LOCAL CLASSIFICATION AND EXAMPLES OF AN IMPORTANT CLASS OF PARACONTACT METRIC MANIFOLDS

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ABSTRACT. We study paracontact metric (κ,μ)-spaces with $\kappa=-1$, equivalent to $h^2=0$ but not $h=0$. In particular, we will give an alternative proof of Theorem 3.2 of [11] and present examples of paracontact metric $(-1,2)$-spaces and $(-1,0)$-spaces of arbitrary dimension with tensor h of every possible constant rank. We will also show explicit examples of paracontact metric $(-1,\mu)$-spaces with tensor h of non-constant rank, which were not known to exist until now.

1. Introduction

Paracontact metric manifolds, the odd-dimensional analogue of paraHermitian manifolds, were first introduced in [10] and they have been the object of intense study recently, particularly since the publication of [14]. An important class among paracontact metric manifolds is that of the (κ,μ)-spaces, which satisfy the nullity condition [5]

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY), \tag{1}$$

for all X,Y vector fields on M, where κ and μ are constants and $h=\frac{1}{2}L_\xi\varphi$.

This class includes the paraSasakian manifolds [10, 14], the paracontact metric manifolds satisfying $R(X,Y)\xi=0$ for all X,Y [15], certain g-natural paracontact metric structures constructed on unit tangent sphere bundles [7], etc.

The definition of a paracontact metric (κ,μ)-space was motivated by the relationship between contact metric and paracontact geometry. More precisely, it was proved in [4] that any non-Sasakian contact metric (κ,μ)-space accepts two paracontact metric $(\widetilde{\kappa},\widetilde{\mu})$-structures with the same contact form. On the other hand, under certain natural conditions, every non-paraSasakian paracontact $(\widetilde{\kappa},\widetilde{\mu})$-space admits a contact metric (κ,μ)-structure compatible with the same contact form ([5]).

Paracontact metric (κ,μ)-spaces satisfy that $h^2=(\kappa+1)\phi^2$ but this condition does not give any type of restriction over the value of κ, unlike in contact metric geometry, because the metric of a paracontact metric manifold is not positive definite. However, it is useful to distinguish the cases $\kappa>-1$, $\kappa<-1$ and $\kappa=-1$. In the first two, equation (1) determines the curvature completely and either the tensor h or φh are diagonalisable [5]. The case $\kappa=-1$ is equivalent to $h^2=0$ but not to $h=0$. Indeed, there are examples of paracontact metric (κ,μ)-spaces with $h^2=0$ but $h\neq 0$, as was first shown in [2, 5, 8, 12].

However, only some particular examples were given of this type of space and no effort had been made to understand the general behaviour of the tensor h of a paracontact metric $(-1,\mu)$-space until the author published [11], where a local classification depending on the rank of h was given in Theorem 3.2. The author also provided explicit examples of all the possible constant values of the rank of h when $\mu=2$. She explained why the values $\mu=0$ and $\mu=2$ are special and studying them is enough. Finally, she showed some paracontact metric $(-1,0)$-spaces of any dimension with

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rank$(h) = 1$ and of paracontact metric $(-1, 0)$-spaces of dimension 5 and 7 for any possible constant rank of h. These were the first examples of this type with $\mu \neq 2$ and dimension greater than 3.

In the present paper, after the preliminaries section, we will give an alternative proof of Theorem 3.2 of [11] that does not use [13] and we will complete the examples of all the possible cases of constant rank of h by presenting $(2n + 1)$-dimensional paracontact metric $(-1, 0)$-spaces with rank$(h) = 2, \dots, n$. Lastly, we will also show the first explicit examples ever known of paracontact metric $(-1, 2)$-spaces and $(-1, 0)$-spaces with h of non-constant rank.

2. Preliminaries

An *almost paracontact structure* on a $(2n + 1)$-dimensional smooth manifold M is given by a $(1, 1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions [10]:

(i) $\eta(\xi) = 1$, $\varphi^2 = I - \eta \otimes \xi$,
(ii) the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of φ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

It follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and rank$(\varphi) = 2n$. If an almost paracontact manifold admits a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all X, Y on M, then $(M, \varphi, \xi, \eta, g)$ is called an *almost paracontact metric manifold*. Then g is necessarily of signature $(n + 1, n)$ and satisfies $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)$.

