On making a graph crossing-critical

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Abstract

A graph is *crossing-critical* if its crossing number decreases when we remove any of its edges. Recently it was proved that if a non-planar graph G is obtained by adding an edge to a cubic polyhedral (planar 3-connected) graph, then G can be made crossing-critical by a suitable multiplication of its edges. Here we show: (i) a new family of graphs that can be transformed into crossing-critical graphs by a suitable multiplication of its edges, and (ii) a family of graphs that cannot be made crossing-critical by any multiplication of its edges.

Introduction

This work is motivated by the following question settled by Beaudou et al. in [1]: to what extent is crossing-criticality a property that is inherent to the structure of a graph, and to what extent can it be induced on a non crossing-critical graph by multiplying (all or some of) its edges? In [1] a family of non crossing-critical graphs are transformed into crossing-critical graphs by *multiplying* (adding parallel) edges.

The use of parallel edges has been essential in many other important results on crossing number. For example, in [2] was proved that for every $a > b > 0$, there exist a graph G with crossing number a in the plane, crossing number b in the torus, and crossing number 0 in the double torus. This is called the orientable crossing sequence of a graph G. In [4] was reported a conjecture of R. B. Richter which states that crossing-critical graphs have bounded maximum degree. This conjecture was disproved by Dvořák and Mohar [3], who exhibited crossing-critical graphs with large maximum degree. In all these papers the use of weighted (also called “thick”) edges is essential.

No simple graph that satisfies the properties as defined in [2] and [3] are not known. On the other hand there exist simple non crossing-critical graphs such that if we multiply its edges (or equivalently, assign weights on them) then they become crossing-critical, as in [1]. Thus, important questions remain: What makes a graph crossing-critical? Does every non-planar graph become crossing-critical by a weight assignment to its edge set?

In this paper we give two infinite families $\mathcal{G}$ and $\mathcal{G}'$ of graphs such that (i) every graph in $\mathcal{G}$ remains non crossing-critical even after any multiplications of its edges, and (ii) every graph in $\mathcal{G}'$ is not crossing-critical, but after a suitable multiplication of its edges, it is transformed into a crossing-critical graph. Let us proceed to define these families.

Let W_n be the wheel with $n+1$ vertices, $n \geq 5$, and let v_0 be the degree n vertex. The remaining n vertices are labeled $v_1, v_2, \dots, v_n$ in the order in which they appear in the n-cycle. We add a new vertex u to W_n which is joined to vertices v_0, v_1 and v_3 and denote by G_n the resulting graph. Let G'_n be the graph that results by removing the edges v_0v_1 and v_0v_3 from G_n. We define $\mathcal{G} := \{G_n : n \geq 5\}$ and $\mathcal{G}' := \{G'_n : n \geq 5\}$. See Figure 1. Note that each graph of $\mathcal{G} \cup \mathcal{G}'$ is 3-connected.

It is not difficult to find non-planar graphs which cannot be made crossing-critical by any multiplication of its edges. For example, if G is a non-planar graph and e is a cut edge of G, then G cannot be made crossing-critical by any multiplication of its edges because e cannot be made critical. Thus, some connectivity assumption is needed in order to guarantee that a non-planar graph can be made crossing-critical. In [1], for example, the graphs under consideration are assumed to be internally 3–connected and such an assumption plays a central role in its work. On the other hand, as we will see in Theorem 1, the family $\mathcal{G}$ is interesting because shows that 3-connectedness property by itself is not sufficient to ensure that a non-planar graph can be made crossing-critical.

Our main results are the following.

Theorem 1 *Any graph $G_n \in \mathcal{G}$ is not crossing-critical. Moreover, G_n cannot be made crossing-critical by any multiplication of its edges.*

Theorem 2 *Every graph $G'_n \in \mathcal{G}'$ is not crossing-critical, but there exists a suitable multiplication of its edges such that the resulting graph $\overline{G'_n}$ is crossing-critical.*

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An edge e of a graph G is a *Kuratowski edge* if there is a subgraph H of G which is isomorphic to a subdivision of a K_5 or $K_{3,3}$ and e lies on H. In [1] was conjectured that a graph whose edges are all Kuratowski becomes crossing-critical after a suitable multiplication of its edges. In this sense, note that the graphs in $\mathcal{G}'$ are consistent with this conjecture because any edge in G'_n is a Kuratowski edge (unlike the edges v_0v_1 and v_0v_3 in G_n).

