dimensional planar assignment problem and the number of Latin squares related to an autotopism*

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Abstract

There exists a bijection between the set of Latin squares of order n and the set of feasible solutions of the 3-dimensional planar assignment problem ($3PAP_n$). In this paper, we prove that, given a Latin square isotopism Θ, we can add some linear constraints to the $3PAP_n$ in order to obtain a $1-1$ correspondence between the new set of feasible solutions and the set of Latin squares of order n having Θ in their autotopism group. Moreover, we use Gröbner bases in order to describe an algorithm that allows one to obtain the cardinal of both sets.

Introduction

A *Latin square* of order n is an $n \times n$ array with elements chosen from a set of n distinct symbols (in this paper, it will be the set $[n] = \{1, 2, ..., n\}$) such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order n is denoted by $LS(n)$. A *partial Latin square* of order n, is a $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. The set of partial Latin squares of order n is denoted by $PLS(n)$.

The permutation group on $[n]$ is denoted by S_n. Every permutation $\delta \in S_n$ can be uniquely written as a composition of $\mathbf{n}_\delta$ pairwise disjoint cycles, $\delta = C_1^\delta \circ C_2^\delta \circ ... \circ C_{\mathbf{n}_\delta}^\delta$, where for all $i \in [\mathbf{n}_\delta]$, one has $C_i^\delta = \left(c_{i,1}^\delta \ c_{i,2}^\delta \ ... \ c_{i,\ \lambda_i^\delta}^\delta\right)$, with $c_{i,1}^\delta = \min_j\{c_{i,j}^\delta\}$. The *cycle structure of* δ is the sequence $\mathbf{l}_\delta = (\mathbf{l}_1^\delta, \mathbf{l}_2^\delta, ..., \mathbf{l}_n^\delta)$, where $\mathbf{l}_i^\delta$ is the number of cycles of length i in δ, for all $i \in [n]$. Thus, $\mathbf{l}_1^\delta$ is the cardinal of the set of *fixed points* of δ, $Fix(\delta) = \{i \in [n] \mid \delta(i) = i\}$. An *isotopism* of a Latin square $L \in LS(n)$ is a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n = S_n \times S_n \times S_n$. Therefore, α, β and γ are permutations of rows, columns and symbols of L, respectively. The *cycle structure* of Θ is the triple $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$.

An isotopism which maps L to itself is an *autotopism*. The possible cycle structures of the set of non-trivial autotopisms of Latin squares of order up to 11 were obtained in [3]. The set of all possible autotopisms of order n is denoted by $\mathfrak{A}_n$. The stabilizer subgroup of L in $\mathfrak{A}_n$ is its *autotopism group* $\mathfrak{A}(L)$. Given $\Theta \in \mathfrak{A}_n$, the set of all Latin squares L such that $\Theta \in \mathfrak{A}(L)$ is denoted by $LS(\Theta)$ and the cardinality of $LS(\Theta)$ is denoted by $\Delta(\Theta)$. Specifically, if Θ_1 and Θ_2 are two autotopisms with the same cycle structure, then $\Delta(\Theta_1) = \Delta(\Theta_2)$. Given $\Theta \in \mathfrak{A}_n$ and $P \in PLS(n)$, the number $c_P = \Delta(\Theta)/|LS_P(\Theta)|$ is called the *P-coefficient of symmetry of* Θ, where $LS_P(\Theta) = \{L \in LS(\Theta) \mid P \subseteq L\}$.

