RECENT ADVANCES ON TSAGAS-SOURLAS-SANTILLI ISOTOPOLOGY

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Abstract

Because the Lie Theory solely applies to linear systems, in 1978 Santilli proposed the isotopic lifting of Lie's theory for nonlinear systems, today known as the Lie-Santilli isotheory, via the reconstruction of linearity on the isotopic lifting of spaces and fields. In order to identify the proper mathematical background of the Lie-Santilli isotheory, Kadeisvili introduced in 1992 the notion of isocontinuity; Tsagas and Sourlas proposed in 1995 a for of isotopology defined over conventional fields; Santilli extended it in 1996 its formulation on isofields; and the authors conducted in 2003 a systematic study of the new isotopology. In this paper we outline the foundation of the new isotopology and present various advances.

Dedication: This paper is dedicated to the memory of Prof. Gr. Tsagas in admiration of his studies on Lie-Santilli isotheory.

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1 Introduction

Since the Lie theory solely applies to linear systems, while systems are generally nonlinear in reality, the physicist Santilli [7] proposed in 1978 the axiom-preserving isotopic lifting of Lie's theory for nonlinear systems, today known ad the *Lie-Santilli isotheory.* The proposal was based on the lifting of enveloping associative algebras, Lie algebras, Lie group, representation theory, as well as the spaces on which they are defined. Consistency was achieved via the reconstruction of linearity on isospaces, with the nonlinearity emerging in the projection of the isotheory into ordinary spaces. Santilli then continued his studies on isotopies in monographs [5] [6] of 1978-1991, the lifting of fields in Ref. [8] of 1993, the introduction of the new isodifferential calculus in Ref. [9] of 1996, and other contributions.

In order to identify the proper mathematical framework of the Lie-Santilli isotheory, the Russian mathematician Kadeisvili [4] introduced in 1992 the notion of isocontinuity. The Greek mathematicians Tsagas and Sourlas presented in monograph [7] a systematic study of the Lie-Santilli isotheory on isospaces over ordinary fields and in the subsequent papers [12] [13] of 1995 they introduced the notion of isomanifold as well as the first known form of isotopology formulated on conventional fields. Subsequently, Santilli [9] presented an extension of the isotopology to isofields and the new topology is today called the *Tsagas-Sourlas-Santilli isotopology.* We should also indicate that the Chinese mathematician Jiang conducted in monograph [3] of 2002 a comprehensive study of Santilli isonumber theory.

More recently, in 2001, the authors have presented in monograph [1] a systematic study of the Lie-Santilli theory within a full isotopic context, including the isotopies of spaces and fields. In the subsequent memoir [2] of 2003, the authors have presented a systematic study of the Tsagas-Sourlas-Santilli isotopology, with numerous advances.

As a result of all these studies there has been the emergence of a new branch of mathematics that applies not only to nonlinear systems, but also to nonlocal and non-Hamiltonian systems occurring in physics, chemistry, biology and other fields.

In this paper we review the foundations of the Tsagas-Sourlas-Santilli isotopology and present various advances.

The fundamental notion is that of *Santilli isoreal isofield* [8] ($\widehat{\mathbb{R}}^m$, $\widehat{+}$, $\widehat{\times}$) of dimension m, based on the isotopic lifting of the unit, called isounit, with expressions of the type $\widehat{I} = diag(n_1^2, n_2^2, ..., n_m^2)$, with $n_k = n_k(x, dx, d^2x, \tau, \delta, ...) \neq 0$, $\forall k \in \{1, ..., m\}$.

Particularly important for these studies is the classification of isofields into those of type I, occurring when the isounit is an arbitrary (non-null) element of the original field, and thsoe of type II, occurring when the isounit is not an element of the original field [8] (see also Jiang [3] for details).

Next, we have the isotopic lifting of metric spaces, for the first time introduced by Santilli [5] and [6]: $M(x, m, \mathbb{R}) \rightarrow \widehat{M}(\widehat{x}, \widehat{m}, \widehat{\mathbb{R}}_{II})$, where $\overline{\widehat{x}} = x \times \widehat{I}$, $\overline{\widehat{m}} = T \times \widehat{m}$ and $T = \widehat{I}^{-1}$ over the isoreal isofield $\widehat{\mathbb{R}}$ of the second type.

