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A PARAMETRIZATION FOR THE SYMBOLS OF A HANKEL TYPE OPERATOR

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Abstract. Hankel operators and their symbols, as generalized by V. Pták and P. Vrbová, are considered. In this more general framework, a linear operator X from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is said to be a Hankel operator for given contractions T_1 on $\mathcal{H}_1$ and T_2 on $\mathcal{H}_2$ if, and only if, $XT_1^* = T_2X$ and X satisfies a boundedness condition that depends on the unitary parts of the minimal isometric dilations V_1 of T_1 and V_2 of T_2. A Hankel symbol of X is a dilation Z of X, with a certain norm constraint, such that $ZV_1^* = V_2Z$. The boundedness condition imposed to X has revealed to be essential, indeed necessary and sufficient, for X to admit Hankel symbols. As for a description of the symbols of X, this work provides a parametric labeling of all of them by means of Schur like formula. As a by-product, a new proof of the existence of Hankel symbols is obtained. The proof is established by associating to X, T_1 and T_2 a suitable isometry V so that there is a bijective correspondence between the symbols of X and the family of all minimal unitary extensions of V.

1. Introduction

Under the commutant perspective, the classical Hankel operators can be characterized by the intertwining relation they satisfy. From the same point of view, other intertwining operators may be thought as abstract or generalized Hankel operators. In this more general setting, which was the one adopted by Pták and Vrbová [7], [8], [9], a Hankel operator is a linear map X from a

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separable Hilbert space $\mathcal{H}_1$ into a separable Hilbert space $\mathcal{H}_2$ such that $XT_1^* = T_2X$, where T_1 on $\mathcal{H}_1$ and T_2 on $\mathcal{H}_2$ are given contraction operators and T_1^* denotes the adjoint of T_1.

If we consider the contractions $T_1 := PS^*|_{H^2}$ and $T_2 := P_-S|_{H^2_-}$, with S the shift operator on L^2, H^2 the Hardy space, H^2_- its orthogonal complement in L^2, and P and P_- the corresponding orthogonal projections, then a classical Hankel operator can be regarded as a linear map $X : H^2 \to H^2_-$ such that $XT_1^* = T_2X$.

In the classical case, the celebrated Nehari's Theorem states that a Hankel operator X is bounded if and only if there exists an L^∞ function Φ, a symbol of X, such that $Xf = P_-\Phi f$ for all $f \in H^2$. Hence, the symbols are either multiplication operators induced by L^∞ functions with prescribed antianalytic part or, from another point of view, operators that commute with S and have the same fixed component from H^2 into H^2_-. Since the unitary operators $V_1 = S^*$ and $V_2 = S$ are the corresponding minimal isometric dilations of the contractions T_1 and T_2 defined above, we can conclude that the symbols of the given Hankel operator X are the intertwining dilations Z of X, namely, those linear operators $Z : L^2 \to L^2$ such that $P_-Z|_{H^2} = X$ and $ZV_1^* = V_2Z$.

In the abstract framework the operators that play the role of symbols might be the solutions Z of the commutant dilation problem $ZV_1^* = V_2Z$, where V_1 and V_2 are the minimal isometric dilations of T_1 and T_2, respectively. The investigations carried on by Pták and Vrbová revealed that the problem is solvable whenever X satisfies certain boundedness condition that depends on the unitary parts of the Wold-von Neumann decompositions of V_1 and V_2. Since the Wold-von Neumann decomposition is trivial in the classical case, for S being unitary, the result includes the classical situation.

In this survey, we deal with the problem of describing the symbols Z of any abstract Hankel operator X, for given contractions T_1 and T_2. We show that there is a bijective correspondence between the symbols of X and the minimal unitary extensions of a Hilbert space isometry V determined by X, T_1 and T_2. Since any Hilbert space isometry has at least one minimal unitary extension, our approach provides a direct proof of the existence of the symbols for the generalized Hankel operator. The Arov-Grossman functional model [1] yields a complete description of the minimal unitary extensions of V, as it associates to each minimal unitary extension U of V a function θ_U in a suitable Schur class of operator valued functions, and to each function θ in the Schur class, an operator model U_θ which gives rise to a minimal unitary extension of V, in such a way that the outlined correspondence is bijective. This method along with the Arov-Grossman model gives, in turn, a bijective correspondence between the symbols of X and the Schur class. As a consequence, the connection between the symbols and the Schur functions can be realized as a parametric description. We also present uniqueness criteria and a Schur like formula.

