Las Trans-S-Variedades y la Clasificación de Gray-Hervella.

Manuel Soriano Trigueros

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Tutelado por

Prof. Luis M. Fernández Fernández

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Abstract

Recently, trans-S manifolds have been defined as a natural generalization of f-Kenmotsu, S-manifolds and C-manifolds. This uses f-structure techniques to extend results for almost contact manifolds, where trans-Sasakian are a generalization of Kenmotsu, Sasakian and cosymplectic manifolds. The defintion of trans-S is formulated using the covariant derivative of the tensor f. Although this formulation coincides with the characterization of trans-Sasakian when $s = 1$, the latter type of manifolds were not initially defined in this way.

Trans-Sasakian manifolds were defined in 1985 using the Gray-Hervella's classification of almost Hermitian manifolds. Therefore, one could ask whether trans-S manifolds could be defined using this classification and how they relate with almost Hermitian manifolds.

The objective of this project is to study how trans-S-manifolds and almost trans-S-manifolds relates with the Gray-Hervella's classification. We will find both similarities and differences with the trans-Sasakian case ($s = 1$).

Resumen

Las trans-S-variedades han sido definidas recientemente como una generalización natural de las f-Kenmotsu, S-variedades y C-variedades. Estos resultados representan una versión para f-structuras de algunos resulados ya conocidos en variedades casi contacto, donde las trans-Sasakianas generalizan variedades como las Kenmotsu, las Sasakianas y las cosimpléticas. La definición de variedad trans-S ha sido formulada en términos de la derivada covariante del tensor f. Aunque esta definición coincide con la caracterización de las trans-Sasakianas cuando $s = 1$, este último tipo de variedad no fue definida de esta manera en un principio.

Las variedades trans-Sasakianas fueron definidas en 1985 usando la clasificación de Gray y Hervella para variedades casi Hermitianas. Por tanto, una pregunta interesante sobre variedades trans-S es si podrían definirse de la misma manera y cómo se relacionan con la clase de variedades Hermitianas.

El objetivo de este proyecto es estudiar como se relacionan las variedades trans-S y casi trans-S con la clasificación de Gray y Hervella. Descubriremos algunas similitudes con el caso trans-Sasakiano ($s = 1$) pero también algunas diferencias.

Introduction

Since its beginning at the end of nineteenth century, manifolds and Riemannian geometry has been an intensive area of research. The discover of general relativity, which would have been impossible without this theory, was the greatest proof of its power to explain phenomena which could not be treated before. While relations of this new geometry with other fields of mathematics and science was found, some structures of study gained importance. For example, complex and symplectic manifolds in even dimensions, and contact manifolds in odd dimensions.

In particular a Kähler manifold is a Riemannian manifold with both complex and symplectic structure. A Sasakian manifold is a contact manifold M such that $M \times \mathbb{R}$ is conformal to a Kähler manifold. Despite all these objects have desirable properties, it is a difficult task to find out when they can be defined in a particular manifold.

This is the reason why almost complex and almost contact structures were found so convenient at first. An almost complex structure is a $(1, 1)$ tensor J which satisfies $J^2X = -X$ and an almost contact structure is a $(1, 1)$ tensor ϕ, a vector field ξ, and a 1-form η such that $\eta(\xi) = 1$ and $\phi^2X = -X + \eta(X)\xi$. Both are much easier to define than a proper complex or contact structure and, if these tensors satisfies some properties, they actually induce not only complex and contact structures, but more rigid ones like Kähler and Sasakian.

Nevertheless, their interest does not finish there. The existence of these two weaker structures provides some properties to the manifolds which make them of interest by their own right. Keeping this idea in mind, Yano introduced in [10] f-structures: tensors of type $(1, 1)$ satisfying $f^3 + f = 0$. In particular, metric f-manifolds satisfy $f^2 = -I + \sum_{i=1}^{s} \eta_i \otimes \xi_i$ where ξ_i are global vector fields and η_i there associates 1-forms. Both, almost contact and almost complex are particular cases of f-structures. The study of these tensors and how they relate with the already known results involving almost contact and almost complex structures is an active area of research.

In [5] Gray and Hervella studied how the covariant derivative of the associated 2-form can be used to create 16 classes for almost complex manifolds. The four basic classes are W_1, W_2, W_3, W_4, the rest are given by their direct sum. Kähler manifolds are precisely the intersection of these four classes. At the end of this introduction we show a chart from the original article where all classes are summarized.

In 1985 Oubiña defined trans-Sasakian manifolds as almost contact manifolds M such that $M \times \mathbb{R}$ was an almost complex manifold in the class W_4, containing locally conformal Kähler manifolds (see [8]). In addition it was shown that being trans-Sasakian was equivalent to being normal and satisfying

$$(\nabla_X f)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(fX,Y)\xi - \eta(Y)fX\} \tag{1}$$

for some differentiable functions α, β on M. Another almost contact manifold defined in the article was almost Sasakian-manifolds. This type was not normal and the product with $\mathbb{R}$ gives an almost complex manifold in the class $W_2 \oplus W_4$.

The objective of Oubiña was to generalize Kenmotsu, cosymplectic, Sasakian and Quasi-Sasakian manifolds. The relation between these manifolds and trans-Sasakian manifolds is given by the following chart where $\Phi(X,Y) = g(X,\phi Y)$ is the fundamental 2-form.

<table>
<tr><td>Kenmotsu:
$d\eta = 0$, normal</td><td rowspan="3">Quasi-Sasakian:
$d\Phi = 0$,
normal</td><td rowspan="3">Trans-Sasakian:
$d\Phi = 2\beta(\Phi \wedge \eta)$,
$d\eta = \alpha\Phi$,
$\phi^*(\delta\Phi) = 0$,
normal</td></tr>
<tr><td>Cosymplectic:
$d\Phi = 0, \quad d\eta = 0$,
normal</td></tr>
<tr><td>Sasakian:
$\Phi = d\eta$, normal</td></tr>
</table>

Analogously, P. Alegre, L. M. Fernández and A. Prieto defined in [1] a new class of metric f-manifold called tran-S-manifolds. They are characterized as normal manifolds satisfying

$$(\nabla_X f)Y = \sum_{i=1}^{s} \left(\alpha_i\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\} + \beta_i\{g(fX, Y)\xi_i - \eta_i(Y)fX\}\right), \quad (2)$$

where α_i, β_i are differentiable functions on the manifold. If the normality condition is removed, the manifold is called almost trans-S. We can resume the situation showed in [1] using the following chart.

<table>
<tr>
<td>f-Kenmotsu:
definition
$dF = 2\sum_{i=1}^{s} \eta_i \wedge F,\ \forall i\ d\eta_i = 0,$
normal</td>
<td rowspan="3">K-manifold:
definition
$dF = 0,$
normal</td>
<td rowspan="3">Trans-S-manifold:
necessary condition
$dF = 2F \wedge \sum_{i=1}^{s} \beta_i\eta_i,$
$\forall i\ d\eta_i = \alpha_i F,$
$f^*(\delta F) = 0,$
normal</td>
</tr>
<tr>
<td>C-manifold:
definition
$dF = 0, \quad \forall i\ d\eta_i = 0,$
normal</td>
</tr>
<tr>
<td>S-manifold:
definition
$\forall i\ F = d\eta_i,$ normal</td>
</tr>
</table>

Where $F(X, Y) = g(X, fY)$ is the fundamental 2-form. It should be noted that it is necessary, but not sufficient, for a trans-S-manifold to satisfy the equations given in the right cell. In fact, there exist K-manifolds which are not trans-S. In other words, trans-S manifolds cannot be defined using $dF, d\eta_i$ and $f^*(\delta F)$. On the other hand; f-Kenmotsu, S-manifolds and C-manifolds are always trans-S.

One question remains:

Can we use the Gray-Hervella's classification for defining (almost) trans-S-manifolds?

This project tries to clarify this question and compare the results with the trans-Sasakian case. In Chapter one we will remember some results about f-structures and tensor calculus. In the first Section of Chapter two we study the covariant derivative of the 2-form associated to the trans-S. In Section 2.2 we will prove the following result.

Theorem.- *If (M, f, η_i, ξ_i, g) is a trans-S-manifold of dimension $2r + s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ is an Hermitian manifold and*

1. *If $s = 1$, $\bar{M}$ lies in W_4.*
2. *If $s > 1$, $\bar{M}$ lies in $W_3 \oplus W_4$.*

We will continue in Section 2.3 showing there are manifolds in $W_3 \oplus W_4$ of the form $M \times \mathbb{R}^s$ where M is a metric f-manifold but not trans-S. Finally, in Chapter 3 we will find examples of manifolds $M \times \mathbb{R}^s$ in the most general class $W_1 \oplus W_2 \oplus W_3 \oplus W_4$, where M is almost trans-S.