This last construction is also important due to its practical applications. As an example, we have that the valence bond characterizing every molecule is, structurally, non local-integral, due to the penetration of the packages of waves of the valence electrons, which requires the use of a non local-integral topology to be studied in this paper. In fact, Santilli proved in [10] that the use of both a new integer-differentiable isotopology and isotopic methods related for a Santilli's isounit $\widehat{I} = O \times exp^{-3r} \int \psi^{\dagger}(r) \times \psi(r)$, where O is an operator and $\psi(r)$ is the wave function of the degree electron has allowed, for the first time, to obtain an exact and invariant representation of all molecular characteristics. In this sense, it is necessary to use Santilli isospaces and isofields of the second type to get the results obtained in [7].

Tsagas and Sourlas [12] [13] introduced the isotopic lifting of topology over ordinary fields, subsequently extended by Santilli [9] to the isofields, by constructing an isotopy of the conventional space $\mathbb{R}^m$, defined by $T = \{\emptyset, \mathbb{R}^m, \cup_{i \in I} B_i\}$, where each of B_i is:

$$B_i = \{P = (P_1, ..., P_m) : \alpha_{i_k} < P_k < \beta_{i_k}; \alpha_{i_k}, \beta_{i_k} \in \mathbb{R}, \forall k \in \{1, ..., m\}\},$$

which can be considered as the Cartesian product of the topology of open intervals on the straight line, m times. The topological space $\{\mathbb{R}^m, T\}$ is denoted by $T^m(\mathbb{R})$ and it is called *real Cartesian topological space*.

In the isospace $\widehat{\mathbb{R}}^m$, they defined the isotopic lifted of the topology T as: $\widehat{T} = \{\emptyset, \widehat{\mathbb{R}}^m, \cup_{i \in I} \widehat{B}_i\}$, where each of $\widehat{B}_i$ is

$$\widehat{B}_i = \{\widehat{P} = (\widehat{P}_1, ..., \widehat{P}_m) : \widehat{\alpha}_{i_k} < \widehat{P}_k < \widehat{\beta}_{i_k}; \widehat{\alpha}_{i_k}, \widehat{\beta}_{i_k} \in \widehat{\mathbb{R}}_{n_k^2}, \forall k \in \{1, ..., m\}\}.$$

When $\overline{\widehat{\mathbb{R}}} = \mathbb{R}$, Tsagas and Sourlas pointed out that $\overline{\widehat{\mathbb{R}}}_{n_k^2} \simeq \mathbb{R}$ and that $\overline{\widehat{\mathbb{R}}}^m \simeq \mathbb{R}^m$. For this reason, they called the pair $\{\widehat{\mathbb{R}}^m, \widehat{T}\}$ as *real Cartesian isotopological space*, and they denoted it by $\widehat{T}^m(\widehat{\mathbb{R}})$. They also pointed out that $T^m(\mathbb{R}) \equiv \widehat{T}^m(\widehat{\mathbb{R}})$, which involves the coincidence between that new topology on $\widehat{\mathbb{R}}^m$ and the conventional one on $\mathbb{R}^m$, with the exception of $\widehat{I}$, which incorporates integrals terms. The resulting structure is actually known as *Tsagas-Sourlas Isotopology* or *Integro-differentiable topology*.

All the previous studies finally allowed to generalize in 2003 [2] the Tsagas-Sourlas-Santilli Isotopology for isofields of the types I and II, by making use of the isotopic construction model MCIM, introduced by the authors in 2001 [1], which generalizes in turn the model by Santilli in 1978 [7]. In particular, we provide an alternative formulation of Kadeisvili isocontinuity [4] from an analytic and a topological point of view.

Note finally that some results appearing in this paper will not be proved, due to restrictions on length length.

2 Isotopology by using the MCIM isotopic model

The generalization of the Tsagas - Sourlas Isotopology [11] to the case of isofields of the second type, proposed by Santilli in [9] was deeply analyzed by ourselves in a recent paper, appeared in 2003 (see [2]).