We can now define the *fundamental* 2*-form* of the almost paracontact metric manifold by $\Phi(X, Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then η becomes a contact form (i.e. $\eta \wedge (d\eta)^n \neq 0$) and $(M, \varphi, \xi, \eta, g)$ is said to be a *paracontact metric manifold*.

We can also define on a paracontact metric manifold the tensor field $h := \frac{1}{2}L_\xi\varphi$, which is symmetric with respect to g (i.e. $g(hX, Y) = g(X, hY)$, for all X, Y), anti-commutes with φ and satisfies $h\xi = \mathrm{tr}h = 0$ and the identity $\nabla\xi = -\varphi + \varphi h$ ([14]). Moreover, it vanishes identically if and only if ξ is a Killing vector field, in which case $(M, \varphi, \xi, \eta, g)$ is called a *K-paracontact manifold*.

An almost paracontact structure is said to be *normal* if and only if the tensor $[\varphi, \varphi] - 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ [14]:

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal paracontact metric manifold is said to be a *paraSasakian manifold* and is in particular K-paracontact. The converse holds in dimension 3 ([6]) but not in general in higher dimensions. However, it was proved in Theorem 3.1 of [11] that it also holds for $(-1, \mu)$-spaces. Every paraSasakian manifold satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2}$$

for every X, Y on M. The converse is not true, since Examples 3.8–3.11 of [11] and Examples 4.1 and 4.5 of the present one show that there are examples of paracontact metric manifolds satisfying equation (2) but with $h \neq 0$ (and therefore not K-paracontact or paraSasakian). Moreover, it is also clear in Example 4.5 that the rank of h does not need to be constant either, since h can be zero at some points and non-zero in others.

The main result of [11] is the following local classification of paracontact metric $(-1, \mu)$-spaces:

Theorem 2.1 ([11]). *Let M be a $(2n + 1)$-dimensional paracontact metric $(-1, \mu)$-space. Then we have one of the following possibilities:*

(1) *either $h = 0$ and M is paraSasakian,*
(2) *or $h \neq 0$ and rank$(h_p) \in \{1, \dots, n\}$ at every $p \in M$ where $h_p \neq 0$. Moreover, there exists a basis $\{\xi_p, X_1, Y_1, \dots, X_n, Y_n\}$ of $T_p(M)$ such that the only non-vanishing components of g are*

$$g_p(\xi_p, \xi_p) = 1, \quad g_p(X_i, Y_i) = \pm 1,$$

and

$$h_{p|\langle X_i,Y_i\rangle} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad h_{p|\langle X_i,Y_i\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where obviously there are exactly rank(h_p) *submatrices of the first type.*

If $n = 1$*, such a basis* $\{\xi_p, X_1, Y_1\}$ *also satisfies that*

$$\varphi_p X_1 = \pm X_1, \quad \varphi_p Y_1 = \mp Y_1,$$

and the tensor h *can be written as*

$$h_{p|\langle \xi_p, X_1, Y_1\rangle} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Many examples of paraSasakian manifolds are known. For instance, hyperboloids $\mathbb{H}^{2n+1}_{n+1}(1)$ and the hyperbolic Heisenberg group $\mathcal{H}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}$, [9]. We can also obtain (η-Einstein) paraSasakian manifolds from contact (κ,μ)-spaces with $|1-\frac{\mu}{2}| < \sqrt{1-\kappa}$. In particular, the tangent sphere bundle T_1N of any space form $N(c)$ with $c < 0$ admits a canonical η-Einstein paraSasakian structure, [3]. Finally, we can see how to construct explicitly a paraSasakian structure on a Lie group (see Example 3.4 of [11]) or on the unit tangent sphere bundle, [7].

On the other hand, until [11] only some types of non-paraSasakian paracontact metric $(-1,\mu)$-spaces were known:

- $(2n+1)$-dimensional paracontact metric $(-1,2)$-space with rank$(h) = n$, [5].
- 3-dimensional paracontact metric $(-1,2)$-space with rank$(h) = n = 1$, [12].
- 3-dimensional paracontact metric $(-1,0)$-space with rank$(h) = n = 1$. This example is not paraSasakian but it satisfies (2), [8].

