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A ZERO OF A PROPER MAPPING

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Abstract. Sufficient conditions are given to assert that a differentiable proper Fredholm mapping between Banach spaces over $K = R$ or $K = C$ has a zero. The proof of the result is constructive and is based upon continuation methods.

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1. Preliminaries

Throughout this paper we assume that Y and Z are Banach spaces both, being over $K = R$ or $K = C$. If $f : Y \to Z$ is a continuous mapping, then one way of solving the equation

$$f(y) = 0$$

is the continuation method with respect to a specific parameter. It consists of finding a homotopy

$$H(t, y) \quad t \in [0, 1],$$

which, when $t = 0$, verifies $H(t, y) = f(y)$. This method is feasible if there is a known solution of $H(t, y) = 0$ which can be continued to a zero of $H(0, y)$

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[1-5,18-21]. A comprehensive exposition of the continuation method and its applications can be found in [22]-[24].

An existence of zero points in finite dimensional case is proved in [7-17]. In the present paper sufficient conditions implying an existence of zeros for Fredholm mappings in general Banach spaces have been formulated. The proof is based on the classical continuation method [1-6] and Banach fixed point theorem.

We briefly recall some concepts to be used. *Direct sum.* If Y_1 and Y_2 are linear subspaces of linear space Y, we write $Y = Y_1 \bigoplus Y_2$ if and only if every $y \in Y$ can be represented uniquely as $y = y_1 + y_2$, where $y_1 \in Y_1$ and $y_2 \in Y_2$. If Y is a Banach space and the above split is verified and Y_1 and Y_2 are closed linear subspaces of Y, we say $Y = Y_1 \bigoplus Y_2$ is a *topological direct sum.* If there is a continuous linear operator $p_i : Y \to Y, \quad p_i y = y_i, i = 1 \quad \text{or} \quad i = 2$, then the split is a direct sum. We say Y_1 is the *topological complement* of Y_2. Let $f, g : U \subset Y \to Z$ be continuous mappings, where Y, Z are Banach spaces. g is said to be *compact* whenever the image $g(B)$ is relatively compact (i.e. its closure is compact in Z) for every bounded subset $B \subset U$. f is said to be *proper* whenever the pre-image $f^{-1}(C)$ of every compact subset $C \subset Z$ is also a compact subset of U.

We recall some known propositions.

Proposition 1. *If $a \in U$, U is open, $g : U \subset Y \to Z$ is compact and the derivative $g'(a)$ exists, then $g'(a) \in L(Y, Z)$ is also compact (see [18, p. 296]).*

If U is open, then $F : U \subset Y \to Z$ is said to be a *Fredholm* mapping if and only if F is a C^1-mapping, and if and only if $F'(y) : Y \to Z$ is a linear Fredholm operator for all $y \in U$. That $L : Y \to Z$ is a *linear Fredholm* operator means that L is linear and continuous and both the numbers $\dim(\ker(L))$ and $\operatorname{codim}(L(Y))$ are finite. Therefore $\ker(L) = Y_1$ is a Banach space and has topological complement Y_2, since $\dim(Y_1)$ is finite. The number

$$\operatorname{ind}(L) = \dim(\ker(L)) - \operatorname{codim}(L(X))$$

is called the *index* of L. From the continuity of $y \mapsto \|F'(y)\|$ we find that $\operatorname{ind}(F'(x))$ is constant on U if U is connected.

Proposition 2. *For a linear Fredholm operator $L : Y \to Z$, the following is true: the perturbed operator $L+C$ is also a Fredholm operator with $ind(L+C) = ind(L)$, if $C \in L(Y,Z)$ and C is compact (see [18, p 366])*

2. A zero of a proper mapping

By $B_a(y^*)$ we mean a ball with center $y^* \in Y$ and radius $a > 0$, that is, $B_a(y^*) = \{y \in Y : \|y - y^*\| < a\}$. As usually by $\partial B_a(y^*)$ we denote the boundary of $B_a(y^*)$. The main result of the paper is:

Theorem. *Let $f : Y \to Z$ be a C^1 - proper Fredholm mapping and $g : Y \to Z$ be a C^1 - compact mapping. Suppose:*
i) there is $y^* \in Y$ such that $f(y^*) = g(y^*)$;
ii) there is $a > 0$ such that $f(y) \neq tg(y)$ for $t \in [0,1]$ and $y \in \partial B_a(y^*)$;
iii) for $t \in [0,1]$ and $y \in B_a(y^*)$ $f'(y) - tg'(y)$ is surjective.
Then f has at least one zero in $B_a(y^*)$.