We point out that the line of investigations initiated by Pták and Vrbová has been pursued mainly by Mancera and Paúl [5], [6]. It is worth to mention that abstract Hankel operators can be treated as bilinear forms defined in the even more general framework of the algebraic scattering systems as by Cotlar and Sadosky (see, for instance, [3], [4] and further references given therein.) The construction of the isometry V, which plays a key role in the proof of our main result, is in fact inspired by the Cotlar-Sadosky algebraic scattering systems methods.

2. Preliminaries

Throughout this survey, all Hilbert spaces are assumed to be complex and separable. If $\mathcal{G}$ is a closed linear subspace of a Hilbert space $\mathcal{K}$, then $P_{\mathcal{G}}^{\mathcal{K}}$ stands for the orthogonal projection from $\mathcal{K}$ onto $\mathcal{G}$.

We denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the space of all bounded linear operators from the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$. The space $\mathcal{L}(\mathcal{H},\mathcal{H})$ is denoted by $\mathcal{L}(\mathcal{H})$. By 1 we denote either the scalar unit or the identity operator, depending on the context. If $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $\|T\| \leq \beta$, then $D_T^\beta = (\beta^2 - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T^\beta = \overline{D_T^\beta \mathcal{H}}$. For $\beta = 1$, we use the standard notation D_T and $\mathcal{D}_T$ for the defect operator and the defect space of T, respectively.

For an isometric operator $V \in \mathcal{L}(\mathcal{K})$, we denote by $\mathcal{R}$ the closed linear subspace of $\mathcal{K}$ that reduces V to its unitary part in the Wold-von Neumann decomposition. In particular, $\mathcal{R} = \bigcap_{n=0}^{\infty} V^n\mathcal{K}$ and $P_{\mathcal{R}}^{\mathcal{K}} = \lim_{n\to\infty} V^n V^{*n}$.

In what follows, $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ are two given contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively.

As defined by Pták and Vrbová [7], [8], [9], an operator $X \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ is said to be a *Hankel operator* for T_1 and T_2 if and only if $XT_1^* = T_2X$ and, for some $\beta \geq 0$,

$$(1)\qquad |\langle Xh_1, h_2\rangle| \leq \beta\|P_{\mathcal{R}_1}^{\mathcal{K}_1}h_1\|\|P_{\mathcal{R}_2}^{\mathcal{K}_2}h_2\|, \quad \text{for all } h_1 \in \mathcal{H}_1 \text{ and } h_2 \in \mathcal{H}_2,$$

where $\mathcal{R}_j$ is the subspace of $\mathcal{K}_j$ which reduces the minimal isometric dilation V_j of T_j to the unitary part R_j of V_j ($j = 1,2$). We define $\|X\|_{PV} = \inf\beta$, where β varies over all numbers satisfying (1).

Given a Hankel operator X for T_1 and T_2, we say that $Z \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ is a *Hankel symbol* of X if and only if (i) $ZV_1^* = V_2Z$, (ii) $P_{\mathcal{H}_2}^{\mathcal{K}_2}Z|_{\mathcal{H}_1} = X$, and (iii) $\|Z\| = \|X\|_{PV}$.

As already remarked in the introduction, the relation $XT_1^* = T_2X$ alone is not sufficient to guarantee the existence of symbols. This difficulty is overtaken by means of the boundedness condition (1), since it turns out to be necessary and sufficient to ensure that there exist intertwining dilations Z of X ((i) and (ii)), which altogether satisfy (iii). The reader is referred to [7], [8] and [9], as the original sources.

If X is a fixed Hankel operator for T_1 and T_2, then Douglas' Lemma (cf. [8, Proposition 1.4]) yields a unique bounded linear operator $\widetilde{X}$ from $\mathcal{E}_1 := \overline{P_{\mathcal{R}_1}^{\mathcal{K}_1}\mathcal{H}_1}$

into $\mathcal{E}_2 := \overline{P^{\mathcal{K}_2}_{\mathcal{R}_2}\mathcal{H}_2}$ such that

$$X = (P^{\mathcal{K}_2}_{\mathcal{R}_2}|_{\mathcal{H}_2})^* \widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1}|_{\mathcal{H}_1} \quad \text{and} \quad \|\widetilde{X}\| = \|X\|_{PV}. \tag{2}$$

As the minimal unitary extensions of an isometry, on one hand, and the so called Schur functions, on the other, play key roles in the description of the symbols of a given Hankel operator, we conclude this section with a few words about these objects.