1 Multigraphs and weighted graphs

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. An edge e of G is *crossing-critical* if $\mathrm{cr}(G-e) < \mathrm{cr}(G)$, and say that G is *crossing-critical* if all its edges are crossing-critical.

Recall that a *weighted graph* is a pair (G, w) where G is a graph and w is a function (the *weight assignment*) that assigns to each edge e of G a number $w(e)$ (the *weight*). The weight assignment is *positive* (respectively, *integer*) if $w(e)$ is a positive (respectively, integer) number for any edge e of G. We only consider positive integer weight assignments.

We extend the concept of crossing number to weighted graphs (G, w) in an analogous way, just taking to account that a crossing between the edges e and e' contributes $w(e) \cdot w(e')$ to $\mathrm{cr}\,(G, w)$. A drawing $\mathcal{D}$ of (G, w) is *optimal* if $\mathrm{cr}(\mathcal{D}) = \mathrm{cr}(G, w)$.

We now proceed to define what a crossing-critical edge is in a weighted graph. Let (G, w) be a weighted graph, an edge e of (G, w) is *crossing-critical* if $\mathrm{cr}\,(G, w_e) < \mathrm{cr}\,(G, w)$, where the weight assignment w_e is defined by,

$$w_e(f) = \begin{cases} w(f) & \text{if } f \neq e, \\ w(f) - 1 & \text{if } f = e. \end{cases}$$

As usual, (G, w) is *crossing-critical* if all its edges are crossing-critical.

Let G^* be a multigraph, and let G be its underlying simple graph. We define the *associated weighted graph* (G, w^*) of G^* as follows: for every edge e of G, we define $w^*(e)$ as the multiplicity of e in G^*. The following observation is straightforward.

Remark 1 *Let G^* be a multigraph and let (G, w^*) be its associated weighted graph. Then G^* is crossing-critical if and only if (G, w^*) is crossing-critical.*

2 Proofs of Theorems 1 and 2

For brevity, let (i) $\alpha_0 := v_0u$ and $\alpha_i := v_0v_i, i = 1, \dots, n$; (ii) $\beta_i := v_iv_{i+1}, i = 1, \dots, n$ with $v_{n+1} = v_1$; and (iii) $\gamma_1 := uv_1$ and $\gamma_2 := uv_3$. Under this notation, observe that $G'_n = G_n - \{\alpha_1, \alpha_3\}$. See Figure 1.

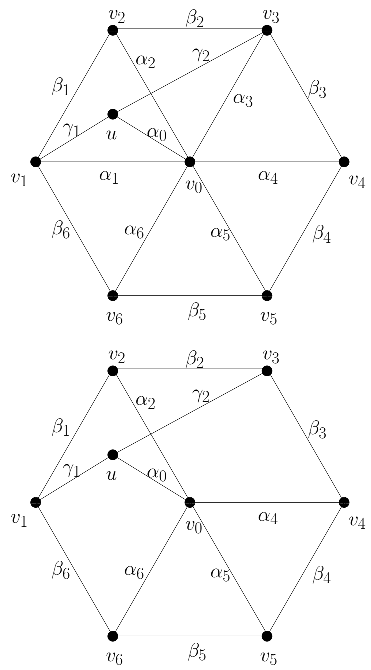

Figure 1: A drawing $\mathcal{D}$ of G_6 (above), and a drawing $\mathcal{D}'$ of G'_6 (below).

Lemma 3 *Let w' be a positive integer weight assignment on G'_n. Then there exists an optimal drawing of (G'_n, w') in which we can add both edges α_1 and α_3 without increasing the crossing number.*

Proof. Let $\mathcal{D}'$ be an optimal drawing of (G'_n, w'). We divide the proof according to the edges that are involved in a crossing in $\mathcal{D}'$.