Therefore, we will conclude that although Gray-Hervella's classification cannot be used to define (almost) trans-S-manifold, there is still an interesting relation in the normal case given by the previous theorem.

The following chart shows the defining conditions of the Gray-Hervella's classes which appeared in [5]. S denotes the Nijenhuis tensor and $\mathfrak{S}$ the cyclic sum.

Class	Defining conditions
$\mathcal{K}$	$\nabla F = 0$
$\mathcal{W}_1 = \mathcal{NK}$	$\nabla_X(F)(X, Y) = 0 \quad (\text{or } 3\nabla F = dF)$
$\mathcal{W}_2 = \mathcal{AK}$	$dF = 0$
$\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$	$\delta F = S = 0 \quad (\text{or } \nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = \delta F = 0)$
$\mathcal{W}_4$	$\nabla_X(F)(Y, Z) = \frac{-1}{2(n-1)}\{\langle X, Y\rangle \delta F(Z) - \langle X, Z\rangle \delta F(Y) - \langle X, JY\rangle \delta F(JZ) + \langle X, JZ\rangle \delta F(JY)\}$
$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{QK}$	$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)(JY, Z) = 0$
$\mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{H}$	$S = 0 \quad (\text{or } \nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0)$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$\nabla_X(F)(X, Y) - \nabla_{JX}(F)(JX, Y) = \delta F = 0$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$dF = F \wedge \theta \quad \left(\text{or } \underset{XYZ}{\mathfrak{S}}\left\{\nabla_X(F)(Y, Z) - \frac{1}{n-1} F(X, Y)\, \delta F(JZ)\right\} = 0\right)$
$\mathcal{W}_1 \oplus \mathcal{W}_4$	$\nabla_X(F)(X, Y) = \frac{-1}{2(n-1)}\{\Vert X\Vert^2 \delta F(Y) - \langle X, Y\rangle \delta F(X) - \langle JX, Z\rangle \delta F(JX)\}$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$\underset{XYZ}{\mathfrak{S}}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = \delta F = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{SK}$	$\delta F = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)\ (JY, Z) = \frac{-1}{n-1}\{\langle X, Y\rangle \delta F(Z) - \langle X, Z\rangle \delta F(Y) - \langle X, JY\rangle \delta F(JZ) + \langle X, JZ\rangle \delta F(JY)\}$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_1$	$\nabla_X(F)(X, Y) - \nabla_{JX}(F)(JX, Y) = 0 \quad (\text{or } \langle S(X, Y), X\rangle = 0)$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{G}_2$	$\underset{XYZ}{\mathfrak{S}}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = 0 \quad \left(\text{or } \underset{XYZ}{\mathfrak{S}} \langle S(X, Y), JZ\rangle = 0\right)$
$\mathcal{W}$	No condition

Figure 1: Gray-Hervella's Classification.

1 Preliminaries

We will assume basic knowledge in manifolds, Riemannian geometry and tensor calculus. Some classic books are [6], [7]. In this chapter we present some results which are not common in standard courses.

1.1 f-structures

Definition 1.1. *Let M be a $2n+s$ dimensional manifold. An f-structure on M is a (1,1) tensor of rank $2n$ satisfying $f^3+f=0$. In particular, M is a metric f-manifold if there exist s global vector fields $\xi_1,\dots,\xi_s$ such that, if $\eta_1,\cdots,\eta_s$ are there associated 1-forms, then*

$$f\xi_i=0;\ \ \eta_i\circ f=0;\ \ f^2=-I+\sum_{i=1}^{s}\eta_i\otimes\xi_i. \tag{1.1}$$

$$g(X,Y)=g(fX,fY)+\sum_{i=1}^{s}\eta_i(X)\eta_i(Y). \tag{1.2}$$

If $s=0$, f is called an almost complex structure and M an almost Hermitian manifold. When $s=1$, f is an almost contact structure and M an almost contact manifold.

Suppose we fix $l=-f^2$ and $m=f^2+I$, it is clear that:

$$l+m=I,\quad l^2=l,\quad m^2=m, \tag{1.3}$$

$$fl=lf=f,\quad mf=fm=0. \tag{1.4}$$

In other words, both operators are projection of two distributions $\mathcal{L}\oplus\mathcal{M}=TM$ such that $\eta_i(\mathcal{L})=0$ and $f(\mathcal{M})=0$. Because of (1.2) these two distributions are complementary and in any neighbourhood we can find an orthonormal local basis of $\mathcal{L}$, $\{X_1, \cdots, X_n, fX_1, \cdots, fX_n\}$ which together with $\{\xi_1, \cdots, \xi_s\}$ form a orthonormal local basis for TM .

The $(2,0)$ tensor $F(X,Y) = g(X, fY)$ gives us important information about the manifold. We call it the *associated 2-form* of the f-structure. This name is justified by the following result

Lemma 1.1. F is skew-symmetric.

Proof. Using the definition of metric f-manifold

$$
\begin{aligned}
F(X,Y) &= g(X, fY) = g(fX, f^2Y) + \sum_{i=1}^{s} \eta_i(X)\eta_i(fY) \\
&= g(fX, -Y + \sum_{i=1}^{s} \eta_i(Y)\xi_i) = g(fX, -Y) = -F(Y,X). \qquad (1.5)
\end{aligned}
$$

Another important tensor is the *Nijenhuis tensor* N_f,

$$N_f = [fX, fY] - f[fX, Y] - f[X, fY] - l[X, Y]. \qquad (1.6)$$

where $[,]$ is the lie bracket. The normality condition is related with integrability conditions for $\mathcal{L}$ and $\mathcal{M}$, see [11] Chapter seven. In particular, we have the following two definitions

Definition 1.2. *When f is an almost complex structure ($s = 0$) and $N_f = 0$ we say f is a complex structure and M an Hermitian manifold.*

Definition 1.3. *We say the f-structure is normal if it satisfies*

$$N_f + 2\sum_{i=1}^{s} \xi_i \otimes d\eta_i = 0 \qquad (1.7)$$

The first one is a classic result which relates almost complex and complex structures, (see for example [11]) but we have chosen to give it as a definition. The reader can find a explanation of why normal structures are interesting in [3], Chapter six.

1.2 Tensor Calculus

Before continue, we need to introduce a couple of concepts which are not usually taught in standard courses of differential geometry. In Section 2.2.2 of [9] the covariant derivative or connection of a tensor is defined as follows

Definition 1.4. *Let S be a (h,t) tensor field, with $t = 0, 1$ then we can define a covariant derivative $\nabla_X S$ as*

$$(\nabla_X S)(Y_1, \cdots, Y_t) = \nabla_X(S(Y_1, \cdots, Y_t)) - \sum_{i=1}^{t} S(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_t). \tag{1.8}$$

It can be checked the operator ∇_X is $\mathbb{R}$-linear and satisfies the Leibniz's Rule. Define now a inner product of forms of the same rank by

$$(\alpha, \beta) = \int_M \alpha \wedge \star\beta \tag{1.9}$$

where $\star$ is the Hodge operator. Then,

Definition 1.5. *Let F be a r-form. The codifferential δF is a (r-1)-form which coincide with the adjoint of the exterior derivative for the inner product $(,)$.*

It can be proven that if $\{X_1, \cdots X_n\}$ is an orthonormal frame, the codifferential is equal to

$$\delta F(Y_2 \cdots, Y_r) = \sum_{i=1}^{n} \nabla_{X_i} T(X_i, Y_2, \cdots, T_r). \tag{1.10}$$

The codifferential is also $\mathbb{R}$-linear.

We will also use product manifolds, that is manifolds which can be written as $M = M_1 \times M_2$. Remember all Riemannian manifold had a unique connection which was metric compatible and torsion free called the Levi-Civita connection. A natural question is: How does the connection of M relate with the connections of M_1, M_2? The following lemma appears as an exercise in [4].

Lemma 1.2. Let M_1 and M_2 be two Riemannian manifold and consider their product manifold $M_1 \times M_2$ with the product metric. The Levi-Civita connection ∇ of $M_1 \times M_2$ is given by

$$\nabla_Y X = \iota^1_* \left(\nabla^1_{\iota_1^* Y} \iota_1^* X \right) + \iota^2_* \left(\nabla^2_{\iota_2^* Y} \iota_2^* X \right), \tag{1.11}$$

where ι^1, ι^2 are the inclusion of M_1 and M_2 into $M_1 \times M_2$, ι_1^*, ι_2^* their pull-back and ι_*^1, ι_*^2 their push-forward.

