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INVARIANT SUBSPACES OF TRANSLATION SEMIGROUPS

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In these lectures I shall give an account of some recent results and open problems relating to subspaces of square integrable functions on the real line which are jointly invariant for a pair of semigroups of unitary operators. These semigroups are quite fundamental, namely, translation semigroups, Fourier translation semigroups, and dilation semigroups. A celebrated theorem of Beurling gives a description of the closed subspaces that are simply invariant for translations (or Fourier translations) and as a result these subspaces are in bijective correspondence with the set of all unimodular functions. In contrast, subspaces that are invariant for two of these semigroups turn out to be finitely parametrised by a family of specific unimodular functions.

I shall indicate how one goes about the identification of these sets of invariant subspaces and how, with the natural topology, they are identifiable as Euclidean manifolds. Also I shall discuss aspects of the relatively novel nonselfadjoint operator algebras that are associated with them. These algebras are generated by two non-commuting copies of $H^\infty(\mathbb{R})$.

To identify the topology on the set of invariant subspaces it turns out that one needs to establish some essentially function theoretic assertions, in which a limit of a sequence of (projections onto) purely invariant subspaces is a particular reducing subspace (projection). We sketch below how one can obtain such "strange limits".

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Lecture I. Translation invariance

We start with some background by considering subspaces that are invariant for a single unitary semigroup on $L^2(\mathbb{R})$, by reviewing Beurling's theorem in the context of $H^\infty(\mathbb{R})$, and by proving a theorem of Sarason to the effect that $H^\infty(\mathbb{R})$ is determined by its invariant subspaces. Subspaces are always considered to be closed and invariant subspaces are often denoted as K. A set of operators that is closed in the weak operator topology will be said to be WOT-closed.

Reflexive operator algebra

An operator algebra $\mathcal{A}$ on a Hilbert space is *reflexive* if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$ where $\operatorname{Lat}\mathcal{A}$ is the collection of all invariant subspaces,

$$\operatorname{Lat}\mathcal{A} = \{K : AK \subseteq K, \text{ for all } A \in \mathcal{A}\}$$

and where, for a set $\mathcal{S}$ of (closed) subspaces of the Hilbert space,

$$\operatorname{Alg}\mathcal{S} = \{B : BK \subseteq K, \text{ for all } K \in \mathcal{S}\}.$$

A reflexive algebra is thus determined by its invariant subspaces. If, at the outset, $\mathcal{A}$ is specified as the WOT-closed algebra generated by some family of operators, that is, if $\mathcal{A}$ is specified intrinsically by generators, then by establishing reflexivity we obtain an alternative, extrinsic description of the algebra as the set of all operators with specified invariant subspaces. Looked at the other way round, we see that an operator which has these invariant subspaces can be approximated, or synthesized, from polynomials in the generators.

Examples

For any family of subspaces S the set of operators $\operatorname{Alg}\mathcal{S}$ is an operator algebra and it is easily checked that it is reflexive algebra. Also, $\operatorname{Alg}\mathcal{S} = \operatorname{Alg}\hat{\mathcal{S}}$ where $\hat{\mathcal{S}} = \operatorname{Lat}\operatorname{Alg}\mathcal{S}$. Generally $\hat{\mathcal{S}}$ contains $\mathcal{S}$ and is referred to as the reflexive closure, or reflexive hull, of $\mathcal{S}$.

By the *Volterra nest* I mean the family of subspaces

$$\mathcal{N}_v = \{L^2[t,\infty) : t \in \mathbb{R}\} \cup \{0, L^2(\mathbb{R})\}.$$

This is a totally ordered set (nest) of subspaces and its operator algebra $\operatorname{Alg}\mathcal{N}_v$, *the Volterra nest algebra*, can be naturally thought of as the algebra of all "lower triangular operators" on $L^2(\mathbb{R})$. This idea becomes precise when one characterises the kernel functions $k(x,y)$ of the Hilbert-Schmidt operators in the algebra. In this case $k(x,y)$ is supported on the triangular set $\{(x,y) : x \leq y\}$.

An elementary example of an operator algebra that is not reflexive is the algebra on $\mathbb{C}^2$ of 2×2 complex matrices which are lower triangular and have equal entries on the diagonal. The Alg Lat algebra is bigger and consists of all lower triangular matrices.

The examples above are straightforward exercises. Deeper is Sarason's theorem which asserts that $H^\infty(\mathbb{R})$, as an operator algebra on $L^2(\mathbb{R})$, or on $H^2(\mathbb{R})$, is reflexive.