(A) Suppose that α_0 is crossed in $\mathcal{D}'$ (analogously for α_2).

Since $\mathcal{D}'$ is an optimal drawing, α_0 does not cross with adjacent edges, therefore α_0 crosses with a β-edge, say β_j. Let $\mathcal{D}^*$ be the drawing defined as follows. Draw a simple regular n-polygon such that $v_1, v_2, \dots, v_n$ (in that order) are its vertices, place v_0 in the center of such a polygon and add the α_i-edge, $i = 2, 4, 5, \dots, n$, as straight line segments. Now draw α_0 as an straight line segment crossing the edge β_j. Finally the edges γ_1 and γ_2 can be added around the n-polygon boundary without introducing new crossings. So $\mathcal{D}^*$ has just one crossing and therefore $\mathrm{cr}(\mathcal{D}^*) \leq \mathrm{cr}(\mathcal{D}')$. But since $\mathcal{D}'$ is an optimal drawing, we must have that $\mathrm{cr}(\mathcal{D}^*) = \mathrm{cr}(G', w')$. Moreover, we can add the edges α_0 and α_3 as straight line segments without adding a new crossing, as required.

(B) Both edges in any of $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$, $\{\beta_1, \gamma_2\}$, $\{\gamma_1, \beta_2\}$ are not crossed in $\mathcal{D}'$.

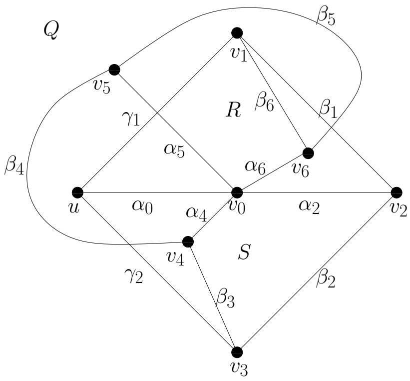

Figure 2: Drawing of G'_6 where γ_1, γ_2, β_1, β_2, α_0 and α_2 form a plane $K_{2,3}$

We only analyze the case in which both edges of $\{\beta_1, \gamma_2\}$ are not crossed in $\mathcal{D}'$ (the rest of the cases can be verified similarly). By (A) we can assume that neither α_0 nor α_2 are crossed in $\mathcal{D}'$. Because none of β_1 and γ_2 has a cross in $\mathcal{D}'$, we can add α_1 (respectively α_3) following the uncrossed path $\alpha_2\beta_1$ (respectively $\alpha_0\gamma_2$) without introducing new crossings.

(C) Suppose β_1 crosses γ_2 in $\mathcal{D}'$ (analogously, β_2 crosses γ_1).

Draw a plane wheel with vertices $v_1, \ldots, v_n$ forming a n-cycle and the vertex v_0 as the center. All edges α_i, $i \neq 1, 3$ are the spokes of this wheel. Then draw the path $\alpha_0\gamma_1$ as a spoke. Finally add the edge γ_2 only crossing the edge β_1. Since the number of crossings of such a drawing is less than or equal to the number of crossings of $\mathcal{D}'$, it is optimal. In this new drawing we can add α_1 and α_3 without increasing the crossing number.

By (A) and (C) we can assume that neither γ_1, γ_2, β_1, β_2, α_0 nor α_2 cross each other. Thus γ_1, γ_2, β_1, β_2, α_0 and α_2 form a plane $K_{2,3}$. Without any loss of generality, we may assume that v_0 is in the bounded region defined by the cycle $\gamma_1\beta_1\beta_2\gamma_2$. Let R, S and Q denote the disjoint regions bounded by the cycles $\alpha_0\alpha_2\beta_1\gamma_1$, $\alpha_0\gamma_2\beta_2\alpha_2$ and $\gamma_1\gamma_2\beta_2\beta_1$, respectively. (See Figure 2)

Let P be the path $\beta_3\beta_4 \ldots \beta_n$. Let μ, ν, and ρ be the element of $\{\beta_1, \gamma_1\}$, $\{\beta_2, \gamma_2\}$, and $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ with less weight, respectively.

