Two stochastic models related with an example coming from group representation theory[1]

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[1] Joint work with Pablo Román

Outline

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One dimensional Markov processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, a (1-D) Markov process with state space $\mathcal{S} \subset \mathbb{R}$ is a collection of $\mathcal{S}$-valued random variables $\{X_t : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) such that

$$\mathbb{P}(X_{t+s} \leq y | X_s = x, X_\tau, 0 \leq \tau < s) = \mathbb{P}(X_{t+s} \leq y | X_s = x)$$

for all $s, t > 0$. This is what is called the Markov property.

The main goal is to find a description of the transition probabilities (discrete case) or the transition density (continuous case)

$$P_{x,y}(t) \equiv \mathbb{P}(X_t = y | X_0 = x), \quad x, y \in \mathcal{S} \subset \mathbb{Z}$$

$$p(t; x, y) \equiv \frac{\partial}{\partial y} \mathbb{P}(X_t \leq y | X_0 = x), \quad x, y \in \mathcal{S} \subset \mathbb{R}$$

Define the transition operator

$$(T_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x], \quad t \geq 0, \quad f \in \mathfrak{B}(\mathcal{S})$$

The family $\{T_t, t > 0\}$ has the *semigroup property* $T_{s+t} = T_s T_t$ and it is completely determined by its infinitesimal operator $\mathcal{A}$ given by

$$(\mathcal{A} f)(x) = \lim_{s \downarrow 0} \frac{(T_s f)(x) - f(x)}{s}$$

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Examples related to OP

There are 3 important cases related to OP:

1. Random walks: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$.
Transitions are only allowed between adjacent states. Therefore the infinitesimal operator can be written as a semi-infinite tridiagonal matrix P which coincides with the one-step transition probabilities

$$\mathcal{A}f(i) = Pf(i) = a_i f(i+1) + b_i f(i) + c_i f(i-1), \quad f \in \mathfrak{B}(\mathcal{S})$$

The n-step transition probability matrix is then given by $P^{(n)} = P^n$.
Some examples related to OP are the gambler's ruin, urn models, the Ehrenfest model or the Laplace-Bernoulli model.

2. Birth and death processes: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = [0, \infty)$.
Again, the transitions are only allowed between adjacent states, but now time is continuous. The transition times are exponentially distributed.
The infinitesimal operator is now a semi-infinite tridiagonal matrix $\mathcal{A}$

$$\mathcal{A}f(i) = \lambda_i f(i+1) - (\lambda_i + \mu_i) f(i) + \mu_i f(i-1), \quad f \in \mathfrak{B}(\mathcal{S})$$

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$$P'(t) = \mathcal{A}P(t), \quad P'(t) = P(t)\mathcal{A}, \quad P(0) = I$$

Some examples of birth-and-death processes related to OP are the $M/M/k$ queue ($k \geq 1$) or linear birth-and-death processes.

3. Diffusion processes: $\mathcal{S} = (a, b), -\infty \leq a < b \leq \infty$, $\mathcal{T} = [0, \infty)$. Starting at $X_0 = x$, the expected value of a small displacement $X_t - X_0$ is approximately $t\mu(x)$ (drift coefficient) while the second moment or variance is approximately $t\sigma^2(x)$ (diffusion coefficient). The infinitesimal operator is now a second-order differential operator

$$\mathcal{A}_x f = \frac{1}{2}\sigma^2(x)f''(x) + \tau(x)f'(x), \quad f \in \mathfrak{B}(\mathcal{S}) \cap C^2(\mathcal{S})$$

The transition density $p(t; x, y)$ satisfies the Kolmogorov equations (backward and forward) with initial conditions

$$\frac{\partial}{\partial t}p(t; x, y) = \mathcal{A}_x p(t; x, y), \quad \frac{\partial}{\partial t}p(t; x, y) = \mathcal{A}_y^* p(t; x, y)$$

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Spectral methods

Given a infinitesimal operator $\mathcal{A}$, if we can find a spectral measure $\omega(x)$ associated with $\mathcal{A}$, and a set of orthogonal eigenfunctions $f(i,x)$ such that

$$\mathcal{A}f(i,x) = \lambda(i,x)f(i,x)$$

then it is possible to find spectral representations of the transition probabilities:

1. Random walks: $f(i,x) = q_i(x), \lambda(i,x) = x, i \in \mathcal{S}, x \in [-1,1]$.

$$\mathbb{P}(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2}\int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

2. Birth-and-death processes: $f(i,x) = q_i(x), \lambda(i,x) = -x, i \in \mathcal{S}, x \in [0,\infty]$.

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Bivariate Markov processes

Now consider a bivariate or 2-component Markov process of the form

$$\{(X_t, Y_t) : t \in \mathcal{T}\}, \quad X_t \in \mathcal{S} \subset \mathbb{R}, \quad Y_t \in \{1, 2, \dots, N\}$$

The first component is the level and the second component is the phase. Now the transition probabilities can be written in terms of an $N \times N$ matrix-valued function $\boldsymbol{P}(t; x, A)$ whose entry (i, j) gives

$$\boldsymbol{P}_{ij}(t; x, A) = \mathbb{P}(X_t \in A, Y_t = j | X_0 = x, Y_0 = i)$$

The transition operator is now matrix-valued and acts on all column vector-valued functions

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The infinitesimal operator $\mathcal{A}$ is also *matrix-valued*

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Ideas behind: *random evolutions*
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Processes related to MVOP

As in the scalar case, there are two situations where matrix-valued orthogonal polynomials (MVOP) can play an important role:

1. Quasi-birth-and-death processes: $\mathcal{S}$ discrete. The infinitesimal operator is now a block-tridiagonal matrix $\mathcal{A}$

$$\mathcal{A}\boldsymbol{f}(i) = \boldsymbol{A}_n\boldsymbol{f}(i+1) + \boldsymbol{B}_n\boldsymbol{f}(i) + \boldsymbol{C}_n\boldsymbol{f}(i-1), \quad \boldsymbol{f} \in \mathfrak{B}(\mathcal{S}^N)$$

where each block $\boldsymbol{A}_n, \boldsymbol{B}_n, \boldsymbol{C}_n$ is a $N \times N$ matrix with the probabilistic properties depending on the case (discrete or continuous time). The transition probabilities and the Kolmogorov equations can be derived from $\mathcal{A}$.

2. Switching diffusion processes: $\mathcal{S}$ continuous. The infinitesimal operator $\mathcal{A}$ is now a second-order matrix-valued differential operator (Berman, 1994)

$$\mathcal{A}_x = \frac{1}{2}\boldsymbol{A}(x)\partial_x^2 + \boldsymbol{B}(x)\partial_x^1 + \boldsymbol{Q}(x)\partial_x^0$$

where $\boldsymbol{A}(x)$ and $\boldsymbol{B}(x)$ are diagonal matrices and $\boldsymbol{Q}(x)$ is the infinitesimal operator of a finite continuous-time Markov chain. Again, the transition density and the Kolmogorov equations can be derived from $\mathcal{A}$.

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The main tool to study spectral methods will be the theory of MVOP.

Spectral methods

Given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a spectral weight matrix $\boldsymbol{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\boldsymbol{F}(i,x)$ such that

$$\mathcal{A}\boldsymbol{F}(i,x) = \boldsymbol{\Lambda}(i,x)\boldsymbol{F}(i,x),$$

then it is possible to find spectral representations of the transition probabilities:

1. Quasi-birth-and-death processes:
$\boldsymbol{F}(i,x) = \boldsymbol{\Phi}_i(x), \boldsymbol{\Lambda}(i,x) = x\boldsymbol{I}, i \in \{0,1,2,\ldots\}, x \in [-1,1]$ (Grünbaum and Dette-Reuther-Studden-Zygmunt, 2007).

$$\boldsymbol{P}_{ij}^n = \left(\int_{-1}^1 x^n \boldsymbol{\Phi}_i(x) \mathrm{d}\boldsymbol{W}(x)\boldsymbol{\Phi}_j^*(x)\right)\left(\int_{-1}^1 \boldsymbol{\Phi}_j(x) \mathrm{d}\boldsymbol{W}(x)\boldsymbol{\Phi}_j^*(x)\right)^{-1}$$

Same result if time is continuous (Dette-Reuther, 2010).

