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Computing the sets of totally symmetric and totally conjugate orthogonal partial Latin squares by means of a SAT solver

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Abstract

Conjugacy and orthogonality of Latin squares have been widely studied in the literature not only for their theoretical interest in combinatorics, but also for their applications in distinct fields as experimental design, cryptography or code theory, amongst others. This paper deals with a series of binary constraints that characterize the sets of partial Latin squares of a given order for which their six conjugates either coincide or are all of them distinct and pairwise orthogonal. These constraints enable us to make use of a SAT solver to enumerate both sets. As an illustrative application, it is also exposed a method to construct totally symmetric partial Latin squares that gives rise, under certain conditions, to new families of Lie partial quasigroup rings.

Key words: Partial Latin square, conjugacy, orthogonality.
MSC 2000: 05B15, 20N05.

1 Introduction

A *quasigroup* [22] is a pair $(S, \cdot)$ formed by a nonempty set S endowed with a product $\cdot$ such that, if any two of the three symbols a, b and c in the equation $a \cdot b = c$ are given as elements of S, then the third one is uniquely determined. The size of S is the *order* of the quasigroup. The multiplication table of a quasigroup of order n constitutes a *Latin square* of the same order, that is, an $n \times n$ array in which each cell contains one symbol chosen from the set S, such that each symbol occurs exactly once in each row and in each column. The number of Latin squares is known [24, 26, 29, 30] for order up to 11.

Bruck [12] introduced the concept of *totally symmetric quasigroup* as a quasigroup $(S,\cdot)$ for which the equation $a\cdot b=c$ remains valid under every permutation of the three symbols $a,b,c\in S$. There exist six such permutations and each one of them gives rise to a new quasigroup, which is said to be *conjugate* to $(S,\cdot)$. Hence, a quasigroup is totally symmetric if its six conjugates coincide. If besides, the quasigroup is *idempotent*, that is, if $a\cdot a=a$, for all $a\in S$, then this notion is equivalent to that of a *Steiner triple system*. The distribution of totally symmetric quasigroups and Steiner triple systems into isomorphism classes is known [1, 25] for orders up to 10 and 19, respectively.

Two quasigroups of order n are said to be *orthogonal* if the juxtaposition of their corresponding multiplication tables gives rise to an $n\times n$ array containing n^2 distinct ordered pairs. Stein [34] posed the problem of constructing a quasigroup or Latin square that is orthogonal to one of its conjugates. It is known in this regard [6, 7, 11, 31] the existence of quasigroups that are orthogonal to the conjugate under consideration, distinct of themselves, for any order $n\not\in\{2,3,6\}$. Much more recently, Bennett and Zhang [10] dealt with Latin squares for which each one of their conjugates is orthogonal to its transpose. They proved the existence of such Latin squares for all prime powers $n\not\in\{2,3,5\}$. Further, Lindner et al. [28] focused on idempotent Latin squares for which their six conjugates are distinct and pairwise orthogonal. They proved in particular the existence of such Latin squares for every order being a prime power $n\geq 8$ and also for all sufficiently large orders n. Bennett [4] established $n>5594$ as an upper bound for this last condition except possibly $n=6810$, and enumerate a series of smaller orders for which these Latin squares also exist. Four years later, he improved [5] the previous upper bound to $n>5074$. Much more recently, Belyavskaya and Popovich [3] introduced the equivalent notion of *totally conjugate orthogonal quasigroup* as a quasigroup for which its six conjugates are distinct and pairwise orthogonal. They proved the existence of such quasigroups for any order $n\geq 11$ that is relatively prime to 2, 3, 5, and 7. Their motivation to study this kind of quasigroups was mainly based on their application in error detecting codes [2].

The concept of quasigroup is straightforwardly generalized to that of *partial quasigroup* of order n, for which (a) the law $\cdot$ is a partial binary operation on a finite set S of n elements, and (b) if the equations $a\cdot x=b$ and $y\cdot a=b$, with $a,b\in S$, have solutions for x and y in S, then these solutions are unique. The multiplication table of a partial quasigroup of order n constitutes a *partial Latin square* of the same order, that is, an $n\times n$ array in which each cell is either empty or contains one element chosen from S, such that each symbol occurs at most once in each row and in each column. The number of partial Latin squares is known [17, 18, 19, 20] for order up to seven.