2 Almost Complex Manifolds induced by a trans-S

The type of metric f-manifolds we are defining now will be the central object of study in this chapter.

Definition 2.1. *A $(2n+s)$-dimensional metric f-manifold M is said to be an almost trans-S-manifold if it satisfies*

$$(\nabla_X f)Y = \sum_{i=1}^{s} \left(\alpha_i\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\} + \beta_i\{g(fX, Y)\xi_i - \eta_i(Y)fX\}\right), \tag{2.1}$$

where α_i, β_i are smooth functions on M. If M is normal, it is called a trans-S-manifold.

When $s = 1$ trans-S-manifolds are called trans-Sasakian. Some more relations studied in [1] are

1. M is a K-manifold if $\beta_i = 0$ for all i. When $s = 1$ it is called quasi-Sasakian.
2. M is a S-manifold if and only if it is a K-manifold and $\alpha_i = 1$ for all i. When $s = 1$ it is called Sasakian.
3. We have $\alpha_i = \beta_i = 0$ for all i if and only if M is a C-manifold.
4. Generalized Kenmotsu manifolds are trans-S with $\alpha_i = 0$ and $\beta_i = 1$ for all i.

Our goal is to prove this type of manifold can be embedded canonically into an almost complex manifold $(\bar{M}, J, \bar{g})$ which is in the Gray-Hervella's Class $W_3 \oplus W_4$. In order to achieve it we need to compute the Levi-Civita connection (∇) of $\bar{M}$ and check $\nabla_{\bar{X}}\Omega(\bar{Y}, \bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y}, \bar{Z}) = 0$ where $\Omega(X, Y) = g(X, JY)$. This will be done in the following sections.

Consider a trans-S-manifold (M, f, η_i, ξ_i, g) of dimension $2r+s$. We define $\bar{M}$ as the product manifold $\bar{M} = M \times \mathbb{R}^s$ of dimension $2r+2s$ equipped with the product metric $\bar{g}(\bar{X}, \bar{Y}) = g(\iota_1^* \bar{X}, \iota_1^* \bar{Y}) + g_e(\iota_2^* \bar{X}, \iota_2^* \bar{Y})$ where ι_1, ι_2 are the projection, ι_1^*, ι_2^* their pull-back and g_e is the Euclidean metric. From now on, $\bar{X}, \bar{Y}$ and $\bar{Z}$ will be tensor fields of $\bar{M}$.

Definition 2.2. *We extend the tensors in M to $\bar{M}$ as follows:*

$$\bar{\eta}_i = \eta_i \circ \iota_1^*, \tag{2.2}$$
$$\bar{f} = \iota_*^1 \circ f \circ \iota_1^*, \tag{2.3}$$

where ι^1 is the inclusion of M into $\bar{M}$ and ι_*^1, ι_1^* its push-forward and pull-back respectively.

Definition 2.3. *if $\{\frac{\partial}{\partial x_j}\}$ is a basis of $\mathbb{R}^s$ and dx_j their dual, then J is the following (1,1) tensor of $\bar{M}$*

$$J(\bar{X}) = \bar{f}\bar{X} - \sum_i dx_i(\bar{X})\xi_i + \sum_j \bar{\eta}_j(\bar{X})\frac{\partial}{\partial x_j}. \tag{2.4}$$

The next lemma follows directly from the definitions

Lemma 2.1. for all i, j we have

$$\begin{aligned} &\bar{\eta}_i(\frac{\partial}{\partial x_j}) = 0, \quad \bar{f}(\frac{\partial}{\partial x_j}) = 0, \\ &dx_i(\xi_j) = 0, \quad dx_i \circ \bar{f} = 0, \\ &\bar{\eta}_i \circ \bar{f} = 0. \end{aligned} \tag{2.5}$$

Using definition 2.2 we can extend the formulas involving these tensors from M and $\mathbb{R}^s$ to $\bar{M}$. The following is another simple result.

Proposition 2.1. J is an almost complex structure and $\bar{M}$ an almost Hermitian Manifold.

Proof. Evaluating J twice we obtain,

$$\begin{aligned}
J^2(\bar{X}) &= J\left(\bar{f}\bar{X} - \sum_i dx_i(\bar{X})\xi_i + \sum_j \bar{\eta}_j(\bar{X})\frac{\partial}{\partial x_j}\right)\\
&= J(\bar{f}\bar{X}) - \sum_i dx_i J(\xi_i) + \sum_j \bar{\eta}_j(\bar{X}) J(\frac{\partial}{\partial x_j})\\
&= \bar{f}^2\bar{X} - \sum_i dx_i(\bar{X})\frac{\partial}{\partial x_i} - \sum_j \bar{\eta}_j(\bar{X})\xi_j\\
&= -X + \sum_j \bar{\eta}_j(\bar{X})\xi_j - \sum_i dx_i(\bar{X})\frac{\partial}{\partial x_i} - \sum_j \bar{\eta}_j(\bar{X})\xi_j\\
&= -\bar{X}.
\end{aligned} \tag{2.6}$$

Now, using the bilinearity of $\bar{g}$, the orthogonality of $\mathcal{L}, \mathcal{M}$ and $\mathbb{R}^s$, and that $\{\xi_i\}$ and $\{\frac{\partial}{\partial x_i}\}$ are orthonormal basis of $\mathcal{M}$ and $\mathbb{R}^s$:

$$\begin{aligned}
&\bar{g}(J\bar{X}, J\bar{Y}) =\\
&=\bar{g}\left(\bar{f}\bar{X} - \sum_i dx_i(\bar{X})\xi_i + \sum_i \bar{\eta}_i(\bar{X})\frac{\partial}{\partial x_i}, \bar{f}\bar{Y} - \sum_j dx_j(\bar{Y})\xi_j + \sum_j \bar{\eta}_j(\bar{Y})\frac{\partial}{\partial x_j}\right)\\
&=\bar{g}\left(\bar{f}\bar{X}, \bar{f}\bar{Y}\right) + \sum_{ij} dx_i(\bar{X})dx_j(\bar{Y})\bar{g}(\xi_i, \xi_j) + \sum_{ij} \bar{\eta}_i(\bar{X})\bar{\eta}_j(\bar{Y})\bar{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\\
&=\bar{g}\left(\bar{f}\bar{X}, \bar{f}\bar{Y}\right) + \sum_i dx_i(\bar{X})dx_i(\bar{Y}) + \sum_i \bar{\eta}_i(\bar{X})\bar{\eta}_i(\bar{Y}).
\end{aligned} \tag{2.7}$$

Because of (1.2) we have $\bar{g}\left(\bar{f}\bar{X}, \bar{f}\bar{Y}\right) + \sum_i \bar{\eta}_i(\bar{X})\bar{\eta}_i(\bar{Y})$ is equal to $g(\iota_1^*\bar{X}, \iota_1^*\bar{Y})$. Then,

$$\bar{g}(J\bar{X}, J\bar{Y}) = g(\iota_1^*\bar{X}, \iota_1^*\bar{Y}) + g_e(\iota_2^*\bar{X}, \iota_2^*\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}). \tag{2.8}$$

∎

2.1 Computing the associated 2-form.

Our next step is to extend the Levi-Civita connections ∇^1 of M and ∇^2 of $\mathbb{R}^s$ to $\bar{M}$. We can use Lemma 1.2 to conclude the Levi-Civita connection must be defined as in (1.11),

$$\nabla_{\bar{Y}}\bar{X} = \iota_*^1\left(\nabla^1_{\iota_1^*\bar{Y}}\iota_1^*\bar{X}\right) + \iota_*^2\left(\nabla^2_{\iota_2^*\bar{Y}}\iota_2^*\bar{X}\right). \tag{2.9}$$

In other words, the connection of the product is just the sum of the connections ∇^1 and ∇^2. This implies we can extend by linearity the formulas involving derivatives from M and $\mathbb{R}^s$ to $\bar{M}$. For example, remember that a trans-S-manifold satisfies

$$(\nabla^1_X f)(\bar{Y}) = \sum_{i=1}^{s} \left[\alpha_i\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\} + \beta_i\{g(fX,Y)\xi_i - \eta_i(Y)fX\}\right] \tag{2.10}$$

Then, we obtain

Lemma 2.2. The covariant derivative of the f-strucuture in $\bar{M}$ is given by

$$(\nabla_{\bar{X}}\bar{f})(\bar{Y}) = \sum_{i=1}^{s} \left[\alpha_i\{\bar{g}(\bar{f}\bar{X}, \bar{f}\bar{Y})\xi_i + \bar{\eta}_i(\bar{Y})\bar{f}^2\bar{X}\} + \beta_i\{\bar{g}(\bar{f}\bar{X},\bar{Y})\xi_i - \bar{\eta}_i(\bar{Y})\bar{f}\bar{X}\}\right]. \tag{2.11}$$

As we said in the beginning, our objective is to check in which Gray-Hervella's Class lies $\bar{M}$. So as to find it out, we need to study the covariant derivative of the associated 2-form.