Such an analysis is made by using the isotopic model named MCIM, which we also introduced in [1]. Every isotopy can be reduced to this model and it is based on the use of so many isounits and $*$-laws as operations existing in the initial mathematical structure:

Proposition 2.1. *Fixed a mathematical structure* $(E, +, \times, \circ, \bullet, ...)$*, if we construct an isotopic lifting such that:*

a) Both primaries $*$, $\widehat{I}$ *and secondaries* $\star, \widehat{S}$ *elements of isotopy are used.*

b) $(E, \star, *, ...)$ *is a structure of the same type as the initial, which is endowed with isounits* $S, I, ...$*, with respect to* $\star, *, ...$*, respectively.*

c) I *is an unit with respect to* $*$ *in the corresponding general set* V*, being* $T = \widehat{I}^{-I} \in V$ *the associated isotopic element.*

So, by defining in the isotopic level the operations:

$$\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}; \quad \widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; ...$$

And being defined in the projection level:

$$\overline{\widehat{a}} = a * \widehat{I}; \quad \alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}; \quad \alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; ...$$

It is obtained that the isostructure $(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, ...)$ *is of the same type as the initial one.*

The study in [2] is made by taking into consideration both isotopic and projection levels. Equivalent results related to injective isotopies are also obtained. In the first place, it is verified Proposition 2.1 for topological spaces and for their elements and basic properties: isotopologies, isoclosed sets, isoopen sets, T_2, etc:

A *topological isospace* is every isospace endowed with a topological space structure. If, besides, such an isospace is an isotopic projection of a topological space, it is called *isotopological isospace.*

Similarly, they are defined concepts of *(iso)boundary isopoint, closure of a set, closed set, isointerior isopoint, interior of a set, open set, (iso)Hausdorff isospace and second countable isospace, among others.*

Proposition 2.2. *The space from which any topological isospace in the isotopic level is obtained can be endowed with the final topology relative to the mapping* $\mathbf{I}$.

The isotopic projection of a topological space is an isotopological isospace in the projection level. If such a projection is injective, then every topological isospace in such a level is, in fact, isotopological.

Similar results are obtained for the concepts of (iso)boundary isopoint, isointerior isopoint and (iso)Hausdorff isospace.

Next, we try to analyze the concept of isocontinuity of isofunctions, attempting to generalize the Kadeisvili isocontinuity [4]:

Let $\overline{\widehat{U}}$ be a $\overline{\widehat{\mathbb{R}}}$-isonormed vector isospace, where $\overline{\widehat{\mathbb{R}}}$ is an isofield of the type I. Let $\leq$ be the usual order in $\mathbb{R}$ and $\overline{\widehat{f}}$ an isofunction from $\overline{\widehat{U}}$ on $\overline{\widehat{\mathbb{R}}}$. We will say that $\overline{\widehat{f}}$ is a *Kadeisvili isocontinuous isofunction* in $\overline{\widehat{X}} \in \overline{\widehat{U}}$, if for all $\overline{\widehat{\epsilon}} > 0$, there exists $\overline{\widehat{\delta}} > 0$ such that, for all $\overline{\widehat{Y}} \in \overline{\widehat{U}}$ with $\overline{\mathbf{I}\left(\left\|(\pi \circ \mathbf{I})^{-1}\left(\overline{\widehat{X}} - \overline{\widehat{Y}}\right)\right\|\right)} < \overline{\widehat{\delta}}$, it is verified that:

$$\overline{\mathbf{I}\left(\left|(\pi \circ \mathbf{I})^{-1}\left(\overline{\widehat{f}}(\overline{\widehat{X}}) - \overline{\widehat{f}}(\overline{\widehat{Y}})\right)\right|\right)} < \overline{\widehat{\epsilon}}.$$

We will say that $\overline{\widehat{f}}$ is *Kadeisvili isocontinuous* in $\overline{\widehat{U}}$ if it is Kadeisvili isocontinuous in $\overline{\widehat{X}}$, for all $\overline{\widehat{X}} \in \overline{\widehat{U}}$.

The Kadeisvili isocontinuity is defined for isofields of the type I obtained from $\mathbb{R}$, which are endowed with the usual real order $\leq$. For this reason, it was proposed in [2] that the basic isofield can be endowed with an isoorder, according to:

Let $\widehat{K}$ be an isofield associated with a field K, endowed with an order $\leq$, by using an isotopology which preserves the inverse element with respect to the addition. We define the *isoorder* $\widehat{\leq}$ as $\widehat{a}\widehat{\leq}\widehat{b}$ if and only if $a \leq b$. If the isotopy is injective, the isoorder $\overline{\widehat{\leq}}$ en $\overline{\widehat{K}}$ is defined in the same way.