2. Switching diffusion processes:
$\boldsymbol{F}(i,x) = \boldsymbol{\Phi}_i(x), \boldsymbol{\Lambda}(i,x) = \boldsymbol{\Gamma}_i, i \in \{0,1,2,\ldots\}, x \in (a,b)$ (MdI, 2012).

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Outline

Matrix-valued spherical functions

Spherical functions associated with groups of the form G/K where (G, K) is a Gel'fand pair are very much related with OP (Helgason, Vilenkin, Klimyk).
They are eigenfunctions of the Casimir operator associated with the group.
The extension to the matrix-valued case was started by Tirao (1977) but it was not until very recently where the connection with MVOP was discovered by Grünbaum-Pacharoni-Tirao (2003):

1. Complex projective space: $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$. Grünbaum-Pacharoni-Tirao (2002). Later it was found the relation with stochastic processes by Grünbaum-MdI (2008), Grünbaum-Pacharoni-Tirao (2012) and MdI (2012).

2. Complex hyperbolic plane: $H_2(\mathbb{C}) = \mathrm{SU}(2,1)/\mathrm{U}(2)$. Pacharoni-Román-Tirao (2006). Dual to the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$.

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In all cases (and others not mentioned) an explicit expression of the weight matrix, the second-order differential operator, the three-term recurrence relation and other structural formulas were derived for the matrix-valued spherical functions. In most of the cases the relation with MVOP was also given.

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The pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \text{diag } \mathrm{SU}(2))$

Koornwinder (1985) studied spherical functions associated with pairs of the form $(K \times K, K)$, where the subgroup is diagonally embedded and $K = \mathrm{SU}(2)$.
More recently Koelink-van Pruijssen-Román (2012) studied with a different approach this example and give the relation with MVOP (related to S^3).
For $\ell \in \mathbb{N}$ and $N = 2\ell + 1$ they produced a one-parameter family of $N \times N$ MVOP where the weight matrix is

$$W(y) = [y(1-y)]^{\nu-1/2}\Psi_0(y)T(\Psi_0(y))^*, \quad T_{ij} = \delta_{ij}\binom{2\ell}{i}\frac{(\nu)_i}{(\nu+2\ell-i)_i}$$

where $\Psi_0(y)$ is certain matrix-valued function containing spherical functions.
The corresponding symmetric second-order differential operator is given by

$$D = y(1-y)\partial_y^2 + (C + \nu - y(2\ell+2\nu+1))\partial_y - (V - (\nu-1)(2\ell+\nu+1))$$

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$$C = -\sum_0^{2\ell-1}\frac{(2\ell-i+1)}{2}E_{i,i+1} + \sum_0^{2\ell}\frac{(2\ell+3)}{2}E_{ii} - \sum_1^{2\ell}\frac{i+1}{2}E_{i,i-1}, \quad V = -\sum_0^{2\ell} i(2\ell-i)E_{i,i}$$

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Three important facts

1. The structure of the group induces the existence of a constant matrix Y such that we can decompose by blocks the weight matrix W in the form

$$\widetilde{W}(y) = YW(y)Y^* = \left(\begin{array}{c|c} W_1(y) & 0 \\ \hline 0 & W_2(y) \end{array}\right)$$

where W_1 is $(\ell+1)\times(\ell+1)$ and W_2 is $\ell \times \ell$. So we will study the probabilistic aspects of these two independent processes ($\ell = 1$).

2. We look for certain family of MVOP such that the corresponding block tridiagonal Jacobi matrix $\mathcal{A}$ has a "stochastic" interpretation, meaning that the sum of each row of $\mathcal{A}$ is ≤ 0 and the off-diagonal entries of $\mathcal{A}$ are ≥ 0 (therefore the infinitesimal operator of a continuous-time Markov chain).