Definition 2.4. *We define the 2-forms associated to J and $\bar{f}$ as*

$$\Omega(\bar{X},\bar{Y}) = \bar{g}(\bar{X}, J\bar{Y}) \tag{2.12}$$
$$F(\bar{X},\bar{Y}) = \bar{g}(\bar{X}, \bar{f}\bar{Y}). \tag{2.13}$$

Obviously, F restricted to M coincide with the 2-form associated to f and when it is restricted to $\mathbb{R}^s$ it is equivalent to zero. As it is shown in the following proposition, Ω can be expressed as the sum of more simple 2-forms where dx_i are the coordinate 1-forms in $\mathbb{R}^s$.

Proposition 2.2.

$$\Omega(\bar{X},\bar{Y}) = F(\bar{X},\bar{Y}) + \sum_{i=1}^{s} dx_i \wedge \bar{\eta}_i(\bar{X},\bar{Y}) \tag{2.14}$$

Proof. Using the bilinearity of $\bar{g}$ and the definition of J we deduce the result.

$$\begin{aligned}\Omega(\bar{X},\bar{Y}) =&\bar{g}(\bar{X},J\bar{Y}) = \bar{g}\left(\bar{X}, f\bar{Y} - \sum_j dx_j(\bar{Y})\xi_j + \sum_j \bar{\eta}_j(\bar{Y})\frac{\partial}{\partial x_j}\right)\\ =&\bar{g}(\bar{X},f\bar{Y}) - \sum_j dx_j(\bar{Y})\bar{g}(\bar{X},\xi_j) + \sum_j \bar{\eta}_j(\bar{Y})\bar{g}(\bar{X},\frac{\partial}{\partial x_j})\\ =&\bar{g}(\bar{X},f\bar{Y}) - \sum_j dx_j(\bar{Y})\bar{\eta}_j(\bar{X}) + \sum_j \bar{\eta}_j(\bar{Y})dx_j(\bar{X})\\ =&F(\bar{X},\bar{Y}) + \sum_{i=1}^{s} dx_i\wedge\bar{\eta}_i(\bar{X},\bar{Y}). \end{aligned}\tag{2.15}$$

The next lemma is just a note which will simplify the computations.

Lemma 2.3.

$$(\nabla_{\bar{X}}\Omega)(\bar{Y},\bar{Z}) = (\nabla_{\bar{X}}F)(\bar{Y},\bar{Z}) + \sum_i \nabla_{\bar{X}}(dx_i\wedge\bar{\eta}_i)(\bar{Y},\bar{Z}).\tag{2.16}$$

Proof. Just use the linearity of the covariant derivative, see definition 1.4.

Then, if we find an explicit formula of $\nabla_{\bar{X}}F$ and $\nabla_{\bar{X}}dx_i\wedge\bar{\eta}_i$ in terms of the metric, we will have one for $\nabla_{\bar{X}}\Omega$.

On one hand, using metric compatibility of the connection and the formula (1.8),

$$\begin{aligned}(\nabla_{\bar{X}}F)(\bar{Y},\bar{Z}) =&\nabla_{\bar{X}}\bar{g}(\bar{Y},f\bar{Z}) - \bar{g}(\nabla_{\bar{X}}\bar{Y},f\bar{Z}) - \bar{g}(\bar{Y},f\nabla_{\bar{X}}\bar{Z})\\ =&\bar{g}(\nabla_{\bar{X}}\bar{Y},f\bar{Z}) + \bar{g}(\bar{Y},\nabla_{\bar{X}}(f\bar{Z})) - \bar{g}(\nabla_{\bar{X}}\bar{Y},f\bar{Z}) + \bar{g}(\bar{Y},(\nabla_{\bar{X}}f)\bar{Z})\\ &- \bar{g}(\bar{Y},\nabla_{\bar{X}}(f\bar{Z})) = \bar{g}(\bar{Y},(\nabla_{\bar{X}}f)). \end{aligned}\tag{2.17}$$

Now, from (2.11) and using $g(\bar{Y},\xi_i) = \bar{\eta}_i(Y)$.

$$\begin{aligned}(\nabla_{\bar{X}}F)(\bar{Y},\bar{Z}) =&\bar{g}(\bar{Y},(\nabla_{\bar{X}}f))\\ =&\bar{g}\left(\bar{Y},\sum_{i=1}^{s}\left[\alpha_i\{\bar{g}(f\bar{X},f\bar{Z})\xi_i + \bar{\eta}_i(\bar{Z})f^2\bar{X}\} + \beta_i\{\bar{g}(f\bar{X},\bar{Z})\xi_i - \bar{\eta}_i(\bar{Z})f\bar{X}\}\right]\right)\\ =&\sum_{i=1}^{s}\alpha_i\{\bar{\eta}_i(\bar{Y})\bar{g}(f\bar{X},f\bar{Z}) + \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y},f^2\bar{X})\}\\ &+ \beta_i\{\bar{\eta}(\bar{Y})\bar{g}(f\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y},f\bar{X})\}. \end{aligned}\tag{2.18}$$

Finally, using the Lemma 1.1.

$$\begin{aligned}(\nabla_{\bar{X}}F)(\bar{Y},\bar{Z}) = \sum_{i=1}^{s} \alpha_i\{\bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X})\}\\ + \beta_i\{\bar{\eta}(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X})\}.\end{aligned} \tag{2.19}$$

We also have $(\nabla_{\bar{X}}dx_i)(\bar{Y}) = \nabla_{\bar{X}}(dx_i(\bar{Y})) - dx_i(\nabla_{\bar{X}}\bar{Y}) = \frac{\partial \bar{Y}^i}{\partial \bar{X}} - \frac{\partial \bar{Y}^i}{\partial \bar{X}} = 0$ where $\bar{Y}^i$ is the coordinate $\frac{\partial}{\partial x_i}$. Using this equality and the Leibniz's rule, we obtain

$$\begin{aligned}\nabla_{\bar{X}}(dx_i\wedge\bar{\eta}_i)(\bar{Y},\bar{Z}) =&(\nabla_{\bar{X}}dx_i)\wedge\bar{\eta}_i(\bar{Y},\bar{Z}) + dx_i\wedge(\nabla_{\bar{X}}\bar{\eta}_i)(\bar{Y},\bar{Z})\\ =&dx_i\wedge(\nabla_{\bar{X}}\bar{\eta}_i)(\bar{Y},\bar{Z}).\end{aligned} \tag{2.20}$$

From [1] we know that

$$(\nabla_{\bar{X}}\bar{\eta}_i)\bar{Y} = \bar{g}(\nabla_{\bar{X}}\xi_i,\bar{Y}) = \bar{g}(-\alpha_i\bar{f}\bar{X} - \beta_i\bar{f}^2\bar{X},\bar{Y}) = -\alpha_i\bar{g}(\bar{Y},\bar{f}\bar{X}) + \beta_i\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X}). \tag{2.21}$$

Then, using (2.21)

$$\begin{aligned}\nabla_{\bar{X}}(dx_i\wedge\bar{\eta}_i)(\bar{Y},\bar{Z}) =&dx_i\wedge(\nabla_{\bar{X}}\bar{\eta}_i)(\bar{Y},\bar{Z})\\ =&dx_i(\bar{Y})(-\alpha_i\bar{g}(\bar{Z},\bar{f}\bar{X}) + \beta_i\bar{g}(\bar{f}\bar{Z},\bar{f}\bar{X}))\\ &- dx_i(\bar{Z})(-\alpha_i\bar{g}(\bar{Y},\bar{f}\bar{X}) + \beta_i\bar{g}(\bar{f}\bar{Y},f\bar{X}))\\ =&\alpha_i(dx_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X}) - dx_i(\bar{Y})\bar{g}(\bar{Z},\bar{f}\bar{X}))\\ &+ \beta_i(dx_i(\bar{Y})\bar{g}(\bar{f}\bar{Z},f\bar{X}) - dx_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X})).\end{aligned} \tag{2.22}$$

Finally we have found an explicit formula for $\nabla_{\bar{X}}\Omega$ depending on the metric. Putting together (2.16), (2.17) and (2.22) we obtain

Proposition 2.3. $(\nabla_{\bar{X}}\Omega)(\bar{Y},\bar{Z})$ is equal to

$$\begin{aligned}\sum_i \alpha_i\{\bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X}) + dx_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X}) - dx_i(\bar{Y})\bar{g}(\bar{Z},\bar{f}\bar{X})\}\\ \beta_i\{\bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X}) + dx_i(\bar{Y})\bar{g}(\bar{f}\bar{Z},\bar{f}\bar{X}) - dx_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X})\}.\end{aligned} \tag{2.23}$$

Now we can figure out in which class lies $\bar{M}$. This is the aim of the following section.