The answer to why there seems to be only examples of paracontact metric $(-1,\mu)$-spaces with $\mu = 2$ or $\mu = 0$ is a $\mathcal{D}_c$-homothetic deformation, i.e. the following change of a paracontact metric structure (M,φ,ξ,η,g) [14]:

$$\varphi' := \varphi, \quad \xi' := \frac{1}{c}\xi, \quad \eta' := c\eta, \quad g' := cg + c(c-1)\eta\otimes\eta,$$

for some $c \neq 0$.

It is known that (φ',ξ',η',g') is again a paracontact metric structure on M and that K-paracontact and paraSasakian structures are also preserved. However, curvature conditions like $R(X,Y)\xi = 0$ are destroyed, since paracontact metric (κ,μ)-spaces become other paracontact metric (κ',μ')-spaces with

$$\kappa' = \frac{\kappa + 1 - c^2}{c^2}, \quad \mu' = \frac{\mu - 2 + 2c}{c}.$$

In particular, if (M,φ,ξ,η,g) is a paracontact metric $(-1,\mu)$-space, then the deformed manifold is another paracontact metric $(-1,\mu')$-space with $\mu' = \frac{\mu-2+2c}{c}$.

Therefore, given a $(-1,2)$-space, a $\mathcal{D}_c$-homothetic deformation with arbitrary $c \neq 0$ will give us another paracontact metric $(-1,2)$-space. Given a paracontact metric $(-1,0)$-space, if we $\mathcal{D}_c$-homothetically deform it with $c = \frac{2}{2-\mu} \neq 0$ for some $\mu \neq 2$, we will obtain a paracontact metric $(-1,\mu)$-space with $\mu \neq 2$. A sort of converse is also possible: given a $(-1,\mu)$-space with $\mu \neq 2$, a $\mathcal{D}_c$-homothetic deformation with $c = 1 - \frac{\mu}{2} \neq 0$ will give us a paracontact metric $(-1,0)$-space. The case $\mu = 0$, $h \neq 0$ is also special because the manifold satisfies (2) but it is not paraSasakian.

Examples of non-paraSasakian paracontact metric $(-1,2)$-spaces of any possible dimension and constant rank of h were presented in [11]:

Example 2.2 ($(2n+1)$-dimensional paracontact metric $(-1,2)$-space with rank$(h) = m \in \{1,\dots,n\}$)**.** *Let* $\mathfrak{g}$ *be the* $(2n+1)$*-dimensional Lie algebra with basis* $\{\xi, X_1, Y_1, \dots, X_n, Y_n\}$ *such that the only* *non-zero Lie brackets are:*

$$[\xi, X_i] = Y_i, \quad i = 1, \dots, m,$$

$$[X_i, Y_j] = \begin{cases} \delta_{ij}(2\xi + \sqrt{2}(1+\delta_{im})Y_m) & \\ \qquad +(1-\delta_{ij})\sqrt{2}(\delta_{im}Y_j + \delta_{jm}Y_i), & i, j = 1, \dots, m, \\ \delta_{ij}(2\xi + \sqrt{2}Y_i), & i, j = m+1, \dots, n, \\ \sqrt{2}Y_i, & i = 1, \dots, m, \ j = m+1, \dots, n. \end{cases}$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$\varphi\xi = 0, \quad \varphi X_i = X_i, \quad \varphi Y_i = -Y_i, \quad i = 1, \dots, n,$$

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \dots, n.$$

The only non-vanishing components of the metric are

$$g(\xi, \xi) = g(X_i, Y_i) = 1, \quad i = 1, \dots, n.$$

A straightforward computation gives that $hX_i = Y_i$ if $i = 1, \dots, m$, $hX_i = 0$ if $i = m+1, \dots, n$ and $hY_j = 0$ if $j = 1, \dots, n$, so $h^2 = 0$ and rank$(h) = m$. Furthermore, the manifold is a $(-1, 2)$-space.