3. Also, in order to find a "stochastic" second-order differential operator, we will perform a y-dependent transformation $S(y)$ such that $\mathcal{D} = S^{-1}(DS)$. Then

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THE FIRST STOCHASTIC MODEL 3×3 ($\ell = 1$)

Let $W_1(y)$ (2×2) and $w_2(y)$ (scalar) be the corresponding block weight matrices and denote by $Q_{n,1}$ and $q_{n,2}$ the corresponding families of MVOP satisfying $Q_{n,1}(0)\boldsymbol{e}_2 = \boldsymbol{e}_2, \boldsymbol{e}_2 = (1,1)^T$ and $q_{n,2}(0) = 1$.

1. A birth-and-death process: The polynomials $q_{n,2}$ satisfy the three-term recurrence relation

$$-yq_{n,2}(y) = a_n q_{n+1,2}(y) - (a_n + c_n)q_{n,2}(y) + c_n q_{n-1,2}(y)$$

where the coefficients are given by

$$a_n = \frac{2\nu + n + 2}{4(\nu + n + 1)}, \quad c_n = \frac{n}{4(\nu + n + 1)}$$

Therefore the Jacobi matrix is

$$\mathcal{A}_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & & \\ \frac{1}{4(\nu+2)} & -\frac{1}{2} & \frac{2\nu+3}{4(\nu+2)} & 0 & \\ 0 & \frac{1}{2(\nu+3)} & -\frac{1}{2} & \frac{\nu+2}{2(\nu+3)} & 0 \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \nu > -3/2$$

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The potential coefficients (inverse of the norms of $q_{n,2}$) are

$$\pi_0 = 1, \quad \pi_n = \frac{2(\nu+n+1)(2\nu+3)_{n-1}}{n!}, \quad n \geq 1$$

while the (normalized) weight is given by

$$w_2(y) = \frac{4^{\nu+1}\Gamma(\nu+2)}{\sqrt{\pi}\Gamma(\nu+3/2)} \left[y(1-y)\right]^{\nu+1/2}, \quad y \in (0,1), \quad \nu > -3/2$$

Therefore we have the Karlin-McGregor representation

$$\begin{aligned} P_{ij}^{(2)}(t) &= \mathbb{P}(X_t = j | X_0 = i) = \pi_j \int_0^1 e^{-yt} q_{i,2}(y) q_{j,2}(y) w_2(y) dy \\ &= \frac{2(\nu+j+1)(2\nu+3)_{j-1}4^{\nu+1}\Gamma(\nu+2)}{j!\sqrt{\pi}\Gamma(\nu+3/2)} \int_0^1 e^{-yt} q_{i,2} q_{j,2} \left[y(1-y)\right]^{\nu+1/2} dy \end{aligned}$$

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$$P_{ij}^{(2)}(t) = \mathbb{P}(X_t = j | X_0 = i) = \pi_j \int_0^1 e^{-yt} q_{i,2}(y) q_{j,2}(y) w_2(y) dy$$
$$= \frac{2(\nu+j+1)(2\nu+3)_{j-1} 4^{\nu+1}\Gamma(\nu+2)}{j!\sqrt{\pi}\Gamma(\nu+3/2)} \int_0^1 e^{-yt} q_{i,2} q_{j,2} \left[y(1-y)\right]^{\nu+1/2} dy$$

Since we have the explicit expression of the weight $w_2(y)$ we can study the recurrence of the process. For $-3/2 < \nu \leq -1/2$ the process is *null recurrent* (since $\sum \pi_n = \infty$), while if $\nu > -1/2$ then the process is *transient*.

This birth-and-death process can be seen as a *rational variant of the one-server queue* as the length of the queue increases.

The potential coefficients (inverse of the norms of $q_{n,2}$) are

$$\pi_0 = 1, \quad \pi_n = \frac{2(\nu+n+1)(2\nu+3)_{n-1}}{n!}, \quad n \geq 1$$

while the (normalized) weight is given by

$$w_2(y) = \frac{4^{\nu+1}\Gamma(\nu+2)}{\sqrt{\pi}\Gamma(\nu+3/2)}\left[y(1-y)\right]^{\nu+1/2}, \quad y \in (0,1), \quad \nu > -3/2$$

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This birth-and-death process can be seen as a *rational variant of the one-server queue* as the length of the queue increases.