Since Evans [15] introduced the problem of embedding a partial quasigroup of order n into a quasigroup of order $2n$, a wide amount of authors have dealt with the embedding of distinct types of partial quasigroups; particularly, that of a partial totally symmetric quasigroup into a totally symmetric quasigroup [13, 27, 32, 33]. Further, the orthogonality

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among conjugates of a partial Latin square was indirectly contemplated [8, 9, 23] by focusing on the existence of incomplete Latin squares that are orthogonal to one of their conjugates and have an empty subsquare that can be filled by means of a Latin square that is orthogonal in turn to its corresponding conjugate. A more general case was recently proposed by the first author [18], who makes use of computational algebraic geometry to enumerate the set of self-orthogonal partial Latin squares of order $n \leq 4$. This paper delves into this topic by dealing with the sets of partial Latin squares of a given order for which their six conjugates either coincide or are all of them distinct and pairwise orthogonal, respectively. In order to improve the computational efficiency, it is proposed to focus on techniques to solve Boolean satisfiability problems instead of those on algebraic geometry.

As an illustrative application of the exposed study, we also delve into a recent work developed by the authors [16] about the enumeration of partial quasigroup rings over finite fields derived from partial Latin squares. Bruck [12] introduced the concept of *quasigroup ring* related to a quasigroup $(S,\cdot)$ as an algebra of basis $\{e_a \mid a \in S\}$ over a base field $\mathbb{K}$ such that $e_a e_b = e_{a\cdot b}$, for all $a, b \in S$. This concept is straightforwardly generalized to that of *partial quasigroup ring* in case of being the pair $(S,\cdot)$ a partial quasigroup. In this paper, we describe a totally symmetric partial Latin square of order $3n$, derived from a given partial Latin square of order n, that enables us to introduce in turn a Lie partial quasigroup ring over a finite field of characteristic two.

The paper is organized as follows. In Section 2, we expose some preliminary concepts and results on partial Latin squares that are used throughout our study. In Section 3, we introduce a pair of series of binary constraints that characterize, respectively, the sets of totally symmetric and totally conjugate orthogonal partial Latin squares of given order and weight. Finally, Section 5 deals with an illustrative example that enables us to construct a family of Lie partial quasigroup rings from a totally symmetric partial Latin square satisfying certain conditions.

2 Preliminaries

This section deals with some basic concepts and notations on partial Latin squares that are used throughout the paper. We refer the reader to the monographs of Dénes and Keedwell [14] for more details about this topic.

Hereafter, the set of partial Latin squares of order n is denoted as PLS_n, whereas the set of symbols of any such a partial Latin square $P = (p_{ij}) \in \mathrm{PLS}_n$ is assumed to be the set $[n] = \{1, \ldots, n\}$. An *entry* of P is any triple $(i, j, p_{i,j}) \in [n] \times [n] \times [n]$. The partial Latin square P is uniquely determined by the set of all its entries, which is called its *entry set* and denoted as $E(P)$. The size of this set coincides, therefore, with the number of non-empty cells of P, which constitutes its *weight*. From here on, $\mathrm{PLS}_{n;m}$ denotes the set of partial Latin squares of order n having weight m. Thus, for instance, the partial

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Latin square P in Figure 1 belongs to the set $\text{PLS}_{3;4}$ and has as entry set the set $E(P) = \{(1,1,2),(1,2,1),(2,1,1),(3,3,3)\}$.

$P \equiv$

2	1	
1		
		3

Figure 1: Partial Latin square in $\text{PLS}_{3;4}$.

Let S_3 denote the symmetric group of three elements. Let P be a partial Latin square in $\text{PLS}_{n;m}$ and let π be a permutation in S_3. The π-*conjugate* of P is defined as the partial Latin square $P^\pi \in \text{PLS}_{n;m}$ such that $E(P^\pi)\{(p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)})\colon\ (p_1,p_2,p_3) \in E(P)\}$. There exist, therefore, six conjugates: $P^{\text{Id}} = P$, $P^{(12)} = P^t$, $P^{(13)}$, $P^{(23)}$, $P^{(123)} = (P^{(23)})^t$ and $P^{(132)} = (P^{(13)})^t$, where the notation ^{t} denotes the transpose of the corresponding partial Latin square. In order to illustrate these conjugates, let us consider the partial Latin square $P \in \text{PLS}_{3;4}$ in Figure 2. Particularly, $E(P) = \{(1,1,1),(1,2,2),(2,2,3),(3,3,1)\}$ and hence, once the corresponding permutations among the three components of each entry are done, we obtain the conjugates therein exposed. Observe that all of them are distinct. Figure 1 shows instead a partial Latin square in $\text{PLS}_{3;4}$ for which all its conjugates coincide. In this case, the partial Latin square under consideration is said to be *totally symmetric*. Hereafter, the set of totally symmetric partial Latin squares of order n and its subset of partial Latin squares of weight m are respectively denoted as TSPLS_n and $\text{TSPLS}_{n;m}$.