If V is an isometric operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(V)$ and range $\mathcal{R}(V)$, both closed linear subspaces of $\mathcal{H}$, then a minimal unitary extension of V is a unitary operator U acting on a Hilbert space $\mathcal{F}$ that contains $\mathcal{H}$ as closed linear subspace such that $U|_{\mathcal{D}(V)} = V$ and $\mathcal{F} = \bigvee_{n\in\mathbb{Z}} U^n\mathcal{H}$. Two minimal unitary extensions of V, namely $U \in \mathcal{L}(\mathcal{F})$ and $U' \in \mathcal{L}(\mathcal{F}')$, are to be interpreted as indistinguishable whenever there exists an isometric isomorphism $\varphi : \mathcal{F} \to \mathcal{F}'$ such that $\varphi|_{\mathcal{H}} = 1$ and $\varphi U = U'\varphi$. As for the existence of minimal unitary extensions of any given isometry V, we remark that if U_T, acting boundedly on the Hilbert space $\mathcal{F}_T$, is the minimal unitary dilation of the contraction $T := VP^{\mathcal{H}}_{\mathcal{D}(V)}$, then U_T is a minimal unitary extension of V. The defect spaces of the isometry V are $\mathcal{N} = \mathcal{H} \ominus \mathcal{D}(V)$ and $\mathcal{M} = \mathcal{H} \ominus \mathcal{R}(V)$. If either $\mathcal{N} = \{0\}$ or $\mathcal{M} = \{0\}$, then V has a unique (up to isometric isomorphisms) minimal unitary extension.

If $\mathcal{N}$ and $\mathcal{M}$ are Hilbert spaces, then the Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$ is the family of all analytic functions $\theta : \mathbb{D} \to \mathcal{L}(\mathcal{N},\mathcal{M})$ such that $\sup\limits_{z\in\mathbb{D}} \|\theta(z)\| \leq 1$.

The Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$ features the Arov-Grossman functional model. The following theorem can be found in [1].

Theorem 2.1. *Let V be an isometric operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(V)$, range $\mathcal{R}(V)$ and defect spaces $\mathcal{N}$ and $\mathcal{M}$. The map that to each minimal unitary extension $U \in \mathcal{L}(\mathcal{F})$ associates the function*

$$\theta_U(z) := P^{\mathcal{F}}_{\mathcal{M}} U(1 - zP^{\mathcal{F}}_{\mathcal{F}\ominus\mathcal{H}}U)^{-1}|_{\mathcal{N}}, \quad z \in \mathbb{D},$$

establishes a bijection between the family $\mathcal{U}(V)$ of all minimal unitary extensions of V and the Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$.

3. Description of the Hankel symbols of a given Hankel operator

We now turn our attention to the problem of describing the Hankel symbols of a given Hankel operator X. We have the following theorem whose proof can be found in [2].

Theorem 3.1. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. For $j = 1, 2$, let $\mathcal{R}_j$ be the subspace of $\mathcal{K}_j$ which reduces V_j to its unitary part. Given X, a Hankel operator for T_1 and T_2, with $\|X\|_{PV} = 1$, let $\widetilde{X} \in \mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ be the contraction operator uniquely determined by X as in (2). Then there is* *a bijection between the set $\mathcal{HS}(X)$ of all Hankel symbols of X and the Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$, where*

$$\mathcal{N} := \mathcal{D}_{\widetilde{X}} \ominus D_{\widetilde{X}} V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} \mathcal{H}_1$$

and

$$\mathcal{M} := \left\{ (e_1, e_2) \in \mathcal{D}_{\widetilde{X}} \oplus \mathcal{E}_2 : T_2 P_{\mathcal{H}_2}^{\mathcal{K}_2} e_2 = 0 \textit{ and } D_{\widetilde{X}} e_1 + \widetilde{X}^* e_2 = 0 \right\}.$$