Translation-multiplication algebras

In considering subspaces that are invariant for a family of operators, such as multiplication and translation operators, it can be helpful to consider the WOT-closed algebra generated by these operators as this algebra plainly has the same set of invariant subspaces as its generators. Dually, such algebras are of interest in their own right and knowing $\operatorname{Lat}\mathcal{A}$ gives insight into $\mathcal{A}$.

For λ, μ real let M_λ and D_μ be the operators on $L^2(\mathbb{R})$ defined by

$$M_\lambda f = e^{i\lambda x} f,\ D_\mu f(x) = f(x-\mu).$$

Note that these operators satisfy the Weyl Commutation Relations (WCR):

$$M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda.$$

The identification of the weak operator topology closed operator algebra generated by the exponential multiplication operators follows from elementary functional analysis. It consists of all multiplication operators:

$$\text{WOT-alg}(\{M_\lambda\}_{\lambda\in\mathbb{R}}) = \{M_\varphi : \varphi \in L^\infty(\mathbb{R})\}.$$

(Trigonometric polynomials can uniformly approximate continuous functions on any large interval, and this is enough for weak operator topology density.) We also have

$$\text{WOT-alg}(\{M_\lambda\}_{\lambda\geq 0}) = \{M_\varphi : \varphi \in H^\infty(\mathbb{R})\}$$

which is a more challenging exercise. (The standard unitary equivalence of $H^\infty(\mathbb{R})$ and $H^\infty(\mathbb{T})$ provides one route. Alternatively one can make use of the Paley-Wiener theorem.)

The 1-parameter translation group above is in fact unitarily equivalent to the 1-parameter multiplication group via the Fourier-Plancherel transform (see Lecture 2). Thus there are similar function space identifications for the two corresponding algebras generated by the operators D_μ.

Let us now discuss an identification which ties some ideas together:

$$\text{WOT-alg}(\{M_\lambda D_\mu\}_{\lambda\in\mathbb{R},\mu\geq 0}) = \operatorname{Alg}\mathcal{N}_v.$$

Since the Volterra nest algebra is WOT-closed and contains the given generators of the algebra on the left hand side it will suffice to show that an operator X in the nest algebra $\operatorname{Alg}\mathcal{N}_v$ can be synthesised as a WOT-limit of sums of products of the generators. This can be done as follows. From general nest algebra theory ([2], [11]) it is known that the linear span of the rank one operators R in $\operatorname{Alg}\mathcal{N}_v$ is WOT-dense, and so we may as well assume that X is such an operator R. On the other hand, the small algebra (on the left hand side) contains operators which are finite sums of operators of the form $M_\varphi D_\mu, \varphi \in L^\infty, \mu \geq 0$. It can be shown that sums of such operators can approximate a rank one operator in the weak operator topology.

Exactly the same style of constructive proof can be used to show that

$$\text{WOT-alg}(\{M_\lambda D_\mu\}_{\lambda,\mu\in\mathbb{R}}) = \mathcal{B}(L^2(\mathbb{R})).$$

An alternative abstract proof of this identification, based on von Neumann's double commutant theorem, is indicated below.

It is natural for us now to define an algebra intermediate between $H^\infty(\mathbb{R})$ and $\operatorname{Alg}\mathcal{N}_v$, which is generated by two 1-parameter semigroups;

$$\mathcal{A}_p = \text{WOT-alg}(\{M_\lambda D_\mu\}_{\lambda,\mu\geq 0}).$$

The subscript p is for parabolic (as explained in the closing remarks below). This is a curious and interesting operator algebra, generated by two non-commuting copies of $H^\infty(\mathbb{R})$. It is antisymmetric, that is, $\mathcal{A}_p \cap \mathcal{A}_p^* = \mathbb{C}I$, and it can be shown to contain no finite rank operators. We see later that it is a reflexive algebra.

Beurling's theorem

We shall make use of Beurling's theorem in the following form.

Theorem 1. *Suppose that K is a closed subspace of $L^2(\mathbb{R})$ with $M_\lambda K \subseteq K$ for all $\lambda \geq 0$ and that*

$$\bigcap_{\lambda\geq 0} M_\lambda K = \{0\}.$$

(One says that K is purely invariant, or simply invariant.) Then $K = \varphi H^2(\mathbb{R})$ for some unimodular function φ. If, in addition, $K \subseteq H^2(\mathbb{R})$, then φ is in $H^\infty(\mathbb{R})$, that is, φ is an inner function.