$P \equiv$

1	2	
	3	
		1

$P^{(12)} \equiv$

1		
2	3	
		1

$P^{(13)} \equiv$

1		3
	1	
	2	

$P^{(23)} \equiv$

1	2	
		2
3		

$P^{(123)} \equiv$

1		3
2		
	2	

$P^{(132)} \equiv$

1		
	1	2
3		

Figure 2: Partial Latin square in $\text{PLS}_{3;4}$ and its conjugates.

Let $P = (p_{ij})$ and $Q = (q_{ij})$ be two partial Latin squares of order n. They are said to be *orthogonal* if all the ordered pairs on non-empty entries that are obtained when both arrays are superimposed are distinct. Equivalently, given $i, i', j, j' \in [n]$ such that $p_{ij} = p_{i'j'} \in [n]$, then q_{ij} and $q_{i'j'}$ are not the same symbol of $[n]$. Thus, for instance, the partial Latin squares P and $P^{(13)}$ in Figure 2 are orthogonal, but the partial Latin squares P and $P^{(12)}$ in the same figure are not. In this regard, given a permutation $\pi \in S_3 \setminus \{\text{Id}\}$, a partial Latin square $P \in \text{PLS}_n$ is said to be π-*orthogonal* if it is orthogonal to its π-conjugate.

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Particularly, if $\pi = (12)$, then P is said to be *self-orthogonal*. Thus, for instance, the partial Latin square $P^{(23)}$ in Figure 2 is self-orthogonal. Finally, we say that a partial Latin square is *totally conjugate orthogonal* if its six conjugates are distinct and pairwise orthogonal. This is the case, for instance, of the partial Latin square shown in Figure 3. From here on, the set of totally conjugate orthogonal partial Latin squares of order n and its subset of partial Latin squares of weight m are respectively denoted as TCOPLS_n and $\mathrm{TCOPLS}_{n;m}$.

$P \equiv$

		3
		2
1	3	

$P^{(12)} \equiv$

		1
		3
3	2	

$P^{(13)} \equiv$

3		
		2
	3	1

$P^{(23)} \equiv$

		3
	3	
1		2

$P^{(123)} \equiv$

		1
	3	
3		2

$P^{(132)} \equiv$

3		
		3
	2	1

Figure 3: Totally conjugate orthogonal partial Latin square in $\mathrm{PLS}_{3;4}$.

The set PLS_n is identified [18] with the set of zeros of the following system of equations in the set of n^3 variables $\{X\} = \{x_{ijk} \mid i,j,k \in [n]\}$.

$$\begin{cases} x_{ijk}x_{i'jk} = 0, \text{ for all } i,i',j,k \leq n \text{ such that } i \neq i', \\ x_{ijk}x_{ij'k} = 0, \text{ for all } i,j,j',k \leq n \text{ such that } j \neq j', \\ x_{ijk}x_{ijk'} = 0, \text{ for all } i,j,k,k' \leq n \text{ such that } k \neq k', \\ x_{ijk} \in \{0,1\}, \text{ for all } i,j,k \leq n. \end{cases} \tag{1}$$

Specifically, every partial Latin square $P = (p_{ij}) \in \mathrm{PLS}_n$ is uniquely identified with a zero $(x_{111}, \ldots, x_{nnn})$, where $x_{ijk} = 1$ if $p_{ij} = k$ and 0, otherwise. Hereafter, in order to avoid degeneracy, partial Latin squares are assumed to have at least one entry in each row, at least one entry in each column, and at least one copy of each symbol. To get this condition, the following inequations are added to (1)

$$\begin{cases} \sum_{j,k \in [n]} x_{ijk} \geq 1, \text{ for all } i \in [n], \\ \sum_{i,k \in [n]} x_{ijk} \geq 1, \text{ for all } j \in [n], \\ \sum_{i,j \in [n]} x_{ijk} \geq 1, \text{ for all } k \in [n]. \end{cases} \tag{2}$$