Proposition 2.3. *The isoorders $\widehat{\leq}$ and $\overline{\widehat{\leq}}$ are orders over $\widehat{K}$ and $\overline{\widehat{K}}$, of the same type as $\leq$.*

The Kadeosvili isocontinuity was generalized in [2] of the following way:

Let $\widehat{U}$ be a $\widehat{\mathbb{R}}$ isovectorspace with isonorm $\widehat{||}.\widehat{||} \equiv \widehat{||.||}$ and isoorder $\widehat{\leq}$, obtained from an isotopy compatible with respect to each one of the initial operations. It will be said that an isoreal isofunction $\widehat{f}$ of $\widehat{U}$ is *isocontinuous in* $\widehat{X} \in \widehat{U}$, if for all $\widehat{\epsilon}\widehat{>}\widehat{S}$, there exists $\widehat{\delta}\widehat{>}\widehat{S}$ such that for all $\widehat{Y} \in \widehat{U}$ with $\widehat{||}\widehat{X} \widehat{-} \widehat{Y}\widehat{||}\widehat{<}\widehat{\delta}$, it is verified that $\widehat{|}\widehat{f}(\widehat{X}) \widehat{-} \widehat{f}(\widehat{Y})\widehat{|}\widehat{<}\widehat{\epsilon}$. We will say that $\widehat{f}$ is *isocontinuous in* $\widehat{U}$ if it is isocontinuous in $\widehat{X}$, for all $\widehat{X} \in \widehat{U}$. Finally, when dealing with injective isotopies, the isocontinuity in the projection level is defined in a similar way.

Proposition 2.4. *The isocontinuity in $\widehat{U}$ is equivalent to the continuity in U. In the case of injective isotopies, both ones are equivalent to the one in $\overline{\widehat{U}}$.*

We are going to observe in the following example that, indeed, the previous definition of isocontinuity really generalizes the Kadeisvili isocontinuity:

Example 2.5. *Let us consider an isoreal isofield of the type I obtained from an isotopy of the isotopic element $\star \equiv +$, $\widehat{S} = 0$, $* \equiv \times$ and $\widehat{I} \in \mathbb{R}^+$ non null. Such an isotopy is injective and allow to obtain the isofield $\overline{\widehat{\mathbb{R}}} \equiv \mathbb{R}$, due to fixed $a \in \mathbb{R}$ it is $a = \overline{\widehat{a * T}}$, being $T = \widehat{I}^{-1}$.*

Such an isotopy preserves the inverse element and it is compatible with respect to $+, \circ, \bullet$ y $\times$. It is checked that $\overline{\widehat{\leq}} \equiv \leq$, $\overline{\widehat{+}} \equiv +$ and $\overline{\widehat{\circ}} \equiv \circ$.

The Kadeisvili isocontinuity is defined, in this case, of the following way (note that $\leq \equiv \overline{\widehat{\leq}}$):

Let $\widehat{f}$ be an isofunction of $\overline{\widehat{U}}$ on $\overline{\widehat{\mathbb{R}}}$. Then, $\overline{\widehat{f}}$ is Kadeisvili isocontinuous in $\overline{\widehat{X}} \in \overline{\widehat{U}}$ if for all $\overline{\widehat{\epsilon}} > 0$, there exists $\overline{\widehat{\delta}} > 0$ such that, if $\overline{\widehat{Y}} \in \overline{\widehat{U}}$ satisfies $\overline{\widehat{||}}\overline{\widehat{X}} - \overline{\widehat{Y}}\overline{\widehat{||}} < \overline{\widehat{\delta}}$, then $\overline{\widehat{|}}\overline{\widehat{f}}(\overline{\widehat{X}}) - \overline{\widehat{f}}(\overline{\widehat{Y}})\overline{\widehat{|}} < \overline{\widehat{\epsilon}}$. ◁

Proposition 2.6. *Under conditions of the Example 2.5, every (Kadeisvili) isocontinuous isofunction $\overline{\widehat{f}}$ en $\overline{\widehat{X}} \in \overline{\widehat{U}}$ is conventionally continuous in such a point. Consequently, every Kadeisvili isocontinuous isofunction $\overline{\widehat{f}}$ in $\overline{\widehat{U}}$ is conventionally continuous in $\overline{\widehat{U}}$.*

Proof.