Examples of non-paraSasakian paracontact metric $(-1, 0)$-spaces of any possible dimension and $\mathrm{rank}(h) = 1$ were also given in [11]:

Example 2.3 ($(2n+1)$-dimensional paracontact metric $(-1, 0)$-space with $\mathrm{rank}(h) = 1$)**.** *Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, Y_1, \dots, X_n, Y_n\}$ such that the only non-zero Lie brackets are:*

$$\begin{array}{lll} [\xi, X_1] = X_1 + Y_1, & [\xi, Y_1] = -Y_1, & [X_1, Y_1] = 2\xi, \\ [X_i, Y_i] = 2(\xi + Y_i), & [X_1, Y_i] = X_1 + Y_1, & [Y_1, Y_i] = -Y_1, \quad i = 2, \dots, n. \end{array}$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$\varphi\xi = 0, \quad \varphi X_1 = X_1, \quad \varphi Y_1 = -Y_1, \quad \varphi X_i = -X_i, \quad \varphi Y_i = Y_i, \quad i = 2, \dots, n,$$

$$\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \dots, n.$$

The only non-vanishing components of the metric are

$$g(\xi, \xi) = g(X_1, Y_1) = 1, \quad g(X_i, Y_i) = -1, \quad i = 2, \dots, n.$$

A straightforward computation gives that $hX_1 = Y_1$, $hY_1 = 0$ and $hX_i = hY_i = 0$, $i = 2, \dots, n$, so $h^2 = 0$ and rank$(h) = 1$.

Moreover, by basic paracontact metric properties and Koszul's formula we obtain that

$$\begin{gathered} \nabla_\xi X_1 = 0, \quad \nabla_\xi Y_1 = 0, \quad \nabla_\xi X_i = X_i, \quad \nabla_\xi Y_i = -Y_i, \quad i = 2, \dots, n, \\ \nabla_{X_i} Y_1 = \delta_{i1}\xi, \quad \nabla_{X_i} Y_j = \delta_{ij}(\xi + 2Y_i), \quad \nabla_{Y_1} X_1 = -\xi, \quad \nabla_{Y_i} X_j = -\delta_{ij}\xi, \quad i, j = 2, \dots, n, \\ \nabla_{X_1} X_j = 0, \quad \nabla_{Y_1} Y_1 = \nabla_{Y_1} Y_j = 0, \quad \nabla_{Y_j} Y_1 = Y_1, \quad i = 2, \dots, n, \end{gathered}$$

and thus

$$\begin{gathered} R(X_i, \xi)\xi = -X_i, \quad i = 1, \dots, n, \\ R(Y_i, \xi)\xi = -Y_i, \quad i = 1, \dots, n, \\ R(X_i, X_j)\xi = R(X_i, Y_j)\xi = R(Y_i, Y_j)\xi = 0, \quad i, j = 1, \dots, n. \end{gathered}$$

Therefore, the manifold is also a $(-1, 0)$-space.

To our knowledge, the previous example is the first paracontact metric $(-1,\mu)$-space with $h^2=0$, $h\neq 0$ and $\mu\neq 2$ that was constructed in dimensions greater than 3. For dimension 3, Example 4.6 of [8] was already known.

In dimension 5, there also exist examples of paracontact metric $(-1,0)$-space with $\mathrm{rank}(h)=2$ and in dimension 7 of $\mathrm{rank}(h)=2,3$, as shown in [11]. Higher-dimensional examples of paracontact metric $(-1,0)$-spaces with $\mathrm{rank}(h)\geq 2$ were not included, which will be remedied in Example 4.1. We will also see how to construct a 3-dimensional paracontact metric $(-1,0)$-space and $(-1,2)$-space where the rank of h is not constant.

3. New proof of Theorem 2.1

We will now present a revised proof of Theorem 2.1 that does not use [13] when $h\neq 0$ but constructs the basis explicitly.

Proof. Since $\kappa=-1$, we know from [5] that $h^2=0$. If $h=0$, then $R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y)$, for all X,Y on M and ξ is a Killing vector field, so Theorem 3.1 of [11] gives us that the manifold is paraSasakian.