Corollary 2. *If we view $H^\infty(\mathbb{R})$ as an algebra of multiplication operators on $L^2(\mathbb{R})$, then $\operatorname{Lat}(H^\infty(\mathbb{R}))$ is the union*

$$\{\varphi H^2(\mathbb{R}) : \varphi \text{ unimodular}\} \cup \{\chi_E L^2(\mathbb{R}) : E \subseteq \mathbb{R} \text{ measurable}\}.$$

Proof. Exercise. □

Corollary 3. *If we view $H^\infty(\mathbb{R})$ as an algebra of multiplication operators on $H^2(\mathbb{R})$ then*

$$\operatorname{Lat} H^\infty(\mathbb{R}) = \{\phi\, H^2(\mathbb{R}) : \phi \text{ is an inner function }\}.$$

Proof. Exercise. □

The last corollary obtains a complete explicit characterisation of the invariant subspaces for H^∞ multiplication. Nevertheless it is a "wild set" of subspaces parametrised by all unimodular functions. But in the subsequent lectures we shall make further demands on these subspaces - that they are also invariant for right translation for example - and this will result in a finitely parametrised or "tame set".

Theorem 4. (Sarason [14])) $H^\infty(\mathbb{R})$*, as an algebra of multiplication operators on* $H^2(\mathbb{R})$*, is a reflexive operator algebra.*

Remarks.

1. The double commutant theorem of von Neumann asserts that a von Neumann algebra M (i.e. a WOT closed unital self-adjoint operator algebra) is equal to its second commutant. Since the commutant is generated by its subset of projections it is straightforward to deduce from this that

$$M = \operatorname{Alg}\operatorname{Lat} M.$$

Here

$$\operatorname{Lat} M = \{P : P = P^* = P^2, P \in M'\},$$

where M' is the commutant of M. Note that this theorem can be used to give a quick (nonconstructive) proof of the equality

$$\text{WOT-alg}(\{M_\lambda D_\mu\}_{\lambda,\mu\in\mathbb{R}}) = \mathcal{B}(L^2(\mathbb{R})).$$

Sarason's theorem can be viewed as a first move towards generalisations of the double commutant theorem in the direction of non-self-adjoint operator algebras.

2. $L^\infty(\mathbb{R})$, as a von Neumann algebra on $L^2(\mathbb{R})$, is its own commutant and so is a maximal abelian self-adjoint algebra. Likewise, $H^\infty(\mathbb{R})$, as an operator algebra on $H^2(\mathbb{R})$, is maximal abelian. We use this fact in the following proof of Sarason's theorem.

Proof. The operator algebra $H^\infty(\mathbb{R})$ is naturally unitarily equivalent to its counterpart algebra on the Hardy space of the unit circle $\mathbb{T}$ and it is in this setting that we prove reflexivity.

Let $A \in \operatorname{Alg}\operatorname{Lat} H^\infty(\mathbb{T})$. For $\alpha \in \mathbb{C}, |\alpha| < 1$, let $v_\alpha = \Sigma_{n\in\mathbb{Z}_+}\alpha^n z^n$. This element of $H^2(\mathbb{T})$ is an eigenvector for the backward shift operator; $T_z^* v_\alpha = \alpha v_\alpha$. Thus $\{v_\alpha\}^\perp \in \operatorname{Lat} T_z = \operatorname{Lat} H^\infty(\mathbb{T})$, where the last equality of lattices of invariant subspaces holds because $H^\infty(\mathbb{T})$ is the WOT-closed algebra generated by T_z. It follows that $\mathbb{C}v_\alpha \in \operatorname{Lat} A^*$ and so $A^* v_\alpha = \lambda v_\alpha$ for some λ depending on α.

Let $A1 = h$. We shall show that h is in $H^\infty(\mathbb{T})$ and that $A = T_h$ (the Toeplitz operator of multiplication by h) and this will complete the proof.

We have

$$T_z^* A^* v_\alpha = T_z^* \lambda v_\alpha = \lambda\alpha v_\alpha = A^*\alpha v_\alpha = A^* T_z^* v_\alpha.$$

This is true for all v_α and it can be shown that these vectors span the Hardy space. Thus $AT_z = T_zA$ and since $H^\infty(\mathbb{T})$ is a maximal abelian algebra it follows that A is an analytic Toeplitz operator, as desired. $\square$

Remarks

1. In the proof we did not need all the subspaces in $\operatorname{Lat} H^\infty(\mathbb{R})$. This is fairly typical for arguments that establish reflexivity and we see this again in the subsequent lectures.

2. The proof above extends readily to the multivariable case of $H^\infty(\mathbb{T}^n)$ viewed as an operator algebra on the Hardy space of the n-torus.

3. The reflexivity of an operator algebra generated by two commuting pure isometries is investigated in Horák and Müller [4]. They conjecture that all such algebras are reflexive and it could be that this problem is still open.

