Monotone crossing number of complete graphs

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Abstract

In 1958, Hill conjectured that the minimum number of crossings in a drawing of K_n is exactly $Z(n) = \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. Generalizing the result by Ábrego et al. for 2-page book drawings, we prove this conjecture for plane drawings in which edges are represented by x-monotone curves. In fact, our proof shows that the conjecture remains true for x-monotone drawings in which adjacent edges do not cross and we count only pairs of edges which cross odd number of times. We also discuss a combinatorial characterization of these drawings.

1 Introduction

Let G be a graph with no loops and multiple edges. In a *drawing D of a graph G* in the plane, the vertices are represented by distinct points and each edge is represented by a simple continuous arc connecting the images of its endpoints. As usual, we identify the vertices and their images, as well as the edges and the arcs representing them. It is required that the edges pass through no vertices other than their endpoints. We also assume for simplicity that any two edges have only finitely many points in common, no two edges *touch* at an interior point and no three edges meet at a common interior point.

A *crossing in D* is a common interior point of two edges where they properly cross. The *crossing number* $\mathrm{cr}(D)$ *of a drawing D* is the number of crossings in D. The *crossing number* $\mathrm{cr}(G)$ *of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all drawings D of G. A drawing D is called *simple* if no two adjacent edges cross and no two edges have more than one common crossing. It is well known and easy to see that every drawing of G which minimizes the crossing number is simple.

According to the famous conjecture of Hill [7, 8] (also known as Guy's conjecture), the crossing number of the complete graph K_n on n vertices satisfies $\mathrm{cr}(K_n) = Z(n)$, where

$$Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

This conjecture has been verified for $n \leq 12$ [12] and for each n, there are drawings of K_n with $Z(n)$ crossings [6, 7, 8, 9].

A curve α in the plane is *x-monotone* if every vertical line intersects α in at most one point. A drawing of a graph G in which every edge is represented by an x-monotone curve and no two vertices share the same x-coordinate is called *x-monotone* (or *monotone*, for short). The *monotone crossing number mon-cr(G) of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all monotone drawings D of G.

The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ *of a graph G* is the smallest number of crossings in a drawing of G where every edge is represented by a straight-line segment. Since every rectilinear drawing of G in which no two vertices share the same x-coordinate is x-monotone, we have $\mathrm{cr}(G) \leq \text{mon-cr}(G) \leq \overline{\mathrm{cr}}(G)$ for every graph G.

We call a drawing of a graph *semisimple* if adjacent edges do not cross but independent edges may cross more than once. The *monotone semisimple odd crossing number* of G (called *monotone odd +* by Schaefer [14]), denoted by $\text{mon-ocr}_+(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone semisimple drawing of G. Clearly, $\text{mon-ocr}_+(G) \leq \text{mon-cr}(G)$.

The monotone crossing number has been introduced by Valtr [15] and recently further investigated by Pach and Tóth [11], who showed that $\text{mon-cr}(G) < 2\mathrm{cr}(G)^2$ holds for every graph G. On the other hand,

they showed that the monotone crossing number and the crossing number are not always the same: there are graphs G with arbitrarily large crossing numbers such that mon-cr$(G) \geq \frac{7}{6}$cr$(G) - 6$.

We study the monotone crossing numbers of complete graphs. The drawings of complete graphs with $Z(n)$ crossings obtained by Blažek and Koman [6] (see also [9]) are 2-*page book* drawings, which may be considered as a strict subset of x-monotone drawings. Thus we have mon-cr$(K_n) \leq Z(n)$. Ábrego et al. [1] recently proved Hill's conjecture for 2-page book drawings of complete graphs. We generalize their techniques and show that Hill's conjecture holds for all x-monotone drawings of complete graphs, even for the monotone semisimple odd crossing number.

Theorem 1 *For every $n \in \mathbb{N}$, we have*

$$\text{mon-ocr}_+(K_n) = \text{mon-cr}(K_n) = Z(n).$$

The rectilinear crossing number of K_n is known to be asymptotically larger than $Z(n)$: this follows from the best current lower bound $\overline{\text{cr}}(K_n) \geq (277/729)\binom{n}{4} - O(n^3)$ [3, 5] and from the simple upper bound $Z(n) \leq \frac{3}{8}\binom{n}{4} + O(n^3)$.