Based on (1) and (2), we establish in Section 3 some equations to deal, respectively, with the sets TSPLS_n and TCOPLS_n. To this end, let us introduce the following notation

$$x^{\pi}_{i_1 i_2 i_3} := x_{i_{\pi(1)} i_{\pi(2)} i_{\pi(3)}},$$

for all $\pi \in S_3$ and $x_{i_1 i_2 i_3} \in \{X\}$. Besides, we label the six permutations in S_3 as

$$S_3 := \{\pi_1 = \mathrm{Id},\ \pi_2 = (12),\ \pi_3 = (13),\ \pi_4 = (23),\ \pi_5 = (123),\ \pi_6 = (132)\}.$$

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3 Binary constraints related to the sets TSPLS_n and TCOPLS_n

This section deals with a series of binary constraints that characterize the sets of totally symmetric and totally conjugate orthogonal partial Latin squares of given order and weight.

Lemma 3.1. *Let n and m be two positive integers such that $n \leq m \leq n^2$.*

a) If $m > n$, then every pair of orthogonal conjugates of a partial Latin square in the set $\mathrm{TCOPLS}_{n;m}$ are distinct.

b) If $|\mathrm{TCOPLS}_{n;m}| = 0$, then $|\mathrm{TCOPLS}_{n;m'}| = 0$, for all $m' \in \{m+1, \ldots, n^2\}$.

Proof. Let us prove each statement separately.

a) Let $P \in \mathrm{PLS}_{n;m}$ and $\pi, \pi' \in S_3$ be such that $\pi \neq \pi'$ and $P^\pi = P^{\pi'}$. Since $m > n$, there exists one symbol $k \in [n]$ and a distinct pair of elements (i_1, j_1) and, (i_2, j_2) in $[n] \times [n]$ such that $\{(i_1, j_1, k), (i_2, j_2, k)\} \subseteq E(P^\pi) \cap E(P^{\pi'})$. As a consequence, $P^\pi = P^{\pi'}$ is not orthogonal to itself.

b) Otherwise, the partial Latin square that results after emptying any $m' - m$ filled cells of the partial Latin square in $\mathrm{TCOPLS}_{n;m'}$ would be in $\mathrm{TCOPLS}_{n;m}$, which is a contradiction. □

Proposition 3.2. *Let n and m be two positive integers such that $n < m \leq n^2$. Then,*

a) The set TSPLS_n is identified with the set of zeros of (1)–(2) and

$$x_{ijk}^{\pi_s} = x_{ijk}, \text{ for all } i, j, k \in [n] \text{ and } s \in \{1, 2, 3\}. \tag{3}$$

b) The set $\mathrm{TSPLS}_{n;m}$ is identified with the set of zeros of (1)–(3) and

$$\sum_{i,j,k \in [n]} x_{ijk} = m. \tag{4}$$

c) The set TCOPLS_n is identified with the set of zeros of (1)–(2) and

$$x_{ijp}^{\pi_s} x_{klp}^{\pi_s} x_{ijq}^{\pi_t} x_{klq}^{\pi_t} = 0, \text{for all } i, j, k, l, p, q \leq n; s, t \leq 3; \text{ such that } (i, j) \neq (k, l), s \leq t. \tag{5}$$

d) The set $\mathrm{TCOPLS}_{n;m}$ is identified with the set of zeros of (1), (2), (4) and (5).

Proof. The result follows straightforwardly from the definitions exposed in Section 2 once each partial Latin square $P = (p_{ij}) \in \mathrm{PLS}_{r,s,n}$ is identified with a zero $(x_{111}, \ldots, x_{rsn})$ such that $x_{ijk} = 1$ if $p_{ij} = k$ and 0, otherwise. Thus, for instance, if we focus on the proof of statement (c), then, given $1 \leq s < t \leq 3$, the system of equations determined by (5) involves the π_s^{-1}- and π_t^{-1}-conjugates of P to be orthogonal. Besides, from Lemma 3.1.a, both conjugates are distinct. □

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Proposition 3.2 has been implemented in the SAT solver Minion [21] to obtain the numerical data exposed in Table 1. Further, Table 2 indicates the run time that is required in a system with an *Intel Core i7-2600, with a 3.4 GHz processor and 16 GB of RAM* to determine one specific example in the sets $\text{TSPLS}_{n;m}$ and $\text{TCOPLS}_{n;m}$.

