On 4-connected geometric graphs

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Abstract

Given a set S of n points in the plane, in this paper we give a necessary and sometimes sufficient condition to build a 4-connected non-crossing geometric graph on S.

Introduction

Given a set S of n points in the plane, a non-crossing geometric graph on S is a graph in which its vertices are the points of S and its edges are straight-line segments between these points such that no edge passes through a vertex different from its endpoints and any two edges may intersect only at a common endpoint.

Since all the geometric graphs considered in this paper are non-crossing, throughout the paper we will use the term geometric graph, meaning that the geometric graph is non-crossing.

The study of geometric graphs and, in particular, the study of problems on how to embed planar graphs as geometric graphs on given point sets is a very active area of research (for a review on geometric graphs and some related topics, see for example [1, 4]). One of these problems is the problem of building geometric graphs with a certain connectivity on a set of points S.

We will say that a set of points S is k-connectible if it admits a k-connected geometric graph on it. For $k = 1, 2, 3$, it is well-known when S is k-connectible and how to build a k-connected geometric graph (see [2, 3]). Given S, it is enough to build a non-crossing tree on S (for example the minimum spanning tree of

S) when $k = 1$, and it is enough to build a simple polygonization of S when $k = 2$. For the case $k = 3$, the only set of points not admitting a 3-connected geometric graph is the convex case. Otherwise, in [3] the authors give an algorithm to build a 3-connected geometric graph using $\max\{\lceil 3n/2 \rceil, n + m - 1\}$ edges, where m is the number of points on the boundary of the convex hull of S, and they prove that there is no 3-connected plane graph on S with less edges.

However, for $k > 3$, little is known about when a set of points is k-connectible. For $k = 4$, Dey et al. [2] show points sets that do not admit any 4-connected geometric graph on them and they provide a necessary and sufficient condition for point sets whose convex hull consists of exactly three points. A general characterization of 4- or 5-connectible sets of points is not known.

In this paper, we study sets of points that are 4-connectible. We define a condition (the U-condition) that any set of points must satisfy to be 4-connectible and we show that the U-condition is always sufficient for some sets of points. By denoting the convex hull of S by $CH(S)$ and the set of points on the boundary of the convex hull of S by $H(S)$, if $Q = H(S)$, $I = S \setminus Q$ and $P = H(I)$, then the U-condition is sufficient for sets of points in which $Q \cup P$ satisfies the U-condition.

1 The U-condition

In this section, we will define the U-condition and we will see that it is a necessary condition to get 4-connected geometric graphs.

A subset C of points of Q is connected if it consists of consecutive points of Q. We will denote by $h(C)$ the number of connected components of a subset C of Q.

Definition 1 A set S of points satisfies the U-condition if
  i) $|Q| \leq |I|$
  ii) For any set $C \subset Q$, $|H(S \setminus C)| \leq |I| + h(C)$.

Lemma 2 Every 4-connectible set S satisfies the U-condition.

Proof. Let us prove that if $G(S)$ is a 4-connected graph drawn on S, then S has to satisfy i) and ii). In $G(S)$, each vertex must have degree at least 4, and since there are no edges linking non-consecutive points of Q (otherwise $G(S)$ would not be 3-connected), then there are at least $2|Q|$ edges having an endpoint in Q and the other one in I. On the other hand, as $G(S)$ is 4-connected, each point of I can be linked to a maximum of two (and consecutive) points of Q. Hence, at most there are $2|I|$ edges with an endpoint in Q and the other one in I. Therefore, $2|Q| \leq 2|I|$ and the necessity of i) is proved.