See a recent survey by Schaefer [14] for an encyclopedic treatment of all known variants of crossing numbers.

After submitting this extended abstract, we were informed that the authors of [1] achieved the result mon-cr$(K_n) = Z(n)$ already during discussions after their presentation at SoCG 2012, and that it will appear in the proceedings of LAGOS 2013 [2].

2   Monotone Crossing Number

To prove the upper bound on the 2-page crossing number of K_n, Ábrego et al. [1] generalized the notion of k-edges to arbitrary simple drawings of complete graphs. They also introduced the notion of $\leq\leq k$-edges. These capture the essential properties of 2-page book drawings better than $\leq k$-edges, which had been successfully used before for rectilinear and pseudolinear drawings [10, 4, 3]. We show that the approach using $\leq\leq k$-edges can be generalized to arbitrary semisimple x-monotone drawings.

For a semisimple drawing D of K_n and distinct vertices u and v of K_n, let γ be the oriented arc representing the edge $\{u, v\}$. If w is a vertex of K_n different from u and v, then we say that w is *on the left (right) side of γ* if the topological triangle uvw with vertices u, v and w traced in this order is oriented counterclockwise (clockwise, respectively). This generalizes the definition introduced by Ábrego et al. [1] for simple drawings. However, we were not able to find a meaningful generalization of this notion to drawings

that are not semisimple, where the edges of the triangle uvw can cross several times.

A k-*edge* is an edge $\{u, v\}$ of D that has exactly k points on the same side (left or right). Since every k-edge has $n - 2 - k$ points on the other side, every k-edge is also an $(n-2-k)$-edge and so every edge of D is a k-edge for some integer k where $0 \leq k \leq \lfloor n/2 \rfloor - 1$.

An i-edge with $i \leq k$ is called a $\leq k$-*edge*. Let $E_i(D)$ be the number of i-edges and $E_{\leq k}(D)$ the number of $\leq k$-edges of D. Clearly, $E_{\leq k}(D) = \sum_{i=0}^{k} E_i(D)$. Similarly, the number of $\leq\leq k$-edges of D, $E_{\leq\leq k}(D)$, is defined by the following identity.

$$E_{\leq\leq k}(D) = \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{i=0}^{k} (k+1-i)E_i(D) \quad (1)$$

Considering the only three different simple drawings of K_4 up to a homeomorphism of the plane, Ábrego et al. [1] showed that the number of crossings in a simple drawing D of K_n can be expressed in terms of the number of k-edges in the following way.

Lemma 2 ([1]) *For every simple drawing D of K_n we have*

$$\text{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D), \quad (2)$$

which can be equivalently rewritten as

$$\text{cr}(D) = 2\sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor$$

$$- \frac{1}{2}\left(1 + (-1)^n\right)E_{\leq\leq \lfloor n/2 \rfloor - 2}(D).$$

In fact, Lemma 2 can be easily generalized to semisimple drawings of K_n where cr(D) is replaced by ocr(D), which counts the number of pairs of edges that cross an odd number of times in D. The main reason is that the cycle C_4 cannot be drawn in the plane in such a way that both its pairs of opposite edges cross oddly while adjacent edges do not cross.

By Lemma 2, lower bounds on $E_k(D)$ imply lower bounds on cr(D) and ocr(D). Considering $\leq k$-edges, Ábrego and Fernández-Merchant [4] and Lovász et al. [10] proved that for rectilinear drawings of K_n, the inequality $E_{\leq k} \geq 3\binom{k+2}{2}$ together with (2) gives $\overline{\text{cr}}(G) \geq Z(n)$. However, there are simple x-monotone (even 2-page) drawings of K_n where $E_{\leq k} < 3\binom{k+2}{2}$ for $k = 1$ [1]. Ábrego et al. [1] showed that similar inequality for $\leq\leq k$-edges is satisfied by all 2-page book drawings. We show that the same inequality is satisfied by all x-monotone semisimple drawings of K_n.

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Note that we can assume that all vertices in an x-monotone

drawing lie on the x-axis. We also assume that the x-coordinates of the vertices satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$.