2. A quasi-birth-and-death process: The polynomials $Q_{n,1}(y)$ satisfy the three-term recurrence relation

$$-yQ_{n,1}(y) = A_n Q_{n+1,1}(y) + B_n Q_{n,1}(y) + C_n Q_{n-1,1}(y)$$

where the coefficients are given by

$$A_n = \begin{pmatrix} \frac{2\nu+n+2}{4(\nu+n+2)} & 0 \\ 0 & \frac{(n+\nu)(2\nu+n+2)}{4(\nu+n+1)^2} \end{pmatrix}, \quad B_n = \begin{pmatrix} -\frac{1}{2} & \frac{\nu}{2(\nu+n)(\nu+n+2)} \\ \frac{1+\nu}{2(\nu+n+1)^2} & -\frac{1}{2} \end{pmatrix}, \quad C_n = \begin{pmatrix} \frac{n}{4(\nu+n)} & 0 \\ 0 & \frac{n(\nu+n+2)}{4(\nu+n+1)^2} \end{pmatrix}$$

Therefore the Jacobi matrix (pentadiagonal) is

$$\mathcal{A}_1 = \left(\begin{array}{cc|cc|cc|cc|c}
-\frac{1}{2} & \frac{1}{2(\nu+2)} & \frac{\nu+1}{2(\nu+2)} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2(\nu+1)} & -\frac{1}{2} & 0 & \frac{\nu}{2(\nu+1)} & 0 & 0 & 0 & 0 & \cdots \\
\hline
\frac{1}{4(\nu+1)} & 0 & -\frac{1}{2} & \frac{\nu}{2(\nu+1)(\nu+3)} & \frac{2\nu+3}{4(\nu+3)} & 0 & 0 & 0 & \cdots \\
0 & \frac{\nu+3}{4(\nu+2)^2} & \frac{1+\nu}{2(\nu+2)^2} & -\frac{1}{2} & 0 & \frac{(1+\nu)(2\nu+3)}{4(\nu+2)^2} & 0 & 0 & \cdots \\
\hline
0 & 0 & \frac{1}{2(\nu+2)} & 0 & -\frac{1}{2} & \frac{\nu}{2(\nu+2)(\nu+4)} & \frac{\nu+2}{2(\nu+4)} & 0 & \cdots \\
0 & 0 & 0 & \frac{\nu+4}{2(\nu+3)^2} & \frac{1+\nu}{2(\nu+3)^2} & -\frac{1}{2} & 0 & \frac{(2+\nu)^2}{2(\nu+3)^2} & \cdots \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right)$$

and it is the infinitesimal operator of a quasi-birth-and-death process ($\nu \geq 0$).

The (normalized) weight matrix is given by

$$W_1(y) = \frac{4^{\nu+1/2}\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)}\left[y(1-y)\right]^{\nu-1/2}\begin{pmatrix} 1-\frac{2(1+\nu)}{\nu+1/2}y(1-y) & \frac{\nu+1}{\nu+2}(1-2y) \\ \frac{\nu+1}{\nu+2}(1-2y) & \frac{\nu+1}{\nu+2}\left(1-\frac{2\nu}{\nu+1/2}y(1-y)\right)\end{pmatrix}, \quad \nu \geq 0$$

Each block entry (i,j) of $P^{(1)}(t)$ admits a Karlin-McGregor representation

$$P_{ij}^{(1)}(t) = \left(\int_0^1 e^{-yt}Q_{i,1}(y)W_1(y)Q_{j,1}^*(y)dx\right)\Pi_j$$

$$\Pi_0 = I, \quad \Pi_n = \left(\|Q_{n,1}\|_{W_1}^2\right)^{-1} = \frac{2(2\nu+3)_{n-1}}{n!}\begin{pmatrix} \frac{(\nu+1)^2}{\nu+n+1} & 0 \\ 0 & \frac{\nu(\nu+2)(\nu+n+1)}{(\nu+n)(\nu+n+2)}\end{pmatrix}$$