Proof. *First step.*
We prove an existence of a compact set V'' containing all the solutions of $f(y) - tg(y) = 0$, when $y \in B_a(y^*), t \in [0,1]$. We put

$$V = g(B),$$

where $B = \{y \in B_a(y^*) : \text{such that } f(y) = tg(y) \text{ for some } t = t(y) \in [0.1]\}$. B is not empty, because $f(y^*) = 1.g(y^*)$.

Since g is a compact mapping, $g(B_a(y^*))$ is a relatively compact set, and $V = g(B) \subset g(B_a)$, implies V is also a relatively compact.

The set $[0,1] \times \overline{V}$ is a compact in $[0,1] \times Y$ and $V' = \{ty : (t,y) \in [0,1] \times \overline{V}\}$,
is a compact in Y, since it is the image of $[0,1] \times \overline{V}$ under the continuous mapping

$$(t,y) \in [0,1] \times \overline{V} \mapsto ty \in Y.$$

But f is proper and therefore $V'' = f^{-1}(V') \subset Y$ is also a compact set.

Second step. Let us construct the following homotopy

$$H : [0,1] \times Y \to Z, H(t,y) = f(y) - tg(y).$$

By Proposition 1 $g'(y)$ is compact, and so is $tg'(y), \forall t \in [0,1]$. Since f is a Fredholm mapping, then $f'(y) \in L(Y,Z)$ is a Fredholm linear operator. By Proposition 2 $f'(y) - tg'(y)$ is also a Fredholm operator, with $\text{ind}(f'(y) - tg'(y)) = \text{ind}(f'(y))$. Therefore $H(t,y)$ is a Fredholm mapping, and it is an homotopy between f and $f-g$. As noted in Section 1, $\text{ind}(f'(y))$ is constant for all y in Y. Hence

$$\text{ind}(f'(y) - tg'(y)) = \text{const.}$$

for all $(t,y) \in [0,1] \times V''$.

(a). We prove that if $H(t_0,y_0) = 0, (t_0,y) \in [0,1] \times B_a(y^*)$, then $H(t,y) = 0$ has a solution for t near t_0. To this end we will transform this problem into a fixed point problem.

Henceforth, partial derivatives will generally denoted by writing initial spaces as subindices of mappings. Let $t_0 \in [0,1]$ and $y_0 \in B_a(y^*)$. We consider the partial derivative

$$H_Y(t_0,y_0) \in L(Y,Z),$$

which is surjective from hypothesis (iii). Since $H'(t_0,y_0) \in L([0,1] \times Y, Z)$ is a linear Fredholm operator, we can conclude that $\ker H'(t_0,y_0)$ is a closed finite subspace of $[0,1] \times Y$, and so

$$\ker H_Y(t_0,y_0) := Y_1$$

is also a closed finite subspace of Y, since

$$\ker H_Y(t_0,y_0) = \ker H'(t_0,y_0) \circ u$$

$$u : Y \to [0,1] \times Y, u(y) = (0,y).$$

Therefore Y_1 splits Y in the sense that Y is the topological direct sum $Y = Y_1 \oplus Y_2$, with Y_2 the topological complement of Y_1.

By A we denote the restriction of the surjective mapping $H_Y(t_0,y_0)$ to Y_2. It is an isomorphism: Indeed $A(y_2) = 0$, with $y_2 \in Y_2$, the definition of Y_1 implies that $y_2 \in Y_1$. Hence $y_2 = 0$, since $Y = Y_1 \oplus Y_2$, and so,A being linear, we conclude that A is injective. By hypothesis (iii) it is surjective, and therefore it is an isomorphism.

We now set

$$G(t,y_1,y_2) := H(t,y_0 + y_1 + y_2),$$

and we solve the equation

$$G(t,0,y_2)=0, \tag{1}$$

for y_2. Obviously we have

$$G(t_0,0,0)=H(t_0,y_0)=0,$$

and $G_Y(t_0,0,0)\in L(Y_2,Z)$ is verified to be the same as the bijective mapping A.

We now define the two following mappings

$$h(t,0,y_2):=A(y_2)-G(t,0,y_2),$$

$$T_t(y_2):=A^{-1}(h(t,0,y_2)).$$

In this case t is an index of the mapping T_t.