Observation 3 *Let D be a semisimple drawing of K_n, not necessarily x-monotone. Let v be a vertex incident with the outer face of D and let γ_i be the ith edge incident with v in the counter-clockwise cyclic order such that γ_1 and γ_{n-1} are incident with the outer face in a small neighborhood of v. Let v_{k_i} be the other endpoint of γ_i. Then for every i, j, $1 \le i < j \le n - 1$, the triangle $v_{k_i} v v_{k_j}$ is oriented clockwise. Consequently, for every k, $1 \le k \le (n-1)/2$, the edges γ_k and γ_{n-k} are $(k-1)$-edges. For even n, the edge $\gamma_{n/2}$ is a halving edge.* □

For an x-monotone drawing D of K_n, we use Observation 3 directly for the vertex v_n and then for each i, for the vertex v_i and the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$.

The following definitions were introduced by Ábrego et al. [1] for 2-page book drawings. Let D be a semisimple x-monotone drawing of K_n and let D' be a drawing obtained from D by deleting the vertex v_n together with its adjacent edges. A k-edge in D is a (D, D')-*invariant k-edge* if it is also a k-edge in D'. It is easy to see that every $\le k$-edge in D' is also a $\le (k+1)$-edge in D. If $0 \le j \le k \le \lfloor n/2 \rfloor - 1$, then a (D, D')-invariant j-edge is called a (D, D')-*invariant $\le k$-edge*. Let $E_{\le k}(D, D')$ denote the number of (D, D')-invariant $\le k$-edges.

For $i < j$, the edge $v_i v_j$ is called the *right edge at* v_i. The right edges at v_i have a natural vertical order.

Lemma 4 *Let k be a fixed integer such that $0 \le k \le (n-3)/2$. For every $i \in \{1, 2, \ldots, k+1\}$, the $k+2-i$ bottommost and the $k+2-i$ topmost right edges at v_i are $\le k$-edges in D. Moreover, at least $k+2-i$ of these $\le k$-edges are (D, D')-invariant $\le k$-edges.*

Proof. The first part of the lemma follows directly from Observation 3. If the edge $v_i v_n$ is one of the $k+2-i$ topmost right edges at v_i, then the $k+2-i$ bottommost right edges at v_i are (D, D')-invariant $\le k$-edges. Otherwise the $k+2-i$ topmost right edges at v_i are (D, D')-invariant $\le k$-edges. □

Corollary 5 *We have*

$$E_{\le k}(D, D') \ge \sum_{i=1}^{k+1}(k+2-i) = \binom{k+2}{2}.$$

□

The following theorem gives the lower bound on the number of $\le \le k$-edges. The proof is essentially the same as in [1], we only extracted Lemma 4, which needed to be generalized. Together with Lemma 2, Theorem 6 yields Theorem 1.

Theorem 6 *Let $n \ge 3$ and let D be a semisimple x-monotone drawing of K_n. Then for every k, $0 \le k < n/2 - 1$, we have $E_{\le \le k}(D) \ge 3\binom{k+3}{3}$.*

Proof. The proof proceeds by induction on n where the case $n = 3$ is trivially true. Let $n \ge 4$ and let D be a semisimple x-monotone drawing of K_n. For the induction step we remove the point v_n together with its adjacent edges to obtain a drawing D' of K_{n-1}, which is also semisimple and x-monotone.

Using Observation 3 we see that for $0 \le i \le k < n/2 - 1$ there are two i-edges adjacent to v_n in D and together they contribute with $2\sum_{i=0}^{k}(k+1-i) = 2\binom{k+2}{2}$ to $E_{\le \le k}(D)$ by (1).