4. An elaboration of the eigenvalue argument above is given in Kribs and Power [8] to obtain a simple proof of the reflexivity of the so-called noncommutative analytic Toeplitz algebras $\mathcal{L}_n$. These algebras are generated by freely noncommuting copies of H^∞ and so are rather different from our algebras in which generators have commutation relations. Reflexivity here was originally obtained by Arias and Popescu [1]. See also [3], [7] for other considerations of reflexivity in the presence of free generators.

5. Background on invariant subspaces, reflexive algebras and Hardy space function theory can be found in the books of Radjavi and Rosenthal [13], and Davidson [2].

Lecture II. Translation-multiplication invariance

We now describe the closed subspaces on the real line that are invariant for the operators M_λ, D_μ for nonnegative λ, μ. First we note the two obvious classes of invariant subspaces.

Let $K = e^{i\alpha x}H^2(\mathbb{R})$. Then

$$D_\mu K = e^{i\alpha(x-\mu)}D_\mu H^2(\mathbb{R}) = e^{i\alpha x}e^{-i\alpha\mu}H^2(\mathbb{R}) = K$$

for all μ in $\mathbb{R}$. So these subspaces are actually fixed by translations (and are reducing subspaces). Also

$$M_\lambda K = e^{i\lambda x}e^{i\alpha x}H^2(\mathbb{R}) \subseteq K$$

if $\lambda \geq 0$. Thus K is doubly invariant, and $K \in \operatorname{Lat} \mathcal{A}_p$. Note that the collection of these subspaces is totally ordered by inclusion and so gives rise to another nest, $\mathcal{N}_a$ say, which we call the analytic nest.

On the other hand let $K = L^2[t, \infty)$. Then $M_\lambda K = K$, for all λ, so that these subspaces are fixed by the multiplication operators, while $D_\mu K \subseteq K$ if $\mu \geq 0$.

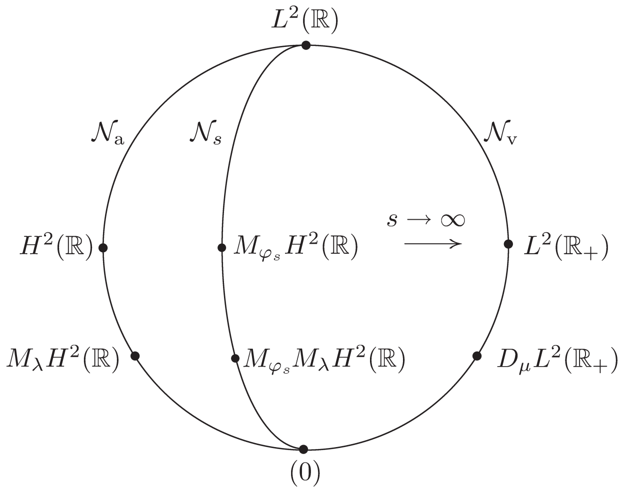

FIGURE 1. The translation-multiplication invariant subspaces.

The following invariant subspaces are more subtle and one can readily check that they are purely invariant for both the multiplication semigroup and the translation semigroup.

Let $\varphi_s(x)$ be the unimodular function

$$\varphi_s(x) = e^{-isx^2/2}.$$

Note that

$$D_\mu \varphi_s = e^{-is(x-\mu)^2/2} = e^{-is\mu^2/2} e^{is\mu x} \varphi_s(x),$$

and so for $s, \mu \geq 0$,

$$D_\mu \varphi_s H^2(\mathbb{R}) = \varphi_s e^{is\mu x} H^2(\mathbb{R}) \subseteq \varphi_s H^2(\mathbb{R}).$$

Thus $\varphi_s H^2(\mathbb{R}) \in \operatorname{Lat} \mathcal{A}_p$. For the same reason we see that $\operatorname{Lat} \mathcal{A}_p$ contains the subspaces

$$\varphi_s e^{i\alpha x} H^2(\mathbb{R}), \quad s \geq 0, \alpha \in \mathbb{R}.$$

We can indicate the parametrisation of all these doubly invariant subspaces with a labeling of the closed unit disc as in the diagram above. The lower point represents the zero space, the north pole represents $L^2(\mathbb{R})$, the Volterra nest is labeled by points on the right boundary, and for fixed s on the equatorial diameter, the nest of subspaces $\varphi_s e^{i\alpha x} H^2(\mathbb{R}), \alpha \in \mathbb{R}$ is labeled by points on a line of longitude passing through s.