If $h\neq 0$, then let us take a point $p\in M$ such that $h_p\neq 0$. We know that ξ is a global vector field such that $g(\xi,\xi)=1$, that $h\xi=0$ and that h is self-adjoint, so $Ker\eta_p$ is h-invariant and $h_p: Ker\eta_p\mapsto Ker\eta_p$ is a non-zero linear map such that $h_p^2=0$. We will now construct a basis $\{X_1,Y_1,\dots,X_n,Y_n\}$ of $Ker\eta_p$ that satisfies all of our requirements.

Take a non-zero vector $v\in Ker\eta_p$ such that $h_pv\neq 0$, which we know exists because $h_p\neq 0$. Then we write $Ker\eta_p=L_1\oplus L_1^{\perp}$, where $L_1=\langle v,h_pv\rangle$. Both L_1 and $L_1^{\perp}$ are h_p-invariant because h_p is self-adjoint. Moreover, $g_p(v,h_pv)\neq 0$ because $g_p(h_pv,h_pv)=0=g_p(h_pv,w)$ for all $w\in L_1^{\perp}$, $h_pv\neq 0$ and g is a non-degenerate metric.We now distinguish two cases:

(1) If $g_p(v,v)=0$, then we can take $X_i=\frac{1}{\sqrt{|g_p(v,h_pv)|}}v$ and $Y_i=\frac{1}{\sqrt{|g_p(v,h_pv)|}}h_pv$, which satisfy that $g_p(X_i,X_i)=0=g_p(Y_i,Y_i)$, $g_p(X_i,Y_i)=\pm 1$ and $h_pX_i=Y_i$.
(2) If $g_p(v,v)\neq 0$, then $v'=v-\frac{g_p(v,v)}{g_p(v,h_pv)}h_pv$ satisfies that $g_p(v',v')=0$, so we can take $X_i=\frac{1}{\sqrt{|g_p(v',h_pv)'|}}v'$, $Y_i=\frac{1}{\sqrt{|g_p(v',h_pv')|}}hv'$. We have again that $g_p(X_i,X_i)=0=g_p(Y_i,Y_i)$, $g_p(X_i,Y_i)=\pm 1$ and $h_pX_i=Y_i$.

In both cases, $L_1=\langle X_i,Y_i\rangle$, so we now take a non-zero vector $v\in L_1^{\perp}$ and check if $h_pv\neq 0$. We know that we can take v such that $h_pv\neq 0$ in this step as many times as the rank of h_p, which is at minimum 1 (since $h_p\neq 0$) and at most n because $\dim Ker\eta_p=2n$ and the spaces L_1 have dimension 2.

If we denote by m the rank of h_p, then we can write $Ker\eta_p$ as the following direct sum of mutually orthogonal subspaces:

$$Ker\eta_p=L_1\oplus L_2\oplus\cdots\oplus L_m\oplus V=\langle X_1,Y_1,\dots,X_m,Y_m\rangle\oplus V,$$

where $h_pv=0$ for all $v\in V$. Each L_i is of signature $(1,1)$ because $\{\tilde{X}_i=\frac{1}{\sqrt{2}}(X_i+Y_i),\tilde{Y}_i=\frac{1}{\sqrt{2}}(X_i-Y_i)\}$ is a pseudo-orthonormal basis such that $g_p(\tilde{X}_i,\tilde{X}_i)=-g_p(\tilde{Y}_i,\tilde{Y}_i)=g_p(X_i,Y_i)=\pm 1$, $g_p(\tilde{X}_i,\tilde{Y}_i)=0$. Therefore, $\langle X_1,Y_1,\dots,X_m,Y_m\rangle$ is of signature (m,m) and, since $Ker\eta_p$ is of signature (n,n), we can take a pseudo-orthonormal basis $\{v_1,\dots,v_{n-m},w_1,\dots,w_{n-m}\}$ of V such that $g_p(v_i,v_j)=\delta_{ij}$ and $g_p(w_i,w_j)=-\delta_{ij}$. Therefore, it suffices to define $X_{m+i}=\frac{1}{\sqrt{2}}(v_i+w_i),Y_{m+i}=\frac{1}{\sqrt{2}}(v_i-w_i)$ to have $g_p(X_i,X_i)=0=g_p(Y_i,Y_i)$, $g_p(X_i,Y_i)=1$ and $h_pX_i=h_pY_i=0$, $i=m+1,\dots,n$.