Remark 3.2. The hypothesis that $\|X\|_{PV} = 1$ can be dropped as long as we deal with symbols Z of X such that $\|Z\| = \|X\|_{PV}$. Clearly, if X is a Hankel operator for T_1 and T_2 with $\|X\|_{PV} = \beta > 0$, then $X' := \frac{1}{\beta} X$ is a Hankel operator for T_1 and T_2 with $\|X'\|_{PV} = 1$. Furthermore, $Z' \in \mathcal{HS}(X')$ if and only if $\beta Z' \in \mathcal{HS}(X)$. Though, Theorem 3.1 can be slightly modified to replace $\mathcal{N}$ and $\mathcal{M}$ by

$$\mathcal{N} := \mathcal{D}_{\widetilde{X}}^{\beta} \ominus D_{\widetilde{X}}^{\beta} V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} \mathcal{H}_1$$

and

$$\mathcal{M} := \left\{ (e_1, e_2) \in \mathcal{D}_{\widetilde{X}}^{\beta} \oplus \mathcal{E}_2 : T_2 P_{\mathcal{H}_2}^{\mathcal{K}_2} e_2 = 0 \text{ and } D_{\widetilde{X}}^{\beta} e_1 + \widetilde{X}^* e_2 = 0 \right\},$$

in order to obtain a bijective correspondence between the set $\mathcal{HS}(X)$ and the corresponding Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$. It can even be considered that β is any fixed nonnegative number such that $\beta \geq \|X\|_{PV}$. In such a case, the bijection is established between $\mathcal{S}(\mathcal{N},\mathcal{M})$ and the larger set $\mathcal{HS}_\beta(X)$ of intertwining dilations Z of X satisfying $\|Z\| \leq \beta$.

Remark 3.3. Set $\mathcal{L}_2 := \overline{(V_2 - T_2)\mathcal{H}_2}$ and, for each $Z \in \mathcal{HS}(X)$, consider the power series

$$S_Z(z) := \sum_{n=0}^{\infty} z^n \widehat{S_Z}(n), \quad z \in \mathbb{D}, \quad \widehat{S_Z}(n) := V_2^{*n} P_{V_2^n \mathcal{L}_2}^{\mathcal{K}_2} Z|_{\mathcal{H}_1}, \quad n \geq 0,$$

so that S_Z is an $\mathcal{L}(\mathcal{H}_1, \mathcal{L}_2)$-valued function defined and analytic on $\mathbb{D}$. Then, for all $z \in \mathbb{D}$,

$$S_Z(z) = a(z) + b(z)\theta(z)(1 - c(z)\theta(z))^{-1} d(z), \tag{3}$$

where a, b, c and d are fixed operator valued functions (determined by the data) and $\theta \in \mathcal{S}(\mathcal{N},\mathcal{M})$. The Schur like formula (3) establishes the direct connection between $\mathcal{S}(\mathcal{N},\mathcal{M})$ and $\mathcal{HS}(X)$, as each $Z \in \mathcal{HS}(X)$ is fully determined by its corresponding power series S_Z.

We last study the problem of determining whether the set $\mathcal{HS}(X)$ has a single element. From Remark 3.2 it is clear that we may assume that $\|X\|_{PV} = 1$. The proof of the following theorem can also be found in [2].

Theorem 3.4. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. For $j = 1, 2$, let $\mathcal{R}_j$ be the subspace of $\mathcal{K}_j$ which reduces V_j to its unitary part. Let X be a Hankel operator for T_1 and T_2 such that $\|X\|_{PV} = 1$. On the Hilbert* *space $\mathcal{H}_1 \oplus \mathcal{H}_2$, with the standard inner product, consider the 2×2 block matrix operators*

$$\widetilde{T_1} := \begin{pmatrix} T_1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{T_2} := \begin{pmatrix} 1 & 0 \\ 0 & T_2 \end{pmatrix}$$

and

$$E := \begin{pmatrix} P^{\mathcal{K}_1}_{\mathcal{H}_1} P^{\mathcal{K}_1}_{\mathcal{R}_1}|_{\mathcal{H}_1} & X^* \\ X & P^{\mathcal{K}_2}_{\mathcal{H}_2} P^{\mathcal{K}_2}_{\mathcal{R}_2}|_{\mathcal{H}_2} \end{pmatrix}.$$

Then X has a unique Hankel symbol if and only if either

(a) $$\text{kernel } (\widetilde{T_1}E) \subseteq \text{kernel } E$$

or

(b) $$\text{kernel } (\widetilde{T_2}E) \subseteq \text{kernel } E.$$

Corollary 3.5. *If either $T_1 P^{\mathcal{K}_1}_{\mathcal{H}_1}|_{\mathcal{E}_1}$ or $T_2 P^{\mathcal{K}_2}_{\mathcal{H}_2}|_{\mathcal{E}_2}$ is injective, any Hankel operator X for T_1 and T_2 has a unique Hankel symbol, say Z_X.*

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