m	$\lvert\text{TSPLS}(n;m)\rvert$				$\lvert\text{TCOPLS}(n;m)\rvert$	
	n				n	
	3	4	5	6	3	4
3	1				36	
4	6	1			216	576
5	6	12	1		12	45168
6	10	24	20	1	0	315048
7	12	64	80	30	0	391824
8	3	60	220	210	0	95028
9	3	100	380	680	0	2616
10		148	910	1980		0
11		72	1010	4380		0
12		90	1630	7660		0
13		72	2740	17820		0
14		36	2040	23370		0
15		16	2784	37476		0
16		16	3395	68850		0
17			2195	68190		
18			2080	96660		
19			2320	145560		
20			900	122040		
21			900	146040		
22			480	196200		
23			240	132480		
24			30	148710		
25			30	157320		
26				101430		
27				81540		
28				86310		
29				35820		
30				33390		
31				20340		
32				11340		
33				4560		
34				3960		
35				720		
36				480		
Total	41	711	24385	1755547	264	850260

Table 1: Distribution of the sets $\text{TSPLS}_{n;m}$ and $\text{TCOPLS}_{n;m}$.

4 Lie partial quasigroup rings derived from the conjugate-extension of a partial Latin square

The inclusion of new binary constraints into (1)–(5) enables us to determine families of partial Latin squares in the sets TSPLS_n and TCOPLS_n with possible applications in distinct fields. As an illustrative example, we conclude this paper by describing in this section a new family of Lie partial quasigroup rings related to a totally symmetric partial Latin square of order $3n$, which is derived in turn from a given partial Latin square of order n. Recall that a *Lie algebra* is an anti-commutative algebra A that holds the so-called *Jacobi identity*

$$J(a,b,c) := (ab)c + (bc)a + (ca)b = 0, \text{ for all } a,b,c \in A. \tag{6}$$

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n	m	Run time (seconds) $\text{TSPLS}_{n;m}$	Run time (seconds) $\text{TCOPLS}_{n;m}$
5	5	0	22
	10	0	3
6	6	0	8561
	12	0	10
	15	0	74
10	10	69	Out of memory
	50	0	”
15	15	> 3 hours	”
	60	2	”
20	100	Out of memory	”

Table 2: Run times required to get exactly one totally symmetric or totally conjugate orthogonal partial Latin square of a given order and weight.

Let $P = (p_{ij}) \in \text{PLS}_{n;m}$. We define the $n \times n$ arrays $P' = (p'_{ij})$ and $P'' = (p''_{ij})$ such that

$$p'_{ij} := \begin{cases} p_{ij} + n, & \text{if } p_{ij} \in [n], \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad p''_{ij} := \begin{cases} p_{ij} + 2n, & \text{if } p_{ij} \in [n], \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Then, we define the partial Latin square $\overline{P} = (\overline{p}_{ij}) \in \text{PLS}_{3n;6m}$ by means of nine $n \times n$ blocks as

$$\overline{P} :\equiv \begin{array}{|c|c|c|} \hline \mathbf{0} & P'' & P'^{(23)} \\ \hline P''^{(12)} & \mathbf{0} & P^{(132)} \\ \hline P'^{(123)} & P^{(13)} & \mathbf{0} \\ \hline \end{array} \tag{8}$$

where $\mathbf{0}$ denotes the $n \times n$ array with all its entries being zero. We call this new partial Latin square the *conjugate-extension* of P. Thus, for instance, Figure 4 shows the conjugate-extension of the partial Latin square exposed in Figure 2.

Lemma 4.1. *If $P \in \text{PLS}_{n;m}$, then $\overline{P} \in \text{TSPLS}_{3n;6m}$.*

Proof. The result follows from the entry set $E(\overline{P})$ once we keep in mind (7) and (8). □

Let $A_{\mathbb{K}}(P)$ denote the partial quasigroup ring over a finite field $\mathbb{K}$ of characteristic two that is related to $\overline{P}$. Particularly, we focus on the case of being $P \in \text{TSPLS}_n$. If this is the case, then the definition (8) of the partial Latin square $\overline{P}$ results

$$\overline{P} \equiv \begin{array}{|c|c|c|} \hline \mathbf{0} & P'' & P' \\ \hline P'' & \mathbf{0} & P \\ \hline P' & P & \mathbf{0} \\ \hline \end{array} \tag{9}$$

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			7	8		4	5	
				9				5
					7	6		
7						1		
8	9						1	2
		7				3		
4		6	1		3			
5				1				
	5			2				

Figure 4: Conjugate-extension of the partial Latin square $P \in \mathrm{PLS}_3$ of Figure 2.