Now, suppose $C \neq \emptyset$ is a subset of points of Q and $C_1, C_2, \ldots, C_{h(C)}$ are its connected components. Let us denote by $\overline{C}_i$ the points of Q placed between component C_i and component C_{i+1}. Thus, Q consists of the points $C_1, \overline{C}_1, C_2, \overline{C}_2, \ldots, C_{h(C)}, \overline{C}_{h(C)}$ in this order. When we remove the points of C, all the points in the subsets $\overline{C}_i$ remain in $H(S \setminus C)$ and perhaps, between $\overline{C}_i$ and $\overline{C}_{i+1}$ (mod $h(C)$), a subset I_{i+1} of points of I appears in $H(S \setminus C)$. Either I_i is empty or it consists of consecutive points of P (see Figure 1). Let $\overline{I}$ be the set $I \setminus (I_1 \cup I_2 \cup \ldots \cup I_{h(C)})$ and let $\overline{C}$ be the set $Q \setminus C$. Observe that proving ii) is equivalent to proving $|\overline{C}| \leq |\overline{I}| + h(C)$.
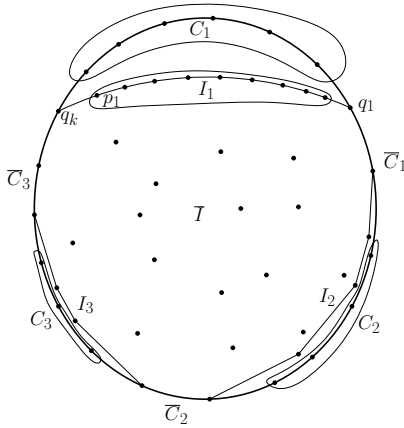

Figure 1: Illustration of Lemma 2.

Suppose that $I_1 = \{p_1, \ldots, p_{k'}\}$ is nonempty and let q_1 and q_k be the points of $\overline{C}$ placed on the boundary of $CH(S \setminus C)$ just after and before the points of I_1, respectively (see Figure 1). Without loss of generality, we can assume that $G(S)$ is a triangulation. So, each point of I_1 must be connected in $G(S)$ to some point of C_1 and, since $G(S)$ is 4-connected, they cannot be connected to a point of $\overline{C}$, except for the edges $p_{k'}q_1$ and q_kp_1. Therefore, for an edge linking a point of $\overline{C}$ with an interior point (at least $2|\overline{C}|$ edges), the interior endpoint must be in $I \setminus I_1$, except for the two mentioned edges $p_{k'}q_1$ and q_kp_1. We can repeat the same reasoning for every subset I_i, obtaining that there must be at least $2|\overline{C}| - 2h(C)$ edges with an endpoint in $\overline{C}$ and the other one in $\overline{I}$. On the other

hand, as before, there are at most $2|\overline{I}|$ connections of this type, so it follows that $2|\overline{C}| - 2h(C) \leq 2|\overline{I}|$. $\quad\square$

Observe that if $|Q| = 3$, then $|I| + 1 = n - 2$. Hence, part ii) can only fail if we remove one point q of Q and the boundary of $CH(S \setminus q)$ contains all the remaining points. In [2], this is the condition that is proved to be necessary and sufficient to build a 4-connected geometric graph (in fact a triangulation) on S.

Lastly, let us point out that, given S, checking whether S satisfies the U-condition or not can be done in $O(|Q| + |P|)$ steps, after calculating Q and P. The algorithm is based in the observation (not easy to prove) that it is not necessary to compute $CH(S \setminus C)$ for all the possible subsets C, but only for a linear number of them.

2   Some 4-connectible sets

In this section, we will give some sets of points for which the U-condition is sufficient. In particular, we will see that if $Q \cup P$ satisfies the U-condition for a set of points S, then S is 4-connectible. We will use $\overline{Q}$ ($\overline{P}$) to refer to the convex polygon defined by the points of Q (P).

Let us start with the case in which S is precisely $Q \cup P$ and $|Q| = |P|$.

Lemma 3 *Let $Q = \{q_1, \ldots, q_n\}$ be a set of points in convex position and let $P = \{p_1, \ldots, p_n\}$ be another set of points in convex position such that $\overline{P}$ is inside $\overline{Q}$. Suppose that the set of points $S = Q \cup P$ satisfies the U-condition. Then S is 4-connectible.*

Proof. Let M be the region $CH(Q) \setminus CH(P)$. To prove the lemma, it is enough to obtain a crossing free zig-zag cycle $Z = p_i q_j p_{i+1} q_{j+1} \ldots p_{i-1} q_{j-1} p_i$ such that its edges are in M, because then the edges of Z and the edges of $\overline{Q}$ and $\overline{P}$ define a 4-connected graph (see Figure 2).
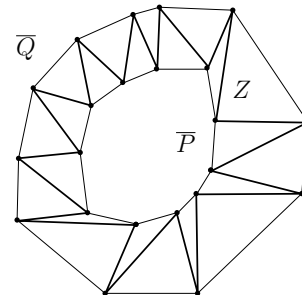

Figure 2: A 4-connected geometric graph when $S = Q \cup P$.