2.2 A Gray-Hervella's Class for $\bar{M}$

After computing the covariant derivative of Ω we can calculate the class of $\bar{M}$. This is done in the following proposition.

Proposition 2.4. If (M, f, η_i, ξ_i, g) is a trans-S-manifold of dimension $2r+s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ has an almost complex structure and lies in $W_3 \oplus W_4$.

Proof. We have already shown in Proposition 2.1 it is almost complex. Remember from [5] that a manifold $\bar{M}$ is in $W_3 \oplus W_4$ if and only if

$$\nabla_{\bar{X}}\Omega(\bar{Y}, \bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y}, \bar{Z}) = 0. \tag{2.24}$$

Therefore, we just have to compute $\nabla_{J\bar{X}}\Omega(J\bar{Y}, \bar{Z})$ and see if it is equal to $\nabla_{\bar{X}}\Omega(\bar{Y}, \bar{Z})$. Using (2.4) we obtain

$$\begin{aligned}
\bar{\eta}_i(J\bar{X}) &= -dx_i(\bar{X}), \\
dx_i(J\bar{X}) &= \bar{\eta}_i(\bar{X}), && (2.25)\\
J\bar{f}\bar{X} &= \bar{f}^2\bar{X}. && (2.26)
\end{aligned}$$

From (2.23) we have $\nabla_{J\bar{X}}\Omega(J\bar{Y}, \bar{Z})$ is equal to

$$\begin{aligned}
\sum_i \alpha_i\{&-dx_i(\bar{Y})\bar{g}(\bar{f}^2\bar{X}, \bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{f}^2\bar{Y}, \bar{f}^2\bar{X}) \\
&+ dx_i(\bar{Z})\bar{g}(J\bar{Y}, \bar{f}^2\bar{X}) - \bar{\eta}_i(\bar{Y})\bar{g}(\bar{Z}, \bar{f}^2\bar{X})\} \\
+ \beta_i\{&-dx_i(\bar{Y})\bar{g}(\bar{f}^2\bar{X}, \bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(J\bar{Y}, \bar{f}^2\bar{X}) \\
&+ \bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{Z}, \bar{f}^2\bar{X}) - dx_i(\bar{Z})\bar{g}(\bar{f}^2\bar{Y}, \bar{f}^2\bar{X})\}.
\end{aligned} \tag{2.27}$$

Now, using

$$\bar{g}(\bar{f}\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, \bar{f}\bar{Y}), \tag{2.28}$$

$$\bar{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \bar{g}(\xi_i, \xi_j) = \delta_i^j, \tag{2.29}$$

$$\bar{g}(\bar{f}\bar{X}, \xi_i) = \bar{g}\left(\bar{f}\bar{X}, \frac{\partial}{\partial x_i}\right) = \bar{g}\left(\xi_i, \frac{\partial}{\partial x_j}\right) = 0, \tag{2.30}$$

(2.31)

together with (1.2) we obtain

$$\begin{aligned}\bar{g}(\bar{f}^2\bar{X}, \bar{f}^2\bar{Y}) &= g(f^2X, f^2Y) = g(fX, fY) - \eta(Y)\eta(X)\\ &= g(fX, fY) = \bar{g}(\bar{f}\bar{X}, \bar{f}\bar{Y}),\end{aligned} \tag{2.32}$$

$$\begin{aligned}\bar{g}(J\bar{Y}, \bar{f}^2\bar{X}) &= \bar{g}(\bar{f}J\bar{Y}, \bar{f}\bar{X}) = -\bar{g}(\bar{f}^2Y, \bar{f}\bar{X}) = -(\bar{f}Y, \bar{X})\\ &+ \sum_i \eta_i(\bar{f}Y)\eta_i(\bar{X}) = \bar{g}(\bar{Y}, \bar{f}\bar{X}).\end{aligned} \tag{2.33}$$

Then, we can reduce (2.27) to

$$\begin{aligned}&\sum_i \alpha_i\{-dx_i(\bar{Y})\bar{g}(\bar{f}\bar{X}, \bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{f}\bar{Y}, \bar{f}\bar{X}) + dx_i(\bar{Z})\bar{g}(\bar{Y}, \bar{f}\bar{X}) + \bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{Z}, \bar{f}\bar{X})\}\\ &\quad \beta_i\{+dx_i(\bar{Y})\bar{g}(\bar{f}\bar{X}, \bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y}, \bar{f}\bar{X}) + \bar{\eta}_i(\bar{Y})\bar{g}(\bar{Z}, \bar{f}\bar{X}) - dx_i(\bar{Z})\bar{g}(\bar{f}\bar{Y}, \bar{f}\bar{X})\}.\end{aligned} \tag{2.34}$$

Sorting the addends we see it is equal to (2.23) and then (2.24) is satisfied.

Note that in Figure 1 there are two equivalent definitions of $W_3 \oplus W_4$. The second one $S = 0$ where S is the Nijenhuis tensor N_J, see equation (1.6). Therefore, we can formulate the following corollary.

Corollary 2.1. If (M, f, η_i, ξ_i, g) is a trans-S-manifold of dimension $2r + s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ is an Hermitian manifold.

Proof. Just remember Definition 1.2.

As $\bar{M}$ lies in $W_3 \oplus W_4$ we could ask if it is actually in W_3 or W_4, the answer is: in general it does not. To verify this assertion we need to compute $\delta\Omega$. As $\delta\Omega = \delta F + \sum_i \delta(dx_1 \wedge \bar{\eta}_i)$. The following lemma will be enough.

Lemma 2.4.

$$\delta F(\bar{X}) = 2n \sum_i \alpha_i \bar{\eta}_i(\bar{X}) \tag{2.35}$$

$$\delta(dx_1 \wedge \bar{\eta}_i)(\bar{X}) = 2n\beta_i dx_i(\bar{X}) \tag{2.36}$$

Proof. The first equality can be found in the article [1].

For the second, remember when we defined metric f-manifolds we pointed out there exist in each neighbourhood vector fields such that

$$\{X_1 \dots X_n, fX_1, \dots, fX_n, \xi_1, \dots, \xi_s\} \tag{2.37}$$

was an orthonormal basis for M. Then, together with $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\}$ we can form an orthonormal basis for $\bar{M}$. Using (1.10) it follows

$$\begin{aligned}\delta(dx_i \wedge \bar{\eta}_i)(\bar{X}) = -\sum_j \Big[& dx_i \wedge (\nabla_{X_j}\bar{\eta}_i)(X_j, \bar{X}) + dx_i \wedge (\nabla_{\bar{f}X_j}\bar{\eta}_i)(\bar{f}X_j, \bar{X}) \\ & + dx_i \wedge (\nabla_{\xi_j}\bar{\eta}_i)(\xi_j, \bar{X}) + dx_i \wedge (\nabla_{\frac{\partial}{\partial x_j}}\bar{\eta}_i)(\frac{\partial}{\partial x_j}, \bar{X}) \Big].\end{aligned} \tag{2.38}$$

Now, using the formula (2.21) and

$$dx_i(X_j) = dx_i(\bar{f}X_j) = dx_i(\xi_j) = 0. \tag{2.39}$$

we have

$$\begin{aligned}\delta(dx_i \wedge \bar{\eta}_i)(\bar{X}) = -\sum_j & -dx_i(\bar{X})\left(-\alpha_i \bar{g}(X_j, \bar{f}X_j) + \beta_i \bar{g}(\bar{f}X_j, \bar{f}X_j)\right) \\ & - dx_i(\bar{X})\left(-\alpha_i \bar{g}(\bar{f}X_j, \bar{f}^2X_j) + \beta_i \bar{g}(\bar{f}^2X_j, \bar{f}^2X_j)\right) \\ & - dx_i(\bar{X})\left(-\alpha_i \bar{g}(\xi_i, \bar{f}\xi_i) + \beta_i \bar{g}(\bar{f}\xi_i, \bar{f}\xi_i)\right) \\ & + dx_i(\frac{\partial}{\partial x_i})\left(-\alpha_i \bar{g}(\bar{X}, \bar{f}\frac{\partial}{\partial x_i}) + \beta_i \bar{g}(\bar{f}\bar{X}, \bar{f}\frac{\partial}{\partial x_i})\right) \\ & - dx_i(\bar{X})\left(-\alpha_i \bar{g}(\frac{\partial}{\partial x_i}, \bar{f}\frac{\partial}{\partial x_i}) + \beta_i \bar{g}(\bar{f}\frac{\partial}{\partial x_i}, \bar{f}\frac{\partial}{\partial x_i})\right).\end{aligned} \tag{2.40}$$