Theorem 5. (Katavolos and Power 1997) *If $K \subseteq L^2(\mathbb{R})$ is a closed subspace with $M_\lambda K \subseteq K, D_\mu K \subseteq K$ for all $\lambda, \mu \geq 0$, then either $K = L^2[t, \infty)$ for some $t \in [-\infty, \infty]$ or $K = \varphi_s e^{i\alpha x} H^2(\mathbb{R})$ for some $s \geq 0, \alpha \in \mathbb{R}$.*

We give the essentials of the proof, omitting various measure theoretic details.

Assume that K is not of the form $L^2[t, \infty)$. Then K must be simply invariant for $\{M_\lambda : \lambda \geq 0\}$, i.e.

$$\bigcap_{\lambda \geq 0} M_\lambda K = \{0\}$$

(or else $K = L^2[E]$ for some $E \subseteq \mathbb{R}$ and then D_μ invariance implies $E = [t, \infty)$). Thus $K = uH^2(\mathbb{R})$ with u unimodular.

The space $D_t K$ is also simply invariant for $\{M_\lambda : \lambda \geq 0\}$ (by the WCR), and $D_t K \subseteq K$ for $t > 0$, by hypothesis, so

$$D_t K = w_t u H^2(\mathbb{R}),$$

where w_t is inner. It follows that w_t divides w_s if $0 < t < s$.

We also have $D_t K = D_t u H^2(\mathbb{R}) = u(x - t) H^2(\mathbb{R})$ so, without loss of generality, we may assume that $u(x - t) = w_t(x) u(x)$. This equation implies that the w_t satisfy the cocycle equation

$$w_{s+t}(x) = w_t(x - s) w_s(x).$$

We shall analyse this equation and find that for some $\rho > 0, \sigma$ real,

$$w_t(x) = e^{i(-\rho \frac{t^2}{2} + \sigma t + \rho t x)}.$$

Thus

$$u(x_0 - t) = u(x_0) e^{i(-\rho \frac{t^2}{2} + \sigma t + \rho t x_0)}$$

or equivalently

$$u(y) = c e^{i(-\rho \frac{y^2}{2} - \sigma y)},$$

as required.

The cocycle equation implies $w_t(x - r) | w_s$, for $0 < r < s - t$. Now, if w_t has a zero in the upper half plane then w_s would have a segment of zeros, so there are no such zeros and each w_s is a singular inner function, with singular measure μ_s say. It can be shown that the divisibility condition $w_t(x - r) | w_s, 0 < r < s - t$, forces each μ_s to be a mass at infinity, that is,

$$w_t(x) = \alpha(t) e^{i\beta(t) x},$$

where $\beta(t)$ is increasing.

The cocycle identity becomes

$$\alpha(s + t) e^{i\beta(s+t) x} = \alpha(t) e^{i\beta(t)(x - s)} \alpha(s) e^{i\beta(s) x}.$$

In particular $\beta(s+t) = \beta(s) + \beta(t)$ and hence $\beta(t) = \rho t$ for some $\rho \geq 0$.

Define $\gamma(t) = \alpha(t)e^{i\rho\frac{t^2}{2}}$. Then

$$\begin{aligned}
\gamma(s+t) &= \alpha(s+t)\left[e^{i\rho\frac{s^2}{2}}\, e^{i\rho\frac{t^2}{2}}\, e^{i\rho st}\right] \\
&= \left(\alpha(t)\ \alpha(s)e^{-ipts}\right)\left[e^{i\rho\frac{s^2}{2}}\, e^{i\rho\frac{t^2}{2}}\, e^{i\rho st}\right] \\
&= \gamma(s)\gamma(t),
\end{aligned}$$

and so $\gamma(t) = e^{i\sigma t}$ for some σ, and the desired form follows.

Further Facts

1. Let $(Ff)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}f(y)dy$ for a set of appropriate functions f dense in $L^2(\mathbb{R})$. Then F extends to a unitary which we refer to as the Fourier-Plancherel transform. We have

$$FM_\lambda F^* = D_\lambda, \quad \lambda \in \mathbb{R},$$

so we define

$$D_\psi = FM_\psi F^*$$

for $\psi \in L^\infty(\mathbb{R})$. Thus $\mathcal{A}_p$ contains the operators $M_\phi D_\psi$ for ϕ, ψ in $H^\infty(\mathbb{R})$. Such an operator has the integral operator representation

$$f \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x)\psi(y)e^{-ixy}\left(F^*f\right)(y)\,dy$$

and so is a Hilbert-Schmidt operator if, in addition, $\phi, \psi \in H^2(\mathbb{R}) \cap H^\infty(\mathbb{R})$. In particular, if

$$\phi_n(x) = \psi_n(x) = \frac{ni}{x+ni}, \quad X_n = M_{\phi_n}D_{\psi_n},$$

then $\{X_n\}$ is a norm bounded sequence of Hilbert-Schmidt operators with $X_n \to I$ (SOT) as $n \to \infty$.