We will say that a triangle $q_j p_i p_{i+1}$ is legal if it is contained in region M. Proving the lemma is equivalent to proving that there is a sequence of n

consecutive legal triangles $q_j p_i p_{i+1}$, $q_{j+1} p_{i+1} p_{i+2}$, ..., $q_{j+n-1} p_{i+n-1} p_{i+n}$, where points with equal subscripts modulo n are considered identical.

Let us assume that $\{q_1, \ldots, q_m\}$ is the set of clockwise points of Q on the left of the line $p_1 p_2$. The following algorithm computes a sequence of n consecutive legal triangles.

Begin
Do $i = j = 2$
While $(i \leq n)$ **Do**
(* **Invariant (1)**: Triangles of the sequence $q_{j-1} p_{i-1} p_i, q_{j-2} p_{i-2} p_{i-1}, \ldots, q_{j-(i-1)} p_1 p_2$ are legal.*)
If (Triangle $q_j p_i p_{i+1}$ is legal) **then**
Do $\{i = i + 1; j = j + 1\}$
Else
(* **Invariant (2)**: q_j is between q_{j-1} and the first crossing of line $p_i p_{i+1}$ with $\overline{Q}$ *)
Do $j = j + 1$
End of While
(* After finishing the algorithm, all the triangles of the sequence $q_{j-n} p_1 p_2, q_{j-(n-1)} p_2 p_3, \ldots, q_{j-1} p_n p_1$ are legal.*)

Let us see that assertions (1) and (2) are always true, so they are invariant in the algorithm.

Trivially, assertion (1) is true the first time because $q_1 p_1 p_2$ is legal by hypothesis. Now, suppose that assertions (1) and (2) are true in the iterations $1, 2, \ldots, k$ of the loop and let us prove that (1) is still true in the following iteration. If we begin the $k+1$ iteration after exploring a legal triangle in the iteration k, then clearly (1) is still true (because we are adding the last explored triangle to a previous legal sequence). If we begin iteration $k+1$ after exploring an illegal triangle $q_j p_i p_{i+1}$ in iteration k, then we need to check that the new sequence $ST = q_j p_{i-1} p_i, \ldots, q_{j-i+2} p_1 p_2$ of triangles (where j has been increased by one) is legal. Assume to the contrary that in this sequence ST a first illegal triangle $q_{j-h} p_{i-h-1} p_{i-h}$ appears, so q_{j-h} is the first clockwise point of Q on the right of line $p_{i-h-1} p_{i-h}$. By removing the points of Q from q_{j+1} to q_{j-h-1} (see Figure 3 left), then the points of P from p_i to p_{i-h} ($n-h+1$ points) and the points of Q from q_{j-h} to q_j ($h+1$ points) appear in the boundary of the new convex hull, contradicting the U-condition (at most $n+1$ points can appear in the boundary of the new convex hull). Therefore (1) is invariant.

For assertion (2), the first time that the algorithm goes to the else branch, we are exploring the illegal triangle $q_j p_j p_{j+1}$, being the triangles $q_{j-1} p_{j-1} p_j, \ldots, q_1 p_1 p_2$ legal. If q_j was on the right side of $p_i p_{i+1}$ and after the second crossing point of that line with $\overline{Q}$, then, by removing the points of Q from q_1 to q_{j-1} (remember that q_1 is the first point of Q to the left of $p_1 p_2$), the U-condition is contradicted because in the boundary of the new convex hull $n+2$
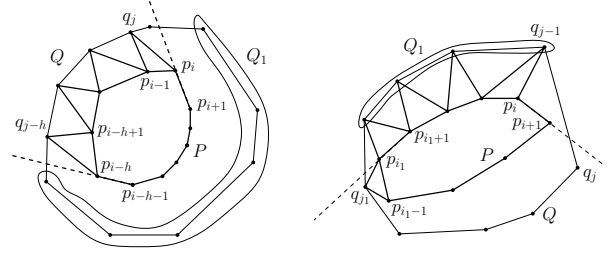