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Gröbner bases were used in [4] to describe an algorithm that allows one to obtain the number $\Delta(\Theta)$ in a computational way. This algorithm was implemented in SINGULAR [7] for Latin squares of order up to 7 [5]. However, after applying it to upper orders, the authors have seen that, in order to improve the time of computation, it is convenient to combine Gröbner bases with some combinatorial tools. In this paper we study, as a possible tool, the 1-1 correspondence between $LS(n)$ and the set of feasible solutions of the *3-dimensional planar assignment problem* ($3PAP_n$) [2]:

$$\min \sum_{i\in I, j\in J, k\in K} w_{ijk}\cdot x_{ijk}, \qquad s.t. \begin{cases} \sum_{i\in I} x_{ijk} = 1, \forall j \in J, k\in K. \\ \sum_{j\in J} x_{ijk} = 1, \forall i \in I, k\in K. \\ \sum_{k\in K} x_{ijk} = 1, \forall i \in I, j\in J. \\ x_{ijk}\in\{0,1\}, \forall i\in I, j\in J, k\in K. \end{cases} \tag{1}$$

where w_{ijk} are real weights and I, J, K are three disjoint n-sets. Thus, any feasible solution of the $3PAP_n$ can be considered as a Latin square $L=(l_{i,j})\in LS(n)$, by taking $I=J=K=[n]$ and $x_{ijk} = \begin{cases} 1, & \text{if } l_{i,j}=k, \\ 0, & \text{otherwise.}\end{cases}$. The reciprocal is analogous.

1 Constraints related to an autotopism of a Latin square

Given a autotopism $\Theta=(\alpha,\beta,\gamma)\in\mathfrak{A}_n$, let $(1)_\Theta$ be the set of constraints obtained by adding to (1) the n^3 constraints $x_{ijk}=x_{\alpha(i)\beta(j)\gamma(k)}$, $\forall i\in I, j\in J, k\in K$.

Theorem 1.1. *There exists a bijection between $LS(\Theta)$ and the set of feasible solutions related to a combinatorial optimization problem having $(1)_\Theta$ as the set of constraints.* □

$(1)_\Theta$ is a system of $3n^2+2n^3$ equations of degrees 1 and 2, in n^3 variables, which can be solved by using Gröbner basis. Thus, if we define $F(x)=x\cdot(x-1)$, then the following result is verified:

Corollary 1.2. *$LS(\Theta)$ corresponds to the set of zeros of the ideal* $I=\langle \sum_{i\in[n]} x_{ijk}-1 \mid j,k\in[n]\rangle + \langle \sum_{j\in[n]} x_{ijk}-1 \mid i,k\in[n]\rangle + \langle \sum_{k\in[n]} x_{ijk}-1 \mid i,j\in[n]\rangle + \langle F(x_{ijk}) \mid i,j,k\in[n]\rangle + \langle x_{ijk}-x_{\alpha(i)\beta(j)\gamma(k)} \mid i,j,k\in[n]\rangle \subseteq \mathbb{Q}[\mathbf{x}]=\mathbb{Q}[x_{111},...,x_{nnn}]$. □

The symmetrical structure of Θ can be used to reduce the number of variables of the previous system. To see it, let us consider $S_\Theta = \left\{(i,j)\mid i\in S_\alpha, j\in \begin{cases}[n], & \text{if } i\notin Fix(\alpha), \\ S_\beta, & \text{if } i\in Fix(\alpha).\end{cases}\right\}$ as a set of $(\mathbf{n}_\alpha-\mathbf{l}^1_\alpha)\cdot n+\mathbf{l}^1_\alpha\cdot\mathbf{n}_\beta$ multi-indices, where $S_\alpha=\{c^\alpha_{i,1}\mid i\in[\mathbf{n}_\alpha]\}$ and $S_\beta=\{c^\beta_{j,1}\mid j\in[\mathbf{n}_\beta]\}$.

Proposition 1.3 (Falcón and Martín-Morales [4])**.** *Let $L=(l_{i,j})\in LS(\Theta)$ be such that all the triples of the Latin subrectangle $R_L=\{(i,j,l_{i,j})\mid (i,j)\in S_\Theta\}$ of L are known. Then, all the triples of L are known.* □

Let φ_Θ be a map in the set of n^3 variables $\mathbf{x}=\{x_{111},...,x_{nnn}\}$ such that $\varphi_\Theta(x_{ijk}) = \begin{cases} x_{ijk}, & \text{if } (i,j)\in S_\Theta, \\ x_{\alpha^m(i)\beta^m(j)\gamma^m(k)}, & \text{otherwise.}\end{cases}$, where $m=\min\{l\in[n]\mid (\alpha^l(i),\beta^l(j))\in S_\Theta\}$.