2. The algebra $\mathcal{A}_p$ is a reflexive operator algebra. In fact, as in the proof of Sarason's theorem, we "do not need all the invariant subspaces". It turns out that

$$\mathcal{A}_p = \mathrm{Alg}\left(\mathcal{N}_v \cup \mathcal{N}_a\right)$$

where $\mathcal{N}_v \cup \mathcal{N}_a$ is the boundary of the lattice in the diagram above. The proof of this can be structured as follows. In view of $\{X_n\}$, the "bounded approximate identity of Hilbert-Schmidt operators", both the smaller algebra $\mathcal{A}_p$ and the containing algebra $\mathrm{Alg}(N_v \cup \mathcal{N}_a)$ are equal to the SOT-closure of their Hilbert-Schmidt operators. Thus it suffices to show that these subspaces of Hilbert-Schmidt operators coincide. It can be shown that a Hilbert-Schmidt operator in the larger algebra has a "doubly triangular" integral operator form, namely

$$f \to \int a(x,y)e^{-ixy}\left(F^*f\right)(y)\ dy$$

with $a(x,y)$ in the Hilbert space tensor product $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$. This two-variable symbol function can be approximated in $L^2(dxdy)$ norm by sums of elementary tensors and these approximating symbol functions correspond to operators of the form $\sum_{i=1}^{N} M_{\phi_i} D_{\psi_i}$, which are in the smaller algebra. The equality of the Hilbert-Schmidt operators follows.

Strange Limits.

The diagram above suggests that perhaps $\operatorname{Lat}\mathcal{A}_p$ with the weak operator topology is homeomorphic to the closed unit disc. The key step to proving this is to show that for the projections $P_{\lambda,s}$ onto the spaces $e^{i\lambda x}\varphi_s H^2$ we have

$$P_{\lambda s,s} \to P_{L^2[\lambda,\infty)}$$

in the weak operator topology, as $s \to \infty$. From this we conclude that the projections corresponding to the boundary of the lattice in the diagram form a topological boundary. This key step may be readily obtained from the following lemma of [5].

Lemma 6. *For $s > 0$, $F(e^{-isx^2/2}H^2(\mathbb{R})) = e^{is^{-1}x^2/2}\overline{H^2(\mathbb{R})}$.*

Formal proofs of the assertions above can be found in [5]. There the lattice $\mathcal{N}_v \cup \mathcal{N}_a$ is referred to as the Fourier binest and its reflexive algebra $\mathcal{A}_{FB} = \operatorname{Alg}(\mathcal{N}_v \cup \mathcal{N}_a)$ is called the Fourier binest algebra. The unitary automorphism group of $\mathcal{A}_p$ is determined in [5] and is shown in [12] to coincide with the isometric automorphism group. Interestingly, this automorphism group is a Lie group of automorphisms $X \to UXU^*$, where U is of the form $M_\lambda D_\mu V_t$ with λ, μ, t in $\mathbb{R}$, where the V_t are the dilation unitaries given below. From this point of view $\mathcal{A}_p$ is closer in mathematical flavour to the operator algebra $H^\infty(\mathbb{R})$ than it is to the nest algebra $\operatorname{Alg}\mathcal{N}_v$.

Lecture III. Translation-dilation invariance

Let V_t be the dilation unitary operator defined by $V_t f(x) = e^{t/2}f(e^t x)$, for each real t. Note that these operators fix the space $H^2(\mathbb{R})$. We now describe the closed subspaces on the real line that are invariant for the operators M_λ, V_t for nonnegative λ, t. First we note the two obvious classes of invariant subspaces.

Let $w \in \mathbb{C}\backslash\{0\}$ and let $u_w(x)$ be the two-valued function which is equal to w for $x < 0$ and equal to 1 otherwise. Then the space $u_w H^2(\mathbb{R})$ is a closed subspace which is fixed by each V_t and which is invariant for $M_\lambda, \lambda \geq 0$.

On the other hand consider, for each $a, b \geq 0$, the subspace $L^2[-a,b]$ of functions supported on the interval $[-a,b]$. For $t > 0$ these spaces are invariant for V_t and of course they are fixed by the multiplication operators M_λ.