If $n=1$, then $\varphi_pX_1=\pm X_1$ and $\varphi_pY_1=\mp Y_1$ follow directly from paracontact metric properties and the definition of the basis $\{X_1,Y_1,\dots,X_n,Y_n\}$. □

It is worth mentioning that Theorem 2.1 is true only pointwise, i.e. $\operatorname{rank}(h_p)$ does not need to be the same for every $p \in M$. Indeed, we will see in Examples 4.3 and 4.5 that we can construct paracontact metric $(-1,\mu)$-spaces such that h is zero in some points and non-zero in others.

4. New examples

We will first present an example of $(2n+1)$-dimensional paracontact metric $(-1,0)$-space with rank of h greater than 1. This means that, together with Examples 2.2 and 2.3, we have examples of paracontact metric $(-1,\mu)$-spaces of every possible dimension and constant rank of h when $\mu=0$ and $\mu=2$.

Example 4.1 ($(2n{+}1)$-dimensional paracontact metric $(-1,0)$-space with $\operatorname{rank}(h)=m\in\{2,\dots,n\}$)**.** *Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Lie algebra with basis $\{\xi,X_1,Y_1,\dots,X_n,Y_n\}$ such that the only non-zero Lie brackets are:*

$$
\begin{aligned}
&[\xi,X_1]=X_1+X_2+Y_1, && [\xi,Y_1]=-Y_1+Y_2,\\
&[\xi,X_2]=X_1+X_2+Y_2, && [\xi,Y_2]=Y_1-Y_2,\\
&[\xi,X_i]=X_i+Y_i,\quad i=3,\dots,m, && [\xi,Y_i]=-Y_i,\quad i=3,\dots,m,
\end{aligned}
$$

$$
[X_i,X_j]=\begin{cases}\sqrt{2}X_1, & \text{if } i=1,\ j=2,\\ -\sqrt{2}X_j & \text{if } i=2,\ j=3\dots,m,\\ \sqrt{2}[\xi,X_i], & \text{if } i=1,\dots,m,\ j=m+1,\dots,n,\end{cases}
$$

$$
[Y_i,Y_j]=\begin{cases}\sqrt{2}(-Y_1+Y_2), & \text{if } i=1,\ j=2,\\ \sqrt{2}Y_j, & \text{if } i=1,2,\ j=3,\dots,m,\end{cases}
$$

$$
[X_i,Y_i]=\begin{cases}2\xi+\sqrt{2}(X_2+Y_2) & \text{if } i=1,\\ -2\xi+\sqrt{2}X_1, & \text{if } i=2,\\ -2\xi+\sqrt{2}(X_1-X_2-Y_2), & \text{if } i=3,\dots,m,\\ -2\xi-\sqrt{2}X_i, & \text{if } i=m+1,\dots,n,\end{cases}
$$

$$
\underset{i\neq j}{[X_i,Y_j]}=\begin{cases}\sqrt{2}(Y_1+X_2) & \text{if } i=1,\ j=2,\\ \sqrt{2}X_1, & \text{if } i=2,\ j=1,\\ \sqrt{2}X_j, & \text{if } i=1,2,\ j=3,\dots,m,\\ \sqrt{2}Y_i, & \text{if } i=3,\dots,m,\ j=2,\\ -\sqrt{2}[\xi,Y_j], & \text{if } i=m+1,\dots,n,\ j=1,\dots,m.\end{cases}
$$

If we denote by G the Lie group whose Lie algebra is $\mathfrak{g}$, we can define a left-invariant paracontact metric structure on G the following way:

$$
\varphi\xi=0,\quad \varphi X_i=X_i,\quad \varphi Y_i=-Y_i,\quad i=1,\dots,n,
$$
$$
\eta(\xi)=1,\quad \eta(X_i)=\eta(Y_i)=0,\quad i=1,\dots,n.
$$

The only non-vanishing components of the metric are

$$
g(\xi,\xi)=g(X_1,Y_1)=1,\quad g(X_i,Y_i)=-1,\quad i=2,\dots,n.
$$

A straightforward computation gives that $hX_i=Y_i$, $i=1,\dots,m$, $hX_i=0$, $i=m+1,\dots,n$ and $hY_i=0$, $i=1,\dots,n$, so $h^2=0$ and rank$(h)=m$.