Theorem 1.4. *$LS(\Theta)$ corresponds to the set of zeros of the ideal* $I'=\langle \sum_{i\in[n]}\varphi_\Theta(x_{ijk})-1 \mid j,k\in[n]\rangle + \langle \sum_{j\in[n]}\varphi_\Theta(x_{ijk})-1 \mid i,k\in[n]\rangle + \langle \sum_{k\in[n]}\varphi_\Theta(x_{ijk})-1 \mid i,j\in[n]\rangle + \langle x_{ijk} \mid \alpha(i)=i,\beta(j)=j,\gamma(k)\neq k\rangle + \langle F(x_{ijk}) \mid (i,j)\in S_\Theta, k\in[n]\rangle = \langle \varphi_\Theta(I)\rangle \subseteq \mathbb{Q}[\varphi_\Theta(\mathbf{x})]$. □

Now, let $P = (p_{i,j}) \in PLS(n)$ be such that $p_{i,j} = \emptyset$, for all $(i,j) \notin S_\Theta$ and let c_P be the P-coefficient of symmetry of Θ. Thus, we know that $\Delta(\Theta) = c_P \cdot |LS_P(\Theta)|$ and we can calculate $|LS_P(\Theta)|$ starting from the set of solutions of an algebraic system of polynomial equations associated with Θ and P. Specifically, we obtain the following algorithm:

Algorithm 1.5 (Computation of $\Delta(\Theta)$)**.**
Input: $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$;
$\mathbf{n}_\alpha$, the number of cycles of α;
$P \in PLS(n)$ such that $p_{i,j} = \emptyset$, for all $(i,j) \notin S_\Theta$;
c_P, the P-coefficient of symmetry of Θ.
Output: $\Delta(\Theta)$, the number of Latin squares having Θ as an autotopism;

$I' := \langle \sum_{i \in [n]} \varphi_\Theta(x_{ijk}) - 1 \mid j,k \in [n] \rangle + \langle \sum_{j \in [n]} \varphi_\Theta(x_{ijk}) - 1 \mid i,k \in [n] \rangle + \langle \sum_{k \in [n]} \varphi_\Theta(x_{ijk}) - 1 \mid i,j \in [n] \rangle + \langle F(x_{ijk}) \mid (i,j) \in S_\Theta, k \in [n] \rangle$;

$I' := I' + \langle x_{ijl} - \delta^l_{p_{i,j}} \mid p_{i,j} \neq \emptyset, l \in [n] \rangle + \langle x_{ilp_{i,j}} - \delta^l_j \mid p_{i,j} \neq \emptyset, l \in [n] \rangle + \langle x_{ljp_{i,j}} - \delta^l_i \mid p_{i,j} \neq \emptyset, l \in [n] \rangle$; $\triangleright$ δ is Kronecker's delta.
$G :=$ Gröbner basis of I' with respect to any term ordering;
Delta $:= \dim_{\mathbb{Q}}(\mathbb{Q}[\varphi_\Theta(\mathbf{x})]/I')$; $\triangleright$ Delta is the cardinality of $V(I')$
RETURN $c_P \cdot$ Delta;

2 Number of Latin squares related to $\mathfrak{A}_8$ and $\mathfrak{A}_9$.

We have implemented Algorithm 1.5 in a Singular procedure [6] which improves running times of [4]. Moreover, we have obtained the number $\Delta(\Theta)$ corresponding to autotopisms of $\mathfrak{A}_8$ and $\mathfrak{A}_9$, as we can see in Table 1. The timing information, measured in seconds, has been taken from an *Intel Core 2 Duo Processor T5500, 1.66 GHz* with *Windows Vista* operating system.