Theorem 4.2. *Let $\mathbb{K}$ be a finite field of characteristic two and let $P \in \mathrm{TSPLS}_n$ be the multiplication table of a quasigroup $([n], \cdot)$ satisfying the* left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \text{ for all } a, b, c \in [n]. \tag{10}$$

Then, the partial quasigroup ring $A_{\mathbb{K}}(P)$ is a Lie algebra.

Proof. The symmetry of the partial Latin square $\overline{P} = (\overline{p}_{ij})$, with $p_{ii} = 0$, for all $i \leq 3n$, together with the fact of being $\mathbb{K}$ a finite field of characteristic two, involves $A_{\mathbb{K}}(P)$ to be anti-commutative. Now, in order to prove that the Jacobi identity (6) holds, suppose $\{e_1, \ldots, e_{3n}\}$ to be the basis of $A_{\mathbb{K}}(P)$, which we partition into the three sets $\{e_1, \ldots, e_n\}$, $\{e_{n+1}, \ldots, e_{2n}\}$ and $\{e_{2n+1}, \ldots, e_{3n}\}$. Let $S(e_i)$ denote which one of these three sets contains each basis vector e_i. From (9), we have that, if $S(e_i) = S(e_j)$, then $e_ie_j = 0$. Besides, if $S(e_i) \neq S(e_j)$ and $e_ie_j \neq 0$, then $S(e_i) \neq S(e_ie_j) \neq S(e_j)$. As a consequence, $J(e_i, e_j, e_k) = 0$, for all $i, j, k \leq 3n$ such that the three sets $S(e_i)$, $S(e_j)$ and $S(e_k)$ either coincide or are pairwise distinct. Then, from the symmetry of the Jacobi identity, it is enough to focus on the expression $J(e_i, e_j, e_k)$ in case of being $S(e_i) = S(e_j) \neq S(e_k)$. If this is the case, $e_ie_j = 0$ and hence, $J(e_i, e_j, e_k) = (e_je_k)e_i + (e_ke_i)e_j = e_{(j\cdot k)\cdot i} + e_{(k\cdot i)\cdot j}$. The result follows from the symmetry of the partial Latin square $\overline{P}$ and the left invertive law. □

Every totally symmetric partial Latin square satisfying (10) constitutes the multiplication table of a partial totally symmetric group. In order to compute this kind of partial Latin squares, we include the following equations to (1)–(4)

$$x_{ijk}x_{kls}x_{ljt}(x_{tis} - 1) = 0, \text{ for all } i, j, k, l, s, t \in [n] \tag{11}$$

$$\left(\sum_{k \leq n} x_{ijk} - 1\right)\left(\sum_{k \leq n} x_{ljk}\right) x_{ljt} \left(\sum_{k \leq n} x_{tik}\right) = 0, \text{ for all } i, j, l, t \in [n] \tag{12}$$

$$x_{ijk}\left(\sum_{s \leq n} x_{kls} - 1\right)\left(\sum_{s \leq n} x_{ljs}\right) x_{ljt} \left(\sum_{s \leq n} x_{tis}\right) = 0, \text{ for all } i, j, k, l, t \in [n] \tag{13}$$

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The implementation of these equations into our SAT solver determines, for instance, the pair of partial Latin squares exposed in Figure 5, which give rise in turn, according to Theorem 4.2, to a pair of Lie partial quasigroup rings as we have previously described.

3		1
	2	
1		3

2	1				
1	2				
		4	3		
		3	4		
				6	5
				5	6

Figure 5: Totally symmetric partial Latin squares satisfying the left invertive law.

5 Conclusion and further studies

We have described in this paper a series of binary constraints that enable us to determine the distribution of the sets TSPLS_n and TCOPLS_n of totally symmetric and totally conjugate partial Latin squares of order n, respectively, according to their weights. By means of the SAT solver Minion, we have computed the former, for all $2 \leq n \leq 6$, and the latter, for all $2 \leq n \leq 4$. A further study to improve the efficiency of the proposed method is required to deal with higher orders. Besides, we have introduced the conjugate-extension of a given partial Latin square, which gives rise to a totally symmetric partial Latin square. Particularly, the description of a family of Lie partial quasigroup rings derived from the conjugate-extension of a totally symmetric partial Latin square that holds the left invertive law has enabled us to delve into the open problem of constructing examples of this type of Lie algebras.

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