Figure 3: Illustration of Lemma 3. Left: Supposing q_{j-h} is on the right of $p_{i-h-1} p_{i-h}$, the U-condition fails if Q_1 is removed. Right: Supposing q_j is on the right of $p_i p_{i+1}$, the U-condition fails if Q_1 is removed.

points appear (the points of P from p_1 to p_{j+1} and the points of Q from q_j to q_n). Therefore, in the first visit to the else branch assertion (2) is true.

Suppose that the last illegal triangle explored is $q_{j_1} p_{i_1} p_{i_1+1}$ and the algorithm is exploring a new illegal triangle $q_j p_i p_{i+1}$. As (2) was true in the previous iterations, point q_{j_1} has to be placed between q_{j_1-1} and the first crossing of line $p_{i_1} p_{i_1+1}$ with $\overline{Q}$. After exploring this triangle, subscript j_1 is increased by one, and then h operations (perhaps $h = 0$) of increasing both subscripts (i and j) are done. Therefore, it must be $j = j_1 + h + 1$ and $i = i_1 + h$, for some $h \geq 0$. Since (1) is invariant, the $h+1$ triangles $q_{j-1} p_{i-1} p_i, q_{j-2} p_{i-2} p_{i-1}, \ldots, q_{j_1} p_{i_1-1} p_{i_1}$ are legal. Therefore, if q_j is placed after the second crossing of line $p_i p_{i+1}$ with $\overline{Q}$, then, by removing the h points of Q from q_{j_1+1} to q_{j-1}, the $h+2$ points of P from p_{i_1+1} to p_{i+1} appear in the boundary of the new convex hull, contradicting the U-condition (see Figure 3 right). Hence, (2) is invariant.

Lastly, since (1) is invariant and the last triangle, $q_{j-(i-1)} p_1 p_2$, is legal, then the subscript $j - i + 1$ has to be between 1 and m. This implies that $j \leq i + m - 1$ in the algorithm. Hence, the algorithm can go to the else branch a maximum of $m - 1$ times, and finishes in a maximum number of $n + m - 1$ steps. When the algorithm finishes, then $i = n + 1$ and the triangles of the sequence $q_{j-1} p_n p_{n+1}, q_{j-2} p_{n-1} p_n, \ldots, q_{j-n} p_1 p_2$ are legal because (1) is invariant. $\square$

Now, assume that $|Q| = |P|$, $Q \cup P$ satisfies the U-condition, there are more points inside $\overline{P}$ and that $|Q| > 3$ (the case $|Q| = 3$ was solved in [2]). To get a 4-connected geometric graph, we proceed as follows. First draw the zig-zag including alternatively the points of P and Q, according to the previous lemma. Then, take a diagonal of $\overline{P}$, for example diagonal $p_1 p_3$. This diagonal divides $\overline{P}$ into two subpolygons $P_1 = \{p_1, p_3, \ldots, p_n, p_1\}$ and $P'_1 = \{p_1, p_2, p_3, p_1\}$. If the interior $I(P_1)$ of P_1 is nonempty, then let P_2 be the convex polygon defined by the points $H(I(P_1) \cup p_1, p_3)$. If the interior $I(P_2)$ of this polygon is again nonempty,