Using the orthonormality of the basis and

$$\bar{f}\xi_i = \bar{f}\frac{\partial}{\partial x_i} = 0, \quad \bar{f}^2\bar{X}_j = -\bar{X}_j. \tag{2.41}$$

We obtain

$$\begin{aligned}\delta(dx_i \wedge \bar{\eta}_i)(\bar{X}) = -\sum_j & -dx_i(\bar{X})\left(-\alpha_i \bar{g}(X_j, \bar{f}X_j) + \beta_i \bar{g}(\bar{f}X_j, \bar{f}X_j)\right) \\ & - dx_i(\bar{X})\left(-\alpha_i \bar{g}(\bar{f}X_j, \bar{f}^2X_j) + \beta_i \bar{g}(\bar{f}^2X_j, \bar{f}^2X_j)\right).\end{aligned} \tag{2.42}$$

Finally, using (1.2) and the orthonormality of the basis

$$\begin{aligned}\bar{g}(\bar{f}^2X_j, \bar{f}^2X_j) = \bar{g}(\bar{f}X_j, \bar{f}X_j) = \bar{g}(X_j, X_j) = 1 \\ \bar{g}(\bar{f}X_j, \bar{f}^2X_j) = \bar{g}(X_j, \bar{f}X_j) = 0.\end{aligned} \tag{2.43}$$

it follows

$$\begin{aligned}\delta(dx_i \wedge \bar{\eta}_i)(\bar{X}) &= -\sum_j dx_i(\bar{X})\beta_i \left(\bar{g}(\bar{f}X_j, \bar{f}X_j) + \bar{g}(X_j, X_j)\right) \\ &= -2n dx_i(\bar{X}). \end{aligned} \tag{2.44}$$

Using the linearity of the codifferential we have

Proposition 2.5.

$$\delta\Omega = -2n \sum_i (\alpha_i \bar{\eta}_i + \beta_i dx_i). \tag{2.45}$$

This allows us to formulate three important corollaries. Two of them are rather trivial, the third one requires more effort.

Corollary 2.2. In general, $\bar{M}$ is not in W_3.

Proof. As it is shown in [5], if $\bar{M}$ is in W_3 then $\delta\bar{M} = 0$.

Corollary 2.3. If $\bar{M}$ is in W_3 then M is a C-manifold.

Proof. $\alpha_i = \beta_i = 0$ is the only possible solution to $-2n\sum_i(\alpha_i\bar{\eta}_i + \beta_i dx_i) = 0$.

Corollary 2.4. If $s > 1$, $\bar{M}$ is not in W_4.

Proof. The condition for being in W_4 which appears in [5] is

$$\begin{aligned}\nabla_{\bar{X}}\Omega(\bar{Y}, \bar{Z}) + \frac{1}{2n}&\left\{\bar{g}(\bar{X}, \bar{Y})\delta\Omega(\bar{Z}) - \bar{g}(\bar{X}, \bar{Z})\delta\Omega(\bar{Y})\right. \\ &\left. -\bar{g}(\bar{X}, J\bar{Y})\delta\Omega(J\bar{Z}) + \bar{g}(\bar{X}, J\bar{Z})\delta\Omega(J\bar{Y})\right\} = 0. \end{aligned} \tag{2.46}$$

Using the definition of $\bar{g}$ and (1.2),

$$\begin{aligned}\bar{g}(\bar{X}, \bar{Y})\delta\Omega(\bar{Z}) = &\left(\bar{g}(\bar{f}\bar{X}, \bar{f}\bar{Y}) + \sum_{i=1}^{s}\left[\bar{\eta}_i(\bar{X})\bar{\eta}_i(\bar{Y}) + dx_i(\bar{X})dx_i(\bar{Y})\right]\right)\cdot \\ &\cdot 2n\sum_{j=1}^{s}\left(\alpha_j\bar{\eta}_j(\bar{Z}) + \beta_j dx_j(\bar{Z})\right) \end{aligned} \tag{2.47}$$

$$\begin{aligned}\bar{g}(\bar{X}, J\bar{Y})\delta\Omega(J\bar{Z}) = &\left(\bar{g}(\bar{X}, \bar{f}\bar{Y}) + \sum_{i=1}^{s}\left[dx_i(\bar{X})\bar{\eta}_i(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_i(\bar{X})\right]\right)\cdot \\ &\cdot 2n\sum_{j=1}^{s}\left(-\alpha_j dx_j(\bar{Z}) + \beta_j\bar{\eta}_j(\bar{Z})\right). \end{aligned} \tag{2.48}$$

Remember we obtained the equation (2.23) for $\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z})$. This transform the sum of (2.46) into

$$\begin{aligned}
&\sum_i \alpha_i\{\bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X}) + dx_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X}) - dx_i(\bar{Y})\bar{g}(\bar{Z},\bar{f}\bar{X})\}\\
&+\sum_i \beta_i\{\bar{\eta}_i(\bar{Y})\bar{g}(\bar{f}\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{g}(\bar{Y},\bar{f}\bar{X}) + dx_i(\bar{Y})\bar{g}(\bar{f}\bar{Z},\bar{f}\bar{X}) - dx_i(\bar{Z})\bar{g}(\bar{f}\bar{Y},\bar{f}\bar{X})\}\\
&+\sum_{i=1}^{s}\left[\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Y}) + \bar{\eta}_i(\bar{X})\ \bar{\eta}_i(\bar{Y}) + dx_i(\bar{X})dx_i(\bar{Y})\right]\sum_{j=1}^{s}\left(\alpha_j\bar{\eta}_j(\bar{Z}) + \beta_j dx_j(\bar{Z})\right)\\
&-\sum_{i=1}^{s}\left[\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Z}) + \bar{\eta}_i(\bar{X})\ \bar{\eta}_i(\bar{Z}) + dx_i(\bar{X})dx_i(\bar{Z})\right]\sum_{j=1}^{s}\left(\alpha_j\bar{\eta}_j(\bar{Y}) + \beta_j dx_j(\bar{Y})\right)\\
&-\sum_{i=1}^{s}\left[\bar{g}(\bar{X},\bar{f}\bar{Y}) + dx_i(\bar{X})\ \bar{\eta}_i(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_i(\bar{X})\right]\sum_{j=1}^{s}\left(-\alpha_j dx_j(\bar{Z}) + \beta_j\bar{\eta}_j(\bar{Z})\right)\\
&+\sum_{i=1}^{s}\left[\bar{g}(\bar{X},\bar{f}\bar{Z}) + dx_i(\bar{X})\ \bar{\eta}_i(\bar{Z}) - dx_i(\bar{Z})\bar{\eta}_i(\bar{X})\right]\sum_{j=1}^{s}\left(-\alpha_j dx_j(\bar{Y}) + \beta_j\bar{\eta}_j(\bar{Y})\right).
\end{aligned} \tag{2.49}$$

Note that the addends where the metric appears cancel. Then, the sum

$$\begin{aligned}
\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z}) + \frac{1}{2n}\big\{&\bar{g}(\bar{X},\bar{Y})\delta\Omega(\bar{Z}) - \bar{g}(\bar{X},\bar{Z})\delta\Omega(\bar{Y})\\
&-\bar{g}(\bar{X},J\bar{Y})\delta\Omega(J\bar{Z}) + \bar{g}(\bar{X},J\bar{Z})\delta\Omega(J\bar{Y})\big\}
\end{aligned} \tag{2.50}$$

is equal to

$$\begin{aligned}
&\sum_{i=1}^{s}\left[\bar{\eta}_i(\bar{X})\ \bar{\eta}_i(\bar{Y}) + dx_i(\bar{X})dx_i(\bar{Y})\right]\sum_{j=1}^{s}\left(\alpha_j\bar{\eta}_j(\bar{Z}) + \beta_j dx_j(\bar{Z})\right)\\
&-\sum_{i=1}^{s}\left[\bar{\eta}_i(\bar{X})\ \bar{\eta}_i(\bar{Z}) + dx_i(\bar{X})dx_i(\bar{Z})\right]\sum_{j=1}^{s}\left(\alpha_j\bar{\eta}_j(\bar{Y}) + \beta_j dx_j(\bar{Y})\right)\\
&-\sum_{i=1}^{s}\left[dx_i(\bar{X})\ \bar{\eta}_i(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_i(\bar{X})\right]\sum_{j=1}^{s}\left(-\alpha_j dx_j(\bar{Z}) + \beta_j\bar{\eta}_j(\bar{Z})\right)\\
&+\sum_{i=1}^{s}\left[dx_i(\bar{X})\ \bar{\eta}_i(\bar{Z}) - dx_i(\bar{Z})\bar{\eta}_i(\bar{X})\right]\sum_{j=1}^{s}\left(-\alpha_j dx_j(\bar{Y}) + \beta_j\bar{\eta}_j(\bar{Y})\right),
\end{aligned} \tag{2.51}$$

which is not zero.