Let us suppose $\overline{\widehat{X}} \in \overline{\widehat{\mathbb{R}}}$ and let $\overline{\widehat{f}}$ be an isocontinuous isofunction in $\overline{\widehat{X}}$. To see that $\overline{\widehat{f}}$ is a conventionally continuous isofunction, we fix $\epsilon > 0$. Let $\epsilon' > 0$ be such that $\overline{\widehat{\epsilon'}} = \epsilon' * \widehat{I} = \epsilon' \times \widehat{I} = \epsilon$. Due to the isocontinuity of $\overline{\widehat{f}}$ there exists $\overline{\widehat{\delta}} > 0$ such that, if $\overline{\widehat{Y}} \in \overline{\widehat{U}}$ is such that $\widehat{\overline{||}}\overline{\widehat{X}} - \overline{\widehat{Y}}\widehat{\overline{||}} < \overline{\widehat{\delta}}$, then $\widehat{\overline{|}}\overline{\widehat{f}}(\overline{\widehat{X}}) - \overline{\widehat{f}}(\overline{\widehat{Y}})\widehat{\overline{|}} < \overline{\widehat{\epsilon'}}$. But then, due to the compatibility with respect to $\circ$ of the isotopy constructing $\overline{\widehat{U}}$ and due to $\widehat{I}$ acts as a constant, which is positive, the previous condition is equivalent to the fact of if $\overline{\widehat{Y}} \in \overline{\widehat{U}}$ is such that $||\overline{\widehat{X}} - \overline{\widehat{Y}}|| = ||\overline{\widehat{X - Y}}|| = ||(X - Y) * \widehat{I}|| = ||X - Y|| \times \widehat{I} = \widehat{\overline{||}}\overline{\widehat{X - Y}}\widehat{\overline{||}} = \widehat{\overline{||}}\overline{\widehat{X}} - \overline{\widehat{Y}}\widehat{\overline{||}} < \overline{\widehat{\delta}}$, then $|\overline{\widehat{f}}(\overline{\widehat{X}}) - \overline{\widehat{f}}(\overline{\widehat{Y}})| = |\overline{\widehat{f(X)}} - \overline{\widehat{f(Y)}}| = |\overline{\widehat{f(X) - f(Y)}}| = |(f(X) - f(Y)) \times \widehat{I}| = |(f(X) - f(Y))| \times \widehat{I} = \widehat{\overline{|}}\overline{\widehat{f(X) - f(Y)}}\widehat{\overline{|}} = \widehat{\overline{|}}\overline{\widehat{f}}(\overline{\widehat{X}}) - \overline{\widehat{f}}(\overline{\widehat{Y}})\widehat{\overline{|}} < \overline{\widehat{\epsilon'}} = \epsilon$. So, it implies that $\overline{\widehat{f}}$ is conventionally continuous in X.

The consequence of the assert is then evident. $\square$

A problem which appears when $\widehat{I}$ is not constant, that is, it depends on external factors, is that the order is not equivalence with the isoorder in the model of Example 2.5. It involves that isocontinuity of [2] in not a particular case of the Kadeisvili isocontinuty. We will see it in the following:

Example 2.7. *Under conditions of Example 2.5, let us consider* $(U, \circ, \bullet) = (\mathbb{R}, +, \times)$, *although we take now as an isounit to:*

$$\widehat{I} = \widehat{I}(x) = \left\{ \begin{array}{c} 1, \text{ if } x = 0 \\ \frac{1}{x^2}, \text{ if } x \neq 0 \end{array} \right\}.$$

Then, $\widehat{\mathbb{R}}$ *is now given by the lifting:*

$$x \to \widehat{\widehat{x}} = x \times \widehat{I} = \left\{ \begin{array}{c} 0, \; if\; x = 0 \\ \frac{1}{x}, \; if\; x \neq 0 \end{array} \right\}.$$