then we define P_3 as the convex polygon defined by the points $H(I(P_2) \cup p_1, p_3)$, and so on, until we obtain an empty convex polygon P_h. Thus, we have a sequence $P_h \subset P_{h-1} \subset \ldots \subset P_2 \subset P_1$ of nested polygons (see Figure 4 left). The same process can be done starting at P'_1, obtaining another sequence $P'_{h'} \subset P'_{h'-1} \subset \ldots \subset P'_2 \subset P'_1$ of nested polygons. Observe that the region M_i (M'_i) bounded by the consecutive polygons P_{i+1} and P_i (P'_{i+1} and P'_i) has the shape of a "half-moon". It is not difficult to prove that M_i (M'_i) can be triangulated such that points p_1 and p_3 are not used, and each one of the added edges has an endpoint in P_i (P'_i) and the other one in P_{i+1} (P'_{i+1}). Then, we triangulate all the half-moons in this way (using edges connecting points placed in different polygons) and we triangulate the convex polygon P_f, formed by concatenating P_h and $P'_{h'}$, such that the only points with degree two are p_1 and p_3. It can be easily checked that the triangulation obtained in this way is 4-connected (see Figure 4 right).
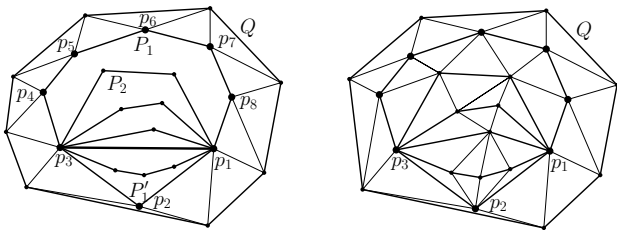

Figure 4: The general construction when $|Q| = |P|$ and there are points inside $\overline{P}$.

The case in which $|Q| < |P|$ and $Q \cup P$ satisfies the U-condition is solved in a similar way. First a graph based on a zig-zag is built, although this starting graph cannot be a zig-zag as in the previous case, because $|Q| < |P|$. Now, the starting graph is a zig-zag connecting the points of Q to some points of P plus some additional edges connecting the points of P not belonging to the zig-zag to some points of Q (bold edges in Figure 5 left). After building this starting graph, we take a diagonal of $\overline{P}$ connecting two points of P, consecutive in the zig-zag but not consecutive in P (points p_{i_1} and p_{i_2} in Figure 5), and we proceed as in the previous case, adding the triangulations of the different half-moons and the final triangulation of the convex polygon P_f (see Figure 5 right). The resulting triangulation is 4-connected.

The U-condition is the key to finding this starting graph (the zig-zag plus some additional edges), although proving the existence of such a graph is not obvious. Due to space limitations, we do not include this proof.

Therefore, we have proved the following theorem.

Theorem 4 *Let S be a set of points. If $Q = H(S)$, $P = H(S \setminus Q)$ and $Q \cup P$ satisfies the U-condition,*
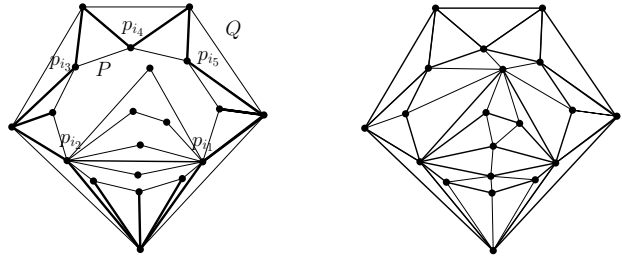

Figure 5: The general construction when $|Q| < |P|$ and there are points inside $\overline{P}$.

then S is 4-connectible.

3 Conclusions

In this paper, we have defined a condition, the U-condition, that any set of points S must satisfy to be 4-connectible. Moreover, we have proved that the U-condition is also sufficient for sets of points in which $Q \cup P$ satisfies the U-condition.

In [2], the case $|Q| = 3$ is completely solved. Given a set S of points such that $|Q| = 3$, the authors show how to build a 4-connected triangulation on S, except for a particular configuration of points. This particular configuration is precisely the only one not satisfying the U-condition, among all the configurations of points such that $|Q| = 3$.

Using different techniques not included in this paper, we can extend the family of 4-connectible sets. For any set S of points satisfying the U-condition (it is not required that $Q \cup P$ satisfies the U-condition) such that $|P| = 3$ or $|P| = 4$, we can built a 4-connected geometric graph on S.

Finally, we conclude with the following conjecture.

Conjecture 1 *If a set of points S satisfies the U-condition, then S is 4-connectible.*

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