Note: when $s = 1$ M is a trans-Sasakian manifold. In this case, we can consider $i = j = 1$ and then (2.51) is zero.

The following theorem summarizes this section.

Theorem 2.1. *If (M, f, η_i, ξ_i, g) is a trans-S-manifold of dimension $2r + s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ is an Hermitian Manifold and*

1. *If $s = 1$, $\bar{M}$ lies in W_4.*
2. *If $s > 1$, $\bar{M}$ lies in $W_3 \oplus W_4$.*

2.3 Trans-S cannot be defined using $W_3 \oplus W_4$

In this section we will prove there are manifolds $M \times \mathbb{R}^s$ in $W_3 \oplus W_4$ with M a metric f-manifold which is not trans-S .

Proposition 2.6. The almost contact manifold induced by a C-manifolds is Kälher.

Proof. In [2] it is shown that a C-manifold satisfies

$$\nabla_1 \xi_i = 0 \tag{2.52}$$

$$\nabla_1 f = 0 \tag{2.53}$$

Remember that

$$(\nabla_{\bar{X}} \Omega)(\bar{Y}, \bar{Z}) = (\nabla_{\bar{X}} F)(\bar{Y}, \bar{Z}) + \sum_{i=1}^{s} dx_i \wedge (\nabla_{\bar{X}} \bar{\eta}_i)(\bar{Y}, \bar{Z}). \tag{2.54}$$

from (2.17) and (2.53)

$$(\nabla_{\bar{X}} F)(\bar{Y}, \bar{Z}) = \bar{g}(\bar{Y}, (\nabla_{\bar{X}} \bar{f})) = 0. \tag{2.55}$$

and from (2.21) and (2.52)

$$(\nabla_{\bar{X}} \bar{\eta}_i)(\bar{Y}) = g(\nabla_{\bar{X}} \xi_i, \bar{Y}) = 0. \tag{2.56}$$

Then

$$(\nabla_{\bar{X}} \Omega)(\bar{Y}, \bar{Z}) = 0. \tag{2.57}$$

and $\bar{M}$ is Kähler.

Note: it is clear from the definition that if the f-structure of a C-manifold is an almost complex structure, then $\nabla_X f = 0$ and the manifold is also Kähler.

Theorem 2.2. *There exists almost complex manifolds $\bar{M} = M \times \mathbb{R}^s$ in $W_3 \oplus W_4$ where M is metric f-manifold but not trans-S.*

Proof. We are proving the result in the more simple case W_3. As $W_3 \subset W_3 \oplus W_4$ this imply the more general assertion. Fix a manifold N with complex structure H in W_3 and suppose it is not Kähler. Therefore, it will not be trans-S because otherwise Corollary 2.3 would imply M to be C-manifold and $\bar{M}$ Kähler. If ∇^1 is its connection, the associated 2-form Φ satisfies

$$\nabla^1_X \Phi(Y, Z) = \nabla^1_{HX} \Phi(HY, Z), \tag{2.58}$$
$$\delta\Phi(X) = 0. \tag{2.59}$$

Now, construct the metric f-manifold $M = N \times \mathbb{R}^s$ with

$$fX = \iota^1_*(H \circ \iota^*_1(X)), \tag{2.60}$$
$$\eta_i(X) = \iota^2_* \frac{\partial}{\partial x_i} \left(\iota^*_2(X)\right). \tag{2.61}$$

Where i_1, i_2 are the inclusions, ι_* their push-forward and ι^* their pull-back. Using Lemma 1.11 we can easily check that

$$\nabla_X f = \nabla^1_{(\iota^*_1 X)} H. \tag{2.62}$$

This formula is independent of ξ_i or η_i and then it cannot be trans-S.

Define now $\bar{M} = M \times \mathbb{R}^s = N \times \mathbb{R}^s \times \mathbb{R}^s$ with complex structure J defined as in (2.4). Again, using Lemma 1.11 we obtain

$$\nabla_X J = \nabla^1_{(\iota^*_1 X)} H \tag{2.63}$$
$$\nabla_X \eta_i = \nabla_X dx_i = 0. \tag{2.64}$$

From (2.16) we have

$$(\nabla_{\bar{X}}\Omega)(\bar{Y},\bar{Z}) = (\nabla_{\bar{X}}F)(\bar{Y},\bar{Z}) + \sum_i \nabla_{\bar{X}}(dx_i \wedge \bar{\eta}_i)(\bar{Y},\bar{Z}) = (\nabla^1_{\iota_1^*\bar{X}}\Phi)(\iota_1^*\bar{Y},\iota_1^*\bar{Z}). \quad (2.65)$$

Finally, using (2.58):

$$(\nabla_{\bar{X}}\Omega)(\bar{Y},\bar{Z}) = (\nabla^1_{\iota_1^*\bar{X}}\Phi)(\iota_1^*\bar{Y},\iota_1^*\bar{Z}) = (\nabla^1_{H\iota_1^*\bar{X}}\Phi)(H\iota_1^*\bar{Y},\iota_1^*\bar{Z}) = (\nabla_{J\bar{X}}\Omega)(J\bar{Y},\bar{Z}). \quad (2.66)$$

This formula means there are manifolds in $W_3 \oplus W_4$ induced by (not trans-S) metric f-manifolds.

∎

Therefore, we cannot use the class $W_3 \oplus W_4$ to define trans-S in contrast to the almost contact case, where the class W_4 can be used to characterize trans-Sasakian manifolds.

3 Almost Trans-S-manifolds and the Gray-Hervella's Classification

Note that $\bar{M} = M \times \mathbb{R}^s$, with M almost trans-S, cannot lie in the classes $W_1 \oplus W_2 \oplus W_3$ and $W_1 \oplus W_2 \oplus W_4$ because trans-S are in $W_3 \oplus W_4$. Then the only two possibilities are the classes $W_2 \oplus W_3 \oplus W_4$ and $W_1 \oplus W_3 \oplus W_4$.

In this section we are going to check $\bar{M}$ does not fit in any of this classes. First we use the following expression for almost trans-S-manifolds appearing in [1].

$$\nabla_X \xi_i = -\alpha_i f X - \beta_i f^2 X + \sum_{j=1}^{s} \eta_j(\nabla_X \xi_i)\xi_j. \tag{3.1}$$

We also have that

$$(\nabla_{\bar{X}} \bar{\eta}_i)(\bar{Y}) = \bar{g}(\nabla_{\bar{X}} \xi_i, \bar{Y}) = -\alpha_i \bar{g}(\bar{f} \bar{X}, \bar{Y}) - \beta_i \bar{g}(\bar{f}^2 \bar{X}, \bar{Y}) + \sum_{j=1}^{s} \eta_j(\nabla_X \xi_i)\bar{\eta}_j(\bar{Y}). \tag{3.2}$$

Therefore,

$$\begin{aligned} \nabla_{\bar{X}} \Omega(\bar{Y}, \bar{Z}) &= \nabla_{\bar{X}} F(\bar{Y}, \bar{Z}) + \sum_{i=1}^{s} dx_i \wedge \nabla_{\bar{X}} \bar{\eta}_i(\bar{Y}, \bar{Z}) \\ &= A(X, Y, Z) + \sum_{i,j=1}^{s} dx_i(\bar{Y})\eta_j(\nabla_X \xi_i)\bar{\eta}_j(\bar{Z}) - dx_i(\bar{Z})\eta_j(\nabla_X \xi_i)\bar{\eta}_j(\bar{Y}) \end{aligned} \tag{3.3}$$

Where $A(\bar{X}, \bar{Y}, \bar{Z})$ is equation (2.27). Remember we have proved in Proposition 2.4

$$A(\bar{X}, \bar{Y}, \bar{Z}) - A(J\bar{X}, J\bar{Y}, \bar{Z}) = 0. \tag{3.4}$$

From [5] the condition an almost Hermitian manifold needed to satisfy for being in $W_2 \oplus W_3 \oplus W_4$ is:

$$\sum_{cyc}(\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y},\bar{Z})) = 0, \tag{3.5}$$

Where $\sum_{cyc}$ is the cyclic sum of (X, Y, Z).