A unimodular description of $u_w H^\infty(\mathbb{R})$ may be derived as follows. Let $w = e^{s\pi}\theta$ where $s \in \mathbb{R}, |\theta| = 1$, and let

$$g_{s,\theta}(x) = \begin{cases} |x|^{is}, & x \geq 0; \\ \theta|x|^{is}, & x < 0. \end{cases}$$

Also consider the function $g_s(z) = z^{is} = \exp(is \log z)$, which is bounded and holomorphic on the upper half plane, where $\log z$ takes its principal value. Then the boundary function $g_s(x)$ is in $H^\infty(\mathbb{R})$ and

$$g_s(x) = \begin{cases} |x|^{is}, & x \geq 0; \\ e^{-s\pi}|x|^{is}, & x < 0. \end{cases}$$

Thus $u_w g_s$ is equal to the unimodular function $g_{s,\theta}$. Since g_s is invertible in $H^\infty(\mathbb{R})$ it follows that $u_w H^2(\mathbb{R}) = g_{s,\theta} H^2(\mathbb{R})$.

We now describe some less obvious invariant subspaces that are purely invariant for both semigroups.

Let $\lambda, \mu \geq 0$ and let $K = e^{i\lambda x} e^{i\mu x^{-1}} H^2(\mathbb{R})$. Note that

$$V_t e^{i\mu x^{-1}} H^2(\mathbb{R}) = e^{i\mu e^{-t} x^{-1}} H^2(\mathbb{R}) = e^{i\mu x^{-1}} [e^{i\mu(e^{-t}-1)x^{-1}} H^2(\mathbb{R})].$$

Since $e^{i\alpha x^{-1}}$ is inner if $\alpha \leq 0$, and $\mu(e^{-t} - 1) \leq 0$ when $\mu, t \geq 0$, it now follows that the subspaces $e^{i\mu x^{-1}} H^2(\mathbb{R})$ are invariant and hence that the subspaces K are invariant.

Theorem 7. (Katavolos-Power 2002) *If $K \subseteq L^2(\mathbb{R})$ is a closed subspace with $M_\lambda K \subseteq K, V_t K \subseteq K$ for all $\lambda, t \geq 0$, then either $K = L^2[-a,b]$ for some $a, b \in [0,\infty]$ or $K = u_w e^{i\lambda x} e^{i\mu x^{-1}} H^2(\mathbb{R})$ for some nonzero $w \in \mathbb{C}$ and $\lambda, \mu \geq 0$.*

Remarks

1. The proof is similar to the proof of Theorem 5.
2. Let us now associate with the lattice $\mathcal{L}_h$ of doubly invariant subspaces K the set of all orthogonal projections P_K endowed with the weak operator topology. Note that the subset

$$\mathcal{L}_L = \{P_{L^2[-a,b]} : 0 \leq a, b \leq \infty\}$$

is homeomorphic to a unit square, realised as $[-\infty, 0] \times [0, \infty]$, while the rest of the lattice,

$$\mathcal{L}_M = \{P_K : K = g_{s,\theta} e^{i(\lambda x + \mu x^{-1})} H^2(\mathbb{R})\}$$

is homeomorphic to $\mathbb{R} \times S^1 \times \mathbb{R}_+ \times \mathbb{R}_+$ (via the $(s, \theta, \lambda, \mu)$ parametrisation). In fact the lattice is a connected topological space by virtue of various "strange limits" of projections, with the unit square providing a compactification of $\mathbb{R} \times S^1 \times \mathbb{R}_+ \times \mathbb{R}_+$. The compactness of $\mathcal{L}_h$ follows in fact from a useful general result of Wogen [16] for reflexive algebras that contain a weak operator topology bounded approximate identity consisting of a sequence of Hilbert-Schmidt operators of bounded operator norm. That such a sequence exists in $\mathcal{A}_h$ is given a direct proof in [10].

3. It has been shown recently in Levene and Power [10] that $\mathcal{A}_h$ is a reflexive operator algebra and in fact

$$\mathcal{A}_h = \mathrm{Alg}(\mathcal{L}_L \cup \mathcal{L}_s)$$

where $\mathcal{L}_s$ is the lattice $\{|x|^{is}H^2(\mathbb{R}) : s \in \mathbb{R}\} \cup \{0, I\}$ consisting of the projections in $\mathcal{L}_h$ which commute with the dilation operators. (This is in parallel with the case for the Fourier binest.)

Furthermore, Levene [9] has studied a superalgebra of $\mathcal{A}_h$ which comes from a (standard) representation of $SL_2(\mathbb{R}_+)$ and has shown that this algebra is also reflexive and that its lattice is $\mathcal{L}_s$. Here one has additional generators that "translate through infinity".

More strange limits.

Let us denote the orthogonal projection onto the closed subspace K as $[K]$.