So, we have $\widehat{\widehat{\mathbb{R}}} \equiv \mathbb{R}$*, and the isotopic lifting used is injective. Moreover, as it preserves the inverse element with respect to the addition, it has a perfect sense to consider the isoorder* $\widehat{\widehat{\leq}}$*, which is not equivalent to the usual one. Indeed, as an example, we have that* $\widehat{\widehat{2}\widehat{\leq}\widehat{3}}$*, due to* $2 \leq 3$*, but on the opposite* $\widehat{\widehat{2}} = \frac{1}{2} \geq \frac{1}{3} = \widehat{\widehat{3}}$*. It cannot be also said that* $\widehat{\widehat{\leq}}$ *is equivalent to the inverse order* $\leq$*, because* $\widehat{\widehat{0}\widehat{\leq}\widehat{2}}$*, due to* $0 \leq 2$*, being* $\widehat{\widehat{0}} = 0 \leq \frac{1}{2} = \widehat{\widehat{2}}$*.* $\triangleleft$

Example 2.8. *Under conditions of Example 2.5 let us consider:*

$$\widehat{I} = \widehat{I}(x) = \left\{ \begin{array}{c} 1, \; if\; x < 1 \\ \frac{x+1}{x}, if\; x \in [1,3) \\ \frac{x-2}{x}, \; if\; x \in [3,4) \\ 1, \; if\; x \geq 4 \end{array} \right\}.$$

Therefore, $\widehat{I}$ *is so positive defined, non singular and invertible, whose inverse is:*

$$T = T(x) = \left\{ \begin{array}{c} 1, \; if\; x < 1 \\ \frac{x+2}{x}, \; if\; x \in [1,2) \\ \frac{x-1}{x}, \; if\; x \in [2,4) \\ 1, \; if\; x \geq 4 \end{array} \right\}.$$

Then, the injective isotopic lifting from $\mathbb{R}$ *to* $\widehat{\widehat{\mathbb{R}}} = \mathbb{R}$*, is defined by:*

$$x \to \widehat{\widehat{x}} = \left\{ \begin{array}{c} x, \; if\; x < 1 \\ x+1, \; if\; x \in [1,3) \\ x-2, \; if\; x \in [3,4) \\ x, \; if\; x \geq 4 \end{array} \right\}.$$

Let us consider the function $f(x) = x - 1$*, which is conventionally continuous. We have then the isofunction:*

$$\overline{\widehat{f}}(x) = \left\{ \begin{array}{c} \overline{\widehat{x}} - 1, \ \textit{if} \ \overline{\widehat{x}} < 1 \\ \overline{\widehat{x}} + 2, \ \textit{if} \ \overline{\widehat{x}} \in [1,2) \\ \overline{\widehat{x}} - 2, \ \textit{if} \ \overline{\widehat{x}} \in [2,3) \\ \overline{\widehat{x}} - 1, \ \textit{if} \ \overline{\widehat{x}} \in [3,4) \\ \overline{\widehat{x}} - 3, \ \textit{if} \ \overline{\widehat{x}} \in [4,5) \\ \overline{\widehat{x}} - 1, \ \textit{if} \ \overline{\widehat{x}} \geq 5 \end{array} \right\}.$$

This isofunction is not isocontinuous in the Kadeisvili's sense, because it is not so, in particular, in $\overline{\widehat{x}} = \overline{\widehat{2}} = 3$. *Indeed, if we take* $\overline{\widehat{\epsilon}} = \frac{1}{2} = \overline{\widehat{\frac{1}{2}}}$, *we can find for each* $\overline{\widehat{\delta}} > 0$, *a certain* $\delta_0 \in (0,1)$, *with* $\delta_0 < \min_{\leq}\{1, \delta\}$ *and* $\overline{\widehat{y}} = \overline{\widehat{2 - \delta_0}}$, *and then* $\widehat{\overline{\widehat{|\overline{\widehat{2}} - \overline{\widehat{y}}|}}} = \widehat{\overline{\widehat{|\overline{\widehat{2}} - \overline{\widehat{2 - \delta_0}}|}}} = \widehat{\overline{|3 - 3 + \delta_0|}} = \widehat{\overline{|\widehat{\delta_0}|}} = \widehat{\overline{\widehat{|\widehat{\delta_0}|}}} = \overline{\widehat{|\widehat{\delta_0}|}} = \overline{\widehat{\delta_0}} = \delta_0 \leq \delta = \overline{\widehat{\delta}}$. *Then,* $\widehat{\overline{\widehat{|\overline{\widehat{f}}(\overline{\widehat{2}}) - \overline{\widehat{f}}(\overline{\widehat{y}})|}}} = \widehat{\overline{\widehat{|f(2) - f(y)|}}} = \widehat{\overline{\widehat{|1 - \widehat{1 - \delta_0}|}}} = \widehat{\overline{|2 - 1 + \delta_0|}} = \widehat{\overline{|1 + \delta_0|}} = \widehat{\overline{|\widehat{3 + \delta_0}|}} = \overline{|\widehat{3 + \delta_0}|} = \overline{\widehat{3 + \delta_0}} = 1 + \delta_0 \geq 1 > \frac{1}{2} = \overline{\widehat{\frac{1}{2}}} = \overline{\widehat{\epsilon}}$. *It implies that* $\overline{\widehat{f}}$ *is not isocontinuous in Kadeisvili's sense.*