Moreover, very long but direct computations give that

$$
\begin{aligned}
R(X_i,\xi)\xi&=-X_i,\quad i=1,\dots,n,\\
R(Y_i,\xi)\xi&=-Y_i,\quad i=1,\dots,n,\\
R(X_i,X_j)\xi=R(X_i,Y_j)\xi&=R(Y_i,Y_j)\xi=0,\quad i,j=1,\dots,n.
\end{aligned}
$$

Therefore, the manifold is also a $(-1,0)$*-space.*

Remark 4.2. *Note that the previous example is only possible when* $n \geq 2$*. If* $n = 1$*, then we can only construct examples of rank*$(h) = 1$*, as in Example 2.3.*

In the definition of the Lie algebra of the previous example, some values of i *and* j *are not possible for* $m = 2$ *or* $m = n$*. In that case, removing the affected Lie brackets from the definition will give us valid examples nonetheless.*

We will present now an example of 3-dimensional paracontact metric $(-1,2)$-space and one of 3-dimensional paracontact metric $(-1,0)$-space, such that $\operatorname{rank}(h_p) = 0$ or 1 depending on the point p of the manifold. These are the first examples of paracontact metric (κ,μ)-spaces with h of non-constant rank that are known.

Example 4.3 (3-dimensional paracontact metric $(-1,2)$-space with $\operatorname{rank}(h_p)$ not constant)**.** *We consider the manifold* $M = \mathbb{R}^3$ *with the usual cartesian coordinates* (x,y,z)*. The vector fields*

$$e_1 = \frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}$$

are linearly independent at each point of M*. We can compute*

$$[e_1,e_2] = 2\,\xi, \quad [e_1,\xi] = -x\,e_2, \quad [e_2,\xi] = 0.$$

We define the semi-Riemannian metric g *as the non-degenerate one whose only non-vanishing components are* $g(e_1,e_2) = g(\xi,\xi) = 1$*, and the 1-form* η *as* $\eta = 2ydx + dz$*, which satisfies* $\eta(e_1) = \eta(e_2) = 0$*,* $\eta(\xi) = 1$*. Let* φ *be the* $(1,1)$*-tensor field defined by* $\varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi\xi = 0$*. Then*

$$\begin{aligned} d\eta(e_1,e_2) &= \frac{1}{2}(e_1(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1,e_2])) = -1 = -g(e_1,e_2) = g(e_1,\varphi e_2),\\ d\eta(e_1,\xi) &= \frac{1}{2}(e_1(\eta(\xi)) - \xi(\eta(e_1)) - \eta([e_1,\xi]) = 0 = g(e_1,\varphi\xi),\\ d\eta(e_2,\xi) &= \frac{1}{2}(e_2(\eta(\xi)) - \xi(\eta(e_2)) - \eta([e_2,\xi]) = 0 = g(e_2,\varphi\xi). \end{aligned}$$

Therefore, (φ,ξ,η,g) *is a paracontact metric structure on* M*.*

Moreover, $h\xi = 0$*,* $he_1 = xe_2$*,* $he_2 = 0$*. Hence,* $h^2 = 0$ *and, given* $p = (x,y,z) \in \mathbb{R}^3$*, rank*$(h_p) = 0$ *if* $x = 0$ *and rank*$(h_p) = 1$ *if* $x \neq 0$*.*

Let ∇ *be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul's formula*

(3) $$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),$$

we can compute

$$\nabla_\xi\xi = 0, \quad \nabla_{e_1}\xi = -e_1 - xe_2, \quad \nabla_{e_2}\xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2,$$
$$\nabla_{e_1}e_1 = x\xi, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_1}e_2 = \xi, \quad \nabla_{e_2}e_1 = -\xi.$$