Equation (1) is equivalent to the following "key equation "

$$y_2=T_t(y_2). \tag{2}$$

The problem (1) is transformed into the fixed point problem (2), which we are going to study. Let

$$|t-t_0|,\|y_2'\|,\|y_2\|\le r,\quad |t-t_0|\le r_0',$$

with r,r_0' which we will fix later. Since
$h_Y(t,0,y_2)=A-G_Y(t,0,y_2)$, then $h_Y(t_0,0,0)=0\in L(Y_2,Z)$. This together with the fact that h_Y is continuous owing to f and g being C^1-mappings, and the application of the mean value theorem implies:

$$\|h(t,0,y_2)-h(t,0,y_2')\|\le \sup\{\|h_Y(t,0,y-2+\theta(y_2'-y_2)\|:\theta\in(0,1)\}.\|y_2'-y_2\|$$

$$=o(1)\|y_2'-y_2\|,\quad o(1)\to 0\quad \text{as}\quad r\to 0. \tag{3}$$

Since $h(t_0,0,0)=0$, and h is continuous, and given equation (3), we also conclude:

$$\|h(t,0,y_2)\|\le\|h(t,0,y_2-h(t,0,0)\|+\|h(t,0,0)\|$$

$$=o(1)\|y_2\|+o'(1),\quad o(1)\to 0\quad\text{as}\quad r\to 0,\quad o'(1)\to 0\quad\text{as}\quad r_0'\to 0. \tag{4}$$

¿From equations (3) and (4) and the definition of T_t we can write

$$\|T_t(y_2)\|\le\|A^{-1}\|\|h(t,0,y_2)\|=\|A^{-1}\|(o(1)\|y_2\|+o'(1))$$

$$o(1)\to 0\quad\text{while}\quad r\to 0,\quad o'(1)\to 0\quad\text{while}\quad r_0'\to 0.$$

We fix r so that $o(1) < \frac{1}{2\|A^{-1}\|}$, and we construct the closed and non-empty set

$$M = \{y_2 \in Y_2 : \|y_2\| \leq r\}.$$

Now we fix r'_0 so that $o'(1) < \frac{r}{2\|A^{-1}\|}$, and we introduce the set

$$M' = \{(t, 0) \in [0, 1] \times Y_1 : |t - t_0| \leq min\{r, r'_0\} = r_0\}.$$

For any $(t, 0) \in M'$ $T_t : M \to M$, and T_t is $\frac{1}{2}-$contractive, are verified since

$$\forall y_2 \in M, \|T_t(y_2)\| \leq r \Rightarrow T_t(y_2) \in M,$$

and

$$\forall y_2, y'_2 \in M \|T_t(y_2) - T_t(y'_2)\| \leq \|A^{-1}\| \frac{1}{2\|A^{-1}\| \|y_2 - y'_2\|}.$$

Then T_t has a unique fixed point $y_2 \in M$ $T_t(y_2(t, 0)) = y_2(t, 0)$, or equivalently $H(t, y_0 + 0 + y_2(t, 0)) = 0$.

(b). We prove here that f has a zero $y^{**} \in B_a(y^*)$ by repeating a narrower version of section (a). Let (t_0, y_0) be any point such that $H(t_0, y_0) = 0$, with $t_0 \in [0, 1], y_0 \in V''$.

In section (a) we took $r > 0$ so that

$$o(1) < \frac{1}{2\|A(t_0, y_0)^{-1}\|}.$$

Given the continuity of $H(\cdot, \cdot)$ and $H_Y(\cdot, \cdot)$ with the compactness of $[0, 1] \times V''$, we can fix r so that

$$o(1) < \frac{1}{2.\max\{\|A(t_0, y_0)^{-1}\| : (t, y) \in [0, 1] \times V''\}} = \frac{1}{2c}$$

and we construct M with this r, therefore r'_0 is selected sot that

$$o'(1) < \frac{r}{2c}.$$

Then we construct M' with $r_0 < \min\{r, r'_0\}$. One can define a path $\Gamma = \{(t, y) \in [0, 1] \times B_a(y^*)\}$ with beginning at $(1, y^*)$. Let us consider the set $\{(t, y) \in [0, 1] \times V''$: such that $H(t, y) = 0\}$, and let $[0, 1] \times V''$ be covered by balls of centres (t, y) and radii r_0. There is a finite subcovering of $[0, 1] \times V''$ by these balls. The path starting at the point $(1, y^*)$ can enter and leave one of the balls of this subcovering at most once and it can always be prolongued in $[0,1] \times V''$ while t decreases. Then (ii) implies that Γ ends at $(0, y^{**})$ for some $y^{**} \in B_a(y^*)$, that is

$$H(0, y^{**}) = 0.$$

Thus $f(y^{**}) = 0$, which proves the Theorem.

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