So, we have found a function f *continuous such that its projection* $\overline{\widehat{f}}$ *is not isocontinuous in Kadeisvili's sense.* $\triangleleft$

Example 2.9. *Under conditions of Example 2.8 let us consider the function:*

$$f(x) = \left\{ \begin{array}{c} x + 1, \ \textit{if} \ x < 0 \\ x + 3, \ \textit{if} \ x \in [0,1) \\ x + 1, \ \textit{if} \ x \in [1,2) \\ x + 2, \ \textit{if} \ x \in [2,3) \\ x - 2, \ \textit{if} \ x \in [3,4) \\ x - 1, \ \textit{if} \ x \geq 4 \end{array} \right\}.$$

This function is not conventionally continuous. Apart from that, the isofunction in $\overline{\widehat{\mathbb{R}}}$ *is obtained:*

$$\overline{\widehat{f}}(x) = \overline{\widehat{x}} + 1 = \left\{ \begin{array}{c} x+1, \text{ if } x < 1 \\ x+2, \text{ if } x \in [1,3) \\ x-1, \text{ if } x \in [3,4) \\ x+1, \text{ if } x \geq 4 \end{array} \right\}.$$

Fixed $\overline{\widehat{\epsilon}} > 0$ and $\overline{\widehat{x}} \in \overline{\widehat{\mathbb{R}}}$, we have then that for all $\overline{\widehat{y}} \in \overline{\widehat{\mathbb{R}}}$ such that $\overline{\widehat{|\overline{\widehat{x}} - \overline{\widehat{y}}|}} < \overline{\widehat{\epsilon}}$, it is verified that $\overline{\widehat{|\overline{\widehat{f}}(\overline{\widehat{x}}) - \overline{\widehat{f}}(\overline{\widehat{y}})|}} = \overline{\widehat{|\overline{\widehat{x}} + 1 - \overline{\widehat{y}} - 1|}} = \overline{\widehat{|\overline{\widehat{x}} - \overline{\widehat{y}}|}} < \overline{\widehat{\epsilon}}$. In this way, as $\overline{\widehat{x}}$ is arbitrary in $\overline{\widehat{\mathbb{R}}}$, we deduce that $\overline{\widehat{f}}$ is Kadeisvili's isocontinuous in the whole of $\overline{\widehat{\mathbb{R}}}$.

So, a function f non conventionally continuous such that its projection $\overline{\widehat{f}}$ Kadeisvili's isocontinuous has been found. $\triangleleft$

The isocontinuity on isotopological isospaces is also analyzed in [2]:

An *isocontinuous isomapping* in the isotopic level between two topological isospaces $\widehat{M}$ and $\widehat{N}$ is every isomapping $\widehat{f} : \widehat{M} \to \widehat{N}$ preserving closures. The definition in the projection level is given in a similar way.

Proposition 2.10. *They are verified that:*

a) $\widehat{f}$ is isocontinuous if and only if the mapping f from which comes from is continuous. That result is similar in the projection level by using injective isotopies.

b) Every isoconstant isomapping is isocontinuous.

c) Isocontinuity is preserved by both topological composition and product.