We can also compute *explicitly* the invariant measure of the process

$$\pi = \left((\Pi_0 e_2)^T; (\Pi_1 e_2)^T; (\Pi_2 e_2)^T; \cdots\right), \quad e_2^T = (1,1),$$
$$= \left(1, 1; \frac{2(\nu+1)^2}{\nu+2}, \frac{2\nu(\nu+2)^2}{(\nu+1)(\nu+3)}; \frac{(2\nu+3)(\nu+1)^2}{\nu+3}, \frac{(2\nu+3)\nu(\nu+3)}{\nu+4}; \cdots\right)$$

In a similar way studied in the scalar case, the process is *null recurrent* for $0 \leq \nu \leq 1/2$, while if $\nu > 1/2$ then the process will be *transient*.

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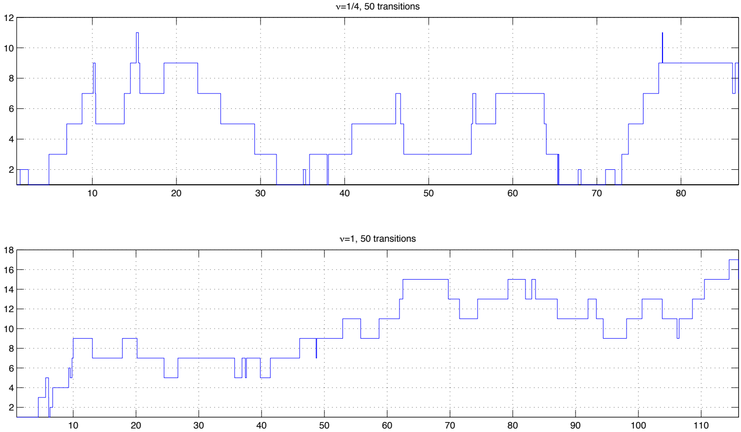
Interpretation: We have a 2 phases quasi-birth-and-death process. If the process moves along any of the phases, then the process can add (or remove) 2 elements to the queue. On the contrary, if the process moves from one phase to another, then the process add (or remove) 1 element to the queue. As the length of the queue increases, it is very unlikely that a transition between phases occurs. Therefore this quasi-birth-and-death process may be viewed as a *rational variation of a couple of one-server queues where the interaction between them is significant in the first states of the queue.*



The second stochastic model 3×3 $(\ell = 1)$

Let $S(y)$ be the transformation matrix

$$S(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $W_1(y)$ (2×2) and $w_2(y)$ (scalar) be the corresponding block weight matrices and denote by $Q_{n,1}$ and $q_{n,2}$ the corresponding families of matrix-valued orthogonal functions (need not to be polynomials any more).

1. A diffusion process with killing: The functions $q_{n,2}$ can be written as

$$q_{n,2}(y) = -2i\sqrt{y(1-y)}\,C_n^{(\nu+1)}(y)$$

where $C_n^{(\lambda)}$ is the family of monic Gegenbauer polynomials.
These are eigenfunctions of the second-order differential operator

$$\mathcal{D}_2 = y(1-y)\partial_y^2 + (\nu+1/2)(1-2y)\partial_y - \frac{\nu(1-2y)^2}{2y(1-y)}, \quad \nu \geq 0$$

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$$\lambda_{n,2} = -1 - n(n+2\nu+2)$$