For $W_1 \oplus W_3 \oplus W_4$ the defining condition is

$$\nabla_{\bar{X}}\Omega(\bar{X},\bar{Y}) - \nabla_{J\bar{X}}\Omega(J\bar{X},\bar{Y}) = 0. \tag{3.6}$$

We will compute first (3.5). From (3.3) it follows

$$\begin{aligned}\sum_{cyc}\left[\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y},\bar{Z})\right] = \sum_{cyc}\Bigg[&A(\bar{X},\bar{Y},\bar{Z}) - A(J\bar{X},J\bar{Y},\bar{Z})\\ &+\sum_{j,i=1}^{s} dx_i(\bar{X})\eta_j(\nabla_X\xi_i)\bar{\eta}_j(\bar{Z}) - dx_i(\bar{Z})\eta_j(\nabla_X\xi_i)\bar{\eta}_j(\bar{Y})\\ &+\sum_{j,i=1}^{s} -dx_i(J\bar{Y})\eta_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(\bar{Z}) + dx_i(\bar{Z})\eta_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(J\bar{Y})\Bigg].\end{aligned} \tag{3.7}$$

Then, using (3.4) and the definition of J:

$$\begin{aligned}\sum_{cyc}(\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y},\bar{Z})) = \sum_{cyc}\sum_{j,i=1}^{s}\big\{&dx_i(\bar{Y})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{Z})\\ -dx_i(\bar{Z})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) - \bar{\eta}_i(\bar{Y})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(\bar{Z}) &- dx_i(\bar{Z})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)dx_j(\bar{Y})\big\},\end{aligned} \tag{3.8}$$

which is not zero. Analogously, the equation (3.6) is equal to

$$\begin{aligned}\nabla_{\bar{X}}\Omega(\bar{X},\bar{Y}) - \nabla_{J\bar{X}}\Omega(J\bar{X},\bar{Y}) = &A(\bar{X},\bar{X},\bar{Y}) - A(J\bar{X},J\bar{X},\bar{Y})\\ &+\sum_{j,i=1}^{s} dx_i(\bar{X})\eta_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) - dx_i(\bar{Y})\eta_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{X})\\ &+\sum_{j,i=1}^{s} -dx_i(J\bar{X})\eta_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) + dx_i(\bar{Y})\eta_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(J\bar{X}).\end{aligned} \tag{3.9}$$

Again, using (3.3), (3.4) and the definition of J.

$$\nabla_{\bar{X}}\Omega(\bar{X},\bar{Y}) - \nabla_{J\bar{X}}\Omega(J\bar{X},\bar{Y}) = \sum_{j,i=1}^{s} \left\{ dx_i(\bar{X})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) \right.$$
$$\left. -dx_i(\bar{Y})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{X}) - \bar{\eta}_i(\bar{X})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)dx_j(\bar{X}) \right\}. \tag{3.10}$$

Actually, we can find an explicit example where these formulas are not zero. Theorem 5 from [1] claims:

Let N be a $(2n+s_1)$-dimensional trans-S-manifold with functions α_i, β_i. Then, if f, ξ_i come from the f-structure of N and $\{\partial/\partial t_i\}$ is a basis for $\mathbb{R}^s$, $\imath$ the inclusion of N into the warped product $M = \mathbb{R}^s \times_h N$ and f^* the following metric f-structure in M.

$$f^*(X) = \imath_*(f(\imath^* X)), \tag{3.11}$$

$$\xi_i^* = \begin{cases} \dfrac{\partial}{\partial t_i} & 1 \le i \le s \\ \dfrac{1}{h}\xi_{i-s} & s+1 \le i \le s+s_1, \end{cases} \tag{3.12}$$

Then, we have that M is a $(2n+s+s_1)$ almost trans-S-manifold with functions

$$\alpha_i^* = \begin{cases} 0 & 1 \le i \le s \\ \dfrac{\alpha_{i-s}}{h} & s+1 \le i \le s+s_1, \end{cases} \tag{3.13}$$

$$\beta_i^* = \begin{cases} \dfrac{h^{i)}}{h} & 1 \le i \le s \\ \dfrac{\beta_{i-s}}{h} & s+1 \le i \le s+s_1, \end{cases} \tag{3.14}$$

and

$$\nabla_X \xi_i^* = \frac{h^{i)}}{h}\imath^* X. \tag{3.15}$$

Now, consider the induced manifold $\bar{M} = M \times \mathbb{R}^{s+s_1} = \mathbb{R}^s \times_h N \times \mathbb{R}^{s+s-1}$. Then if we fix

$$\bar{X} = \bar{Y} = -\frac{\partial}{\partial x_{s+1}} \Rightarrow J\bar{Y} = J\bar{X} = \xi^*_{s+1} = (1/h)\xi_1, \tag{3.16}$$

$$\bar{Z} = -\frac{\partial}{\partial x_1} \Rightarrow J\bar{Y} = \xi^*_1 = \frac{\partial}{\partial t_1}, \tag{3.17}$$

we have

$$\nabla_{\bar{X}}\xi^*_i = \nabla_{\bar{Y}}\xi^*_i = \nabla_{\bar{Z}}\xi^*_i = 0, \tag{3.18}$$

$$\nabla_{J\bar{X}}\xi^*_i = \nabla_{J\bar{Y}}\xi^*_i = \nabla_{\xi^*_{s+1}}\xi^*_i = \frac{h^{i)}}{h}\iota^*\xi^*_{s+1} = \frac{h^{i)}}{h}\xi^*_{s+1}, \tag{3.19}$$

$$\nabla_{J\bar{Z}}\xi^*_i = \frac{h^{i)}}{h}\iota^*\xi^*_1 = 0, \tag{3.20}$$

$$\bar{\eta}_i(\bar{X}) = \bar{\eta}_i(\bar{Y}) = \bar{\eta}_i(\bar{Z}) = 0, \tag{3.21}$$

$$dx_i(\bar{X}) = dx_i(\bar{Y}) = \delta^i_{s+1},\ dx_i(\bar{Z}) = \delta^i_1. \tag{3.22}$$

Finally, evaluating this vector fields at the formula (3.8) we obtain

$$\begin{aligned}\sum_{cyc}(\nabla_{\bar{X}}\Omega(\bar{Y},\bar{Z}) - \nabla_{J\bar{X}}\Omega(J\bar{Y},\bar{Z})) &= -dx_1(\bar{Z})\bar{\eta}_{s+1}(\nabla_{J\bar{X}}\xi_1)dx_{s+1}(\bar{Y}) \\ &- dx_1(\bar{Z})\bar{\eta}_{s+1}(\nabla_{J\bar{Y}}\xi_1)dx_{s+1}(\bar{X}) = -2\frac{h^{1)}}{h} \neq 0,\end{aligned} \tag{3.23}$$

and in (3.10)

$$\begin{aligned}&\nabla_{\bar{X}}\Omega(\bar{X},\bar{Y}) - \nabla_{J\bar{X}}\Omega(J\bar{X},\bar{Y}) = \\ &= \sum_{j,i=1}^{s}\left\{dx_i(\bar{X})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_j(\nabla_{\bar{X}}\xi_i)\bar{\eta}_j(\bar{X})\right. \\ &\qquad \left.-\bar{\eta}_i(\bar{X})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)\bar{\eta}_j(\bar{Y}) - dx_i(\bar{Y})\bar{\eta}_j(\nabla_{J\bar{X}}\xi_i)dx_j(\bar{X})\right\} \\ &= -dx_{s+1}(\bar{Y})\bar{\eta}_{s+1}(\nabla_{J\bar{X}}\xi_i)dx_{s+1}(\bar{X}) = -\frac{h^{s+1)}}{h} \neq 0,\end{aligned} \tag{3.24}$$

as expected.

Conclusion

We have proved that if M is a trans-S-manifold, then the product manifold $M\times\mathbb{R}^s$ is Hermitian and lies in $W_3 \oplus W_4$. This generalizes the trans-Sasakian case where $M \times \mathbb{R}$ lies in W_4. Despite this result, there exist metric f-manifolds M such that $M\times\mathbb{R}^s$ which lies in $W_3 \oplus W_4$ without M being trans-S. Therefore, this class cannot be use to define tran-S manifolds. Finally, when M is an almost trans-S-manifold, $M \times \mathbb{R}^s$ lies in the most general class $W_1 \oplus W_2 \oplus W_3 \oplus W_4$.

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