Finally, the analysis of (iso)(pseudo)metric isospaces is also concreted:

Proposition 2.11. *Let $\widehat{M}$ be a $\widehat{K}$ isovectorspace, isotopic lifting of a vectorspace M, endowed with a (pseudo)metric d defined on an ordered field K, by using an isotopy which preserves the inverse element and compatible with respect to the addition in K. Then, the isofunction $\widehat{d}$ is an iso(pseudo)metric.*

Let $(\widehat{M}, d')$ be an (iso)(pseudo)metric $\widehat{K}$ isovectorspace, endowed with an isoorder $\widehat{\leq}$. $B_{d'}(\widehat{X_0}, \widehat{\epsilon}) = \{\widehat{X} \in \widehat{M} : d'(\widehat{X}, \widehat{X_0}) \widehat{<} \widehat{\epsilon}\}$ is called *metric ball* with center $\widehat{X_0} \in \widehat{M}$ and radius $\widehat{\epsilon} \widehat{>} \widehat{S}$. If M is endowed with a (pseudo)metric d, with $\widehat{d} = d'$, then every metric ball $B_{d'} = B_{\widehat{d}} = \widehat{B_d}$ in $\widehat{M}$, which is isotopic lifting of a metric ball B_d in M, is called *metric isoball* in $\widehat{M}$.

Proposition 2.12. *Under conditions of Proposition 2.11, if $B_d(X_0, \epsilon)$ is a metric ball in M, then $\widehat{B_d(X_0, \epsilon)} = B_{\widehat{d}}(\widehat{X_0}, \widehat{\epsilon})$ is a metric ball in $\widehat{M}$.*

A *metric neighborhood of an isopoint* $\widehat{X} \in \widehat{M}$ is a subset $\widehat{A} \subseteq \widehat{M}$ containing a metric ball centered in $\widehat{X}$. The set of metric neighborhoods of $\widehat{X}$ is denoted by $\widehat{\aleph}^{d'}_{\widehat{X}}$. Finally, if d' is the isoEuclidean isodistance over $\widehat{\mathbb{R}}^n$, the associated metric neighborhoods are called *isoEuclidean neighborhoods.*

Proposition 2.13. *Let d' and d'' two (iso)(pseudo)metrics over an isovectorspace $\widehat{M}$. It is verified that $\widehat{\aleph}^{d'}_{\widehat{X}} = \widehat{\aleph}^{d''}_{\widehat{X}}$ if and only if every metric ball $B_{d'}(\widehat{X}, \widehat{\epsilon})$ contains a ball $B_{d''}(\widehat{X}, \widehat{\rho})$ and every ball $B_{d''}(\widehat{X}, \widehat{\delta})$ contains a ball $B_{d'}(\widehat{X}, \widehat{\mu})$.*

Proposition 2.14. *Every isospace endowed with an (iso)(pseudo)metric is an isotopological isospace.*

So, the isocontinuity among iso(pseudo)metric isospaces generalize rightly the Kadeisvili's one:

Proposition 2.15. *Let $\widehat{f} : (\widehat{M}, d') \to (\widehat{N}, d'')$ be an isomapping between $\widehat{K}$-isospaces endowed with (iso)(pseudo)metric and let us consider $\widehat{X} \in \widehat{M}$. Then, $\widehat{f}$ is isocontinuous in $\widehat{X}$ if and only if for all $\widehat{\epsilon} \widehat{>} \widehat{S}$ there exists $\widehat{\delta} \in \widehat{K}$ such that $\widehat{\delta} \widehat{>} \widehat{S}$, and if $\widehat{Y} \in B_{d'}(\widehat{X}, \widehat{\delta})$, then it is verified that $\widehat{f}(\widehat{Y}) \in B_{d''}(\widehat{f}(\widehat{X}), \widehat{\epsilon})$.*

Proposition 2.16. *Let $\widehat{f} : \widehat{M} \to \widehat{N}$ be an isomapping between two isotopological isospaces $\widehat{M}$ and $\widehat{N}$. If conditions of the definition of isocontinuity are satisfied, then $\widehat{f}$ is isocontinuous if and only if $\widehat{f}^{-1}(\widehat{U})$ is an isoopen of $\widehat{M}$, for all isoopen $\widehat{U}$ of $\widehat{N}$.*

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