Theorem 8. *We have*

$$[e^{i\lambda x}e^{i\lambda x^{-1}}H^2(\mathbb{R})] \to [L^2[-1,1]],$$

as $\lambda \to \infty$, *and*

$$[|x|^{is}H^2(\mathbb{R})] \to [L^2(-\infty, 0]],$$

as $s \to \infty$, *with convergence in the strong operator topology.*

Remark. Note that the orthogonal projections for the spaces $e^{i\lambda x}H^2(\mathbb{R})$ tend to the zero operator 0 as $\lambda \to \infty$. (The Fourier-Plancherel transform maps these spaces to the spaces $L^2[\lambda, \infty)$.) On the other hand, with Z the unitary defined by $Zf(x) = x^{-1}f(-x^{-1})$ we have

$$Z(e^{i\lambda x^{-1}}H^2(\mathbb{R})) = e^{-i\lambda x}ZH^2(\mathbb{R}) = e^{-i\lambda x}H^2(\mathbb{R}),$$

and so the projection for $e^{i\lambda x^{-1}}H^2(\mathbb{R})$ tends to the identity operator I as $\lambda \to \infty$. We can imagine then that in the first limit above, the inner functions $e^{i\lambda x}$ compensate for the co-inner functions $e^{i\lambda x^{-1}}$ and the result is between 0 and I.

We now outline the proof of the first strange limit in the theorem above, as given in [6]. It would be of interest to obtain a direct proof.

Proof. It can be shown that $\mathcal{A}_h$ contains Hilbert-Schmidt operators K_n with $K_n \leq 1$ and $K_n \to I$ (SOT) as $n \to \infty$. (See [10].) It can be shown that this implies that the lattice for the hyperbolic algebra, $\mathcal{L}_h$, is compact in the SOT. (This useful general fact is due to Wagner [15].) Thus, there exists a sequence

$\lambda_n \to \infty$ such that the projections $P_n = [e^{i\lambda_n x} e^{i\lambda_n x^{-1}} H^2(\mathbb{R})]$ converge to $[K]$ for some $K \in \mathcal{L}_h$.

Note that if U is the unitary operator with $Uf(x) = x^{-1}f(x^{-1})$ then $UH^2(\mathbb{R}) = H^2(\mathbb{R})^\perp$ and so

$$Ue^{i\lambda x}e^{i\mu x^{-1}}H^2(\mathbb{R}) = e^{i\lambda x^{-1}}e^{i\mu x}H^2(\mathbb{R})$$

and thus $UP_nU = P_n^\perp$ for each n and so $U[K]U = [K]^\perp$. If $K = L^2[-a,b], a, b \geq 0$ then it follows that $a = b = 1$. Thus it suffices to show that for the remaining invariant subspaces K in $\mathcal{L}_h$, the ones parametrised by unimodular functions, this symmetry equation for U does not prevail and this can be done with a short argument. □

Formal proofs of the assertions above can be found in [6], [10]. The algebra $\mathcal{A}_h$ is referred to as the hyperbolic algebra, while $\mathcal{A}_p$ is called the parabolic algebra. This terminology derives from the nature of the generators; the generators V_t are associated with hyperbolic automorphisms of the upper half plane, while the generators D_μ correspond to parabolic automorphisms.

Finally, let me remark that these operator algebras, through their doubly nonselfadjoint nature, are still rather novel and mysterious and undoubtedly some new ideas and tools are needed to make them less so. For example there should certainly be some more direct (purely function-theoretic) approaches to understanding the "strange limits" that I have highlighted above. Also, one can turn to the well-developed nest algebra theory, as expounded in Davidson's book for example, to quickly come up with natural lines of investigation. Can the detailed knowledge of the Hilbert-Schmidt operators in the algebras $\mathcal{A}_p$ and $\mathcal{A}_h$ really provide an effective substitute for the finite rank operators that repeatedly arise in proofs in the nest algebra theory? Is the compactly perturbed algebra $\mathcal{A}_{FB}+\mathcal{K}$ equal to the intersection of the quasitriangular algebras $\mathrm{Alg}\,\mathcal{N}_v+\mathcal{K}$ and $\mathrm{Alg}\,\mathcal{N}_a+\mathcal{K}$? Are the algebras $\mathcal{A}_p$ and $\mathcal{A}_h$ hyper-reflexive? This last question may be a particularly hard one to resolve!

An indication of the current lack of knowledge of the algebraic structure of these algebras is the following open problem. Is $\mathcal{A}_p$ an integral domain, that is, if X, Y are operators in the algebra with zero product does it follow that X or Y is zero? On aesthetic grounds I would prefer the answer to be "yes", so that, once again, $\mathcal{A}_p$ would have mathematical affinities with $H^\infty(\mathbb{R})$ (rather than with the Volterra nest algebra).

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