Using the following definition of the Riemannian curvature

(4) $$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

we obtain

$$R(e_1,\xi)\xi = -e_1 + 2he_1, \quad R(e_2,\xi)\xi = -e_2 + 2he_2, \quad R(e_1,e_2)\xi = 0,$$

so the paracontact metric manifold M *is also a* $(-1,2)$*-space.*

Remark 4.4. *The previous example does not contradict Theorem 2.1, as we will see by constructing explicitly the basis of the theorem on each point* p *where* $h_p \neq 0$*, i.e., on every point* $p = (x,y,z)$ *such that* $x \neq 0$*.*

Indeed, let us take a point $p = (x,y,z) \in \mathbb{R}^3$*. If* $x \neq 0$*, then we define* $X_1 = \frac{e_{1p}}{\sqrt{|x|}}$*,* $Y_1 = \frac{h_p e_{1p}}{\sqrt{|x|}}$*. We obtain that* $\{\xi_p, X_1, Y_1\}$ *is a basis of* $T_p(\mathbb{R}^3)$ *that satisfies that:*

- *the only non-vanishing components of g are* $g_p(\xi_p, \xi_p) = 1, \quad g_p(X_1, Y_1) = sign(x)$,
- *the tensor h can be written as* $h_{p|\langle \xi_p, X_1, Y_1 \rangle} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,
- $\varphi_p \xi = 0, \quad \varphi_p X_1 = X_1, \quad \varphi_p Y_1 = -Y_1.$

Example 4.5 (3-dimensional paracontact metric $(-1, 0)$-space with rank(h_p) not constant)**.** *We consider the manifold $M = \mathbb{R}^3$ with the usual cartesian coordinates (x, y, z). The vector fields*

$$e_1 = \frac{\partial}{\partial x} + xe^{-2z} \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. We can compute

$$[e_1, e_2] = 2\,\xi, \quad [e_1, \xi] = 2xe^{-2z}\, e_2, \quad [e_2, \xi] = 0.$$

We define the semi-Riemannian metric g as the non-degenerate one whose only non-vanishing components are $g(e_1, e_2) = g(\xi, \xi) = 1$, and the 1-form η as $\eta = 2ydx + dz$, which satisfies $\eta(e_1) = \eta(e_2) = 0$, $\eta(\xi) = 1$. Let φ be the $(1, 1)$-tensor field defined by $\varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi \xi = 0$. Then

$$d\eta(e_1, e_2) = \frac{1}{2}(e_1(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1, e_2])) = -1 = -g(e_1, e_2) = g(e_1, \varphi e_2),$$

$$d\eta(e_1, \xi) = \frac{1}{2}(e_1(\eta(\xi)) - \xi(\eta(e_1)) - \eta([e_1, \xi]) = 0 = g(e_1, \varphi \xi),$$

$$d\eta(e_2, \xi) = \frac{1}{2}(e_2(\eta(\xi)) - \xi(\eta(e_2)) - \eta([e_2, \xi]) = 0 = g(e_2, \varphi \xi).$$

Therefore, (φ, ξ, η, g) is a paracontact metric structure on M.

Moreover, $h\xi = 0$, $he_1 = -2xe^{-2z}e_2$, $he_2 = 0$. Hence, $h^2 = 0$ and, given $p = (x, y, z) \in \mathbb{R}^3$, rank$(h_p) = 0$ if $x = 0$ and rank$(h_p) = 1$ if $x \neq 0$.

Let ∇ be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul's formula (3)*, we can compute*

$$\nabla_\xi \xi = 0, \quad \nabla_{e_1} \xi = -e_1 + 2xe^{-2z} e_2, \quad \nabla_{e_2} \xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2,$$

$$\nabla_{e_1} e_1 = -2xe^{-2z} \xi, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_2 = \xi, \quad \nabla_{e_2} e_1 = -\xi.$$

Using now (4)*, we obtain*

$$R(e_1, \xi)\xi = -e_1, \quad R(e_2, \xi)\xi = -e_2, \quad R(e_1, e_2)\xi = 0,$$

so the paracontact metric manifold M is also a $(-1, 0)$-space.

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