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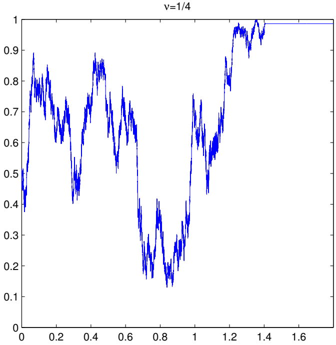
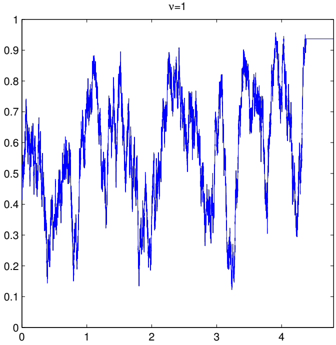
This process can be regarded as a *Wright-Fisher model involving only equal mutation effects with killing*. The behavior of the boundary points can be analyzed in terms of the parameter $\nu \geq 0$. Indeed, 0 and 1 are *regular* boundaries if $0 \leq \nu < 1/2$, while *entrance* boundaries if $\nu \geq 1/2$.
When the process is close to 0 or 1, then almost immediately the process is killed. The closer the trajectory is to $1/2$ the more time it will take to the process to be killed.





2. A switching diffusion process: The functions $Q_{n,1}$ are eigenfunctions of the matrix-valued second-order differential operator ($\nu \geq 0$)

$$\mathcal{D}_1 = y(1-y)\partial_y^2 + \begin{pmatrix} (\nu+1/2)(1-2y) & 0 \\ 0 & (\nu+3/2)(1-2y) - \dfrac{1}{1-2y} \end{pmatrix} \partial_y + \frac{1}{2y(1-y)} \begin{pmatrix} -\nu(1-2y)^2 & \nu(1-2y)^2 \\ 1+\nu & -(1+\nu) \end{pmatrix}$$

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The matrix-valued transition density probability can be written as

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Probabilistic properties:

- 0 and 1 are *regular* boundaries if $0 \leq \nu < 1/2$, while they are *entrance* boundaries if $\nu \geq 1/2$. The important difference now is that in the second phase the drift coefficient tends to ∞ if $y = 1/2$. It turns out that if we approach $1/2$ (on the left or on the right) then, it will always be an *entrance boundary*.
- If the process is near 0 or 1, then the diagonal coefficients of $Q(y)$ are very large, meaning that all phases are instantaneous. We also observe that if the process is near $1/2$ then the entry $(1, 1)$ of $Q(y)$ is very small, meaning that phase 1 is absorbing.
- The process tends to stay more time at phase 1 than in phase 2.

This process can also be regarded as a *variant of the Wright-Fisher model involving only mutation effects with two different phases*. The behavior of the boundaries 0 and 1 in both phases is exactly the same, but, while the process is at phase 2, starting for instance at an interior point of $[0, 1/2^-)$, then there is a force blocking the pass through the threshold located at $1/2$ (same if the interior point is located at $(1/2^+, 1]$). If the process is at phase 1, it can move along the whole state space $[0, 1]$ without any restriction at the point $1/2$.

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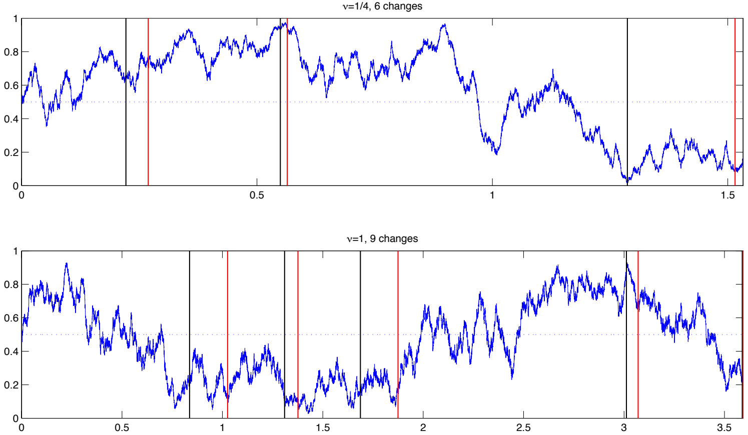
ν=1/4, 6 changes
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
ν=1, 9 changes
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5

The vector-valued invariant distribution (if it exists) is given by

$$\psi(y) = \frac{4^{\nu}\Gamma(\nu+2)[y(1-y)]^{\nu-1/2}}{\sqrt{\pi}(2+\nu)\Gamma(\nu+3/2)}\left(1+\nu\ ,\ \nu(1-2y)^2\right)$$