Let γ be an i-edge in D'. Then γ contributes by $(k-i)$ to the sum $E_{\le \le k-1}(D') = \sum_{i=0}^{k-1}(k-i)E_i(D')$. We already observed that γ is either an i-edge or an $(i+1)$-edge in D. If γ is also an i-edge in D (that is, γ is a (D, D')-invariant i-edge), then it contributes by $(k+1-i)$ to $E_{\le \le k}(D)$. This is a gain of $+1$ towards $E_{\le \le k-1}(D')$. If γ is an $(i+1)$-edge in D, then it contributes only $(k-i)$ to $E_{\le \le k}(D)$. Therefore we have

$$E_{\le \le k}(D) = 2\binom{k+2}{2} + E_{\le \le k-1}(D') + E_{\le k}(D, D').$$

By the induction hypothesis we know that $E_{\le \le k-1}(D') \ge 3\binom{k+2}{3}$ and thus we obtain

$$E_{\le \le k}(D) \ge 3\binom{k+3}{3} - \binom{k+2}{2} + E_{\le k}(D, D').$$

The theorem follows by plugging the lower bound from Corollary 5. □

3 Combinatorial Description

In this section we develop a combinatorial characterization of x-monotone drawings which is based on the signature function introduced by Peters and Szekeres [13] for describing order types of points sets. Let T_n be the set of ordered triples (i, j, k) of the set $[n] = \{1, 2, \ldots, n\}$ and let Σ_n be the set of *signature functions* $\sigma\colon T_n \to \{-, +\}$.

Let D be an x-monotone drawing of the complete graph $K_n = (V, E)$ with vertices $v_1, v_2, \ldots, v_n$ such that their x-coordinates satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$. We assign a signature function $\sigma \in \Sigma_n$ to the drawing D according to the following rule. For each $e = \{v_i, v_k\} \in E$ and every integer j, $i < j < k$, let $\sigma(i, j, k) = -$ if the point v_j lies above the arc representing the edge e and $\sigma(i, j, k) = +$ otherwise.

Note that if the drawing D is also semisimple, then a triangle $v_i v_k v_j$, $j \in (i, k)$, is oriented counterclockwise (clockwise) if and only if $\sigma(i, j, k) = -$ ($\sigma(i, j, k) = +$, respectively). It is easy to see that

for every signature function $\sigma \in \Sigma_n$ there is an x-monotone drawing D which induces σ. However, such a drawing does not have to be semisimple. We show a characterization of simple and semisimple x-monotone drawings by small forbidden configurations in the signature functions.

For $a, b, c, d \in [n]$ with $a < b < c < d$ and a signature function $\sigma \in \Sigma_n$ we say that the 4-tuple (a, b, c, d) is of the form $\xi_1 \xi_2 \xi_3 \xi_4$ in σ if $\sigma(a, b, c) = \xi_1, \sigma(a, b, d) = \xi_2, \sigma(a, c, d) = \xi_3$ and $\sigma(b, c, d) = \xi_4$.

Theorem 7 *A signature function $\sigma \in \Sigma_n$ can be realized by a semisimple x-monotone drawing if and only if each ordered 4-tuple of indices is of one of the forms $++++$, $----$, $++--$, $--++$, $-++-$, $+--+$, $---+$, $+++-$, $+---$, $-+++$ in σ. The signature function σ can be realized by a simple x-monotone drawing if, in addition, there is no 5-tuple (a, b, c, d, e), $a < b < c < d < e$, with*

$$\sigma(a, b, e) = \sigma(a, d, e) = \sigma(b, c, d) = -\sigma(a, c, e).$$

Note that in a simple x-monotone drawing of K_n the crossings can appear only between edges whose endpoints induce a 4-tuple of one of the forms $++++, ----, ++--, --++, -++-, +--+$. Analogously to a similar correspondence in rectilinear drawings of K_n, we may call these 4-tuples *convex*. Then for a simple x-monotone drawing D of K_n the crossing number of D equals the number of convex 4-tuples.

A similar notion of convexity for general k-tuples was used by Peters and Szekeres [13]. This description of crossings is convenient for computer calculations. Using it, we have obtained a complete list of optimal x-monotone drawings of K_n for $n \leq 10$.

4   Concluding remarks

It is an interesting direction of further research to see if similar techniques can be helpful in proving Hill's conjecture for general drawings of complete graphs. We note that the same approach does not generalize to all drawings: for example, a particular planar realization of the so-called *cylindrical drawing* [7, 8] of K_{10}, with crossing number $Z(10)$, does not satisfy the lower bound on $\leq\leq 1$-edges in Theorem 6. It would also be interesting to further generalize Theorem 1 to monotone drawings where also adjacent edges are allowed to cross.

Acknowledgments

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