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n	$\mathbf{l}_\alpha = \mathbf{l}_\beta$	$\mathbf{l}_\gamma$	Δ	r.t. (1.5)
8	(0,0,0,0,0,0,0,1)	(0,0,0,2,0,0,0,0)	1152	16
		(0,2,0,1,0,0,0,0)	1408	12
		(0,4,0,0,0,0,0,0)	3456	10
		(2,1,0,1,0,0,0,0)	1408	14
		(2,3,0,0,0,0,0,0)	3456	13
		(4,0,0,1,0,0,0,0)	3456	15
		(4,2,0,0,0,0,0,0)	8064	21
		(6,1,0,0,0,0,0,0)	17280	34
		(8,0,0,0,0,0,0,0)	40320	12
	(0,0,0,2,0,0,0,0)	(0,0,0,2,0,0,0,0)	106496	945
		(0,2,0,1,0,0,0,0)	188416	1163
		(0,4,0,0,0,0,0,0)	811008	255
		(2,1,0,1,0,0,0,0)	253952	731
		(2,3,0,0,0,0,0,0)	1007616	548
		(4,0,0,1,0,0,0,0)	712704	600
		(4,2,0,0,0,0,0,0)	2727936	660
		(6,1,0,0,0,0,0,0)	7741440	73
		(8,0,0,0,0,0,0,0)	23224320	1
	(0,1,0,0,0,1,0,0)	(2,0,0,0,0,1,0,0)	3456	5
		(2,0,2,0,0,0,0,0)	19008	3
	(1,0,0,0,0,0,1,0)	(1,0,0,0,0,0,1,0)	931	76
	(0,2,0,1,0,0,0,0)	(0,2,0,1,0,0,0,0)	16384	3
		(2,1,0,1,0,0,0,0)	16384	3
		(4,0,0,1,0,0,0,0)	147456	3
	(2,0,0,0,0,1,0,0)	(2,0,0,0,0,1,0,0)	19584	72
	(0,4,0,0,0,0,0,0)	(6,1,0,0,0,0,0,0)	198747095040	6515
		(8,0,0,0,0,0,0,0)	828396011520	9027
	(2,1,0,1,0,0,0,0)	(2,1,0,1,0,0,0,0)	8192	1
	(3,0,0,0,1,0,0,0)	(3,0,0,0,1,0,0,0)	388800	80
	(4,0,0,1,0,0,0,0)	(4,0,0,1,0,0,0,0)	7962624	2
	(4,2,0,0,0,0,0,0)	(4,2,0,0,0,0,0,0)	509607936	10
9	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,1)	2025	50
		(0,0,3,0,0,0,0,0,0)	7128	33
		(3,0,2,0,0,0,0,0,0)	12960	61
		(6,0,1,0,0,0,0,0,0)	71280	221
		(9,0,0,0,0,0,0,0,0)	362880	3
	(0,0,1,0,0,1,0,0,0)	(0,0,1,0,0,1,0,0,0)	15552	46
		(0,3,1,0,0,0,0,0,0)	124416	4
		(3,0,0,0,0,1,0,0,0)	62208	16
		(3,3,0,0,0,0,0,0,0)	1244160	17
	(1,0,0,0,0,0,0,1,0)	(1,0,0,0,0,0,0,1,0)	4096	56
	(0,0,3,0,0,0,0,0,0)	(6,0,1,0,0,0,0,0,0)	403813278720	221
		(9,0,0,0,0,0,0,0,0)	948109639680	1846
	(1,0,0,2,0,0,0,0,0)	(1,0,0,2,0,0,0,0,0)	12189696	11098
	(1,1,0,0,0,1,0,0,0)	(1,1,0,0,0,1,0,0,0)	69120	557
	(2,0,0,0,0,0,1,0,0)	(2,0,0,0,0,0,1,0,0)	438256	615
	(3,0,0,0,0,1,0,0,0)	(3,0,0,0,0,1,0,0,0)	3110400	112
	(4,0,0,0,1,0,0,0,0)	(4,0,0,0,1,0,0,0,0)	199065600	3

Table 1: Number of Latin squares related to $\mathfrak{A}_8$ and $\mathfrak{A}_9$.

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