Now we proceed according to the interior vertices of P that are in each of R, S and Q.

(D) No interior vertices of P are in Q.

Then, all the interior vertices of P are in R, S, or both of them.

(D.i) All interior vertices of P are in R (analogously for S). Since $\mathcal{D}'$ is optimal, β_3 is the only edge that cross the cycle $\alpha_0\alpha_2\beta_1\gamma_1$. By (A), β_3 does not cross neither α_0 nor α_2, thus, by optimality, β_3 only crosses with μ. Then this case follows from (B).

(D.ii) Both R and S contain interior vertices of P. Because Q has no interior vertices of P, at least one β-edge of P goes from R to S. Moreover, such a β-edge is neither β_3 nor β_n. Let β_i be ($i \neq 3, n$), the β-edge of P with less weight. Thus we can get a drawing $\mathcal{D}^*$ where the only crossings are those involving β_i with μ and β_i with ν. Since $\mathcal{D}'$ is optimal, $\mathrm{cr}(\mathcal{D}^*) = \mathrm{cr}(\mathcal{D}')$. By (B), we can add the edges α_1 and α_3 without increasing the crossing number.

(E) Some interior vertices of P are in Q.

In this case we have three possibilities.

(E.i) All interior vertices of P are in Q. In this case we can get a drawing with as many crossings as $\mathcal{D}'$ in which every crossing involves ρ and some α_i, with $i = 4, 5, \ldots, n$. This case follows from (B).

(E.ii) Each region R, S and Q contains interior vertices of P. Then there must be a β-edge, say β_i (respectively β_j), crossing the cycle that bounds the region R (respectively S). By (A), β_i (respectively β_j) must cross either γ_1 or β_1 (respectively γ_2 or β_2). If $w'(\beta_i) \leq w'(\beta_j)$, we can get an optimal drawing $\mathcal{D}^*$ by putting all vertices $v_4, v_5, \ldots, v_i$ in S, all vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ in R. Thus we are back in case (D.ii). If $w'(\beta_j) \leq w'(\beta_i)$ we can proceed analogously.

(E.iii) Both R and Q contain (all the) interior vertices of P (analogously for S and Q). Let v_i be the interior vertex of P in R with the smallest index. If v_4 is in R, then the edge β_3 crosses the cycle $\alpha_0\alpha_2\beta_1\gamma_1$ and, by putting all interior vertices of P in R, we can proceed as in (D.i). Thus v_4 must be in Q and so $i \geq 5$. By the choice of v_i, and (A), β_{i-1} crosses either γ_1 or β_1, and all vertices $v_4, v_5, \ldots, v_{i-1}$ are in Q.

It is easy to get a drawing $\mathcal{D}^*$ whose only crossings are: (i) α_j, $j = 4, \ldots, i-1$ with ρ, (ii) β_{i-1} crosses μ; and in $\mathcal{D}^*$ all the other vertices $v_i, \ldots, v_n$ remain in R. This implies $\mathrm{cr}(\mathcal{D}^*) = \mathrm{cr}(\mathcal{D}')$ and satisfies (B). □

Proof of Theorem 1. Lemma 3 implies that both α_1 and α_3 are not crossing-critical in (G_n, w) for any weight assignment w on G_n. Theorem 1 follows combining this and Remark 1. □

Finally, consider the following weight assignment w'_n on G'_n:

$$w'_n(e) = \begin{cases} 1 & \text{if } e = \alpha_i, i = 4, \ldots, n; \\ n-3 & \text{otherwise.} \end{cases}$$

Lemma 4 $\mathrm{cr}(G'_n, w'_n) = (n-3)^2$ *and* (G'_n, w'_n) *is crossing-critical.*

Proof. The proof for the crossing number is essentially that given in Lemma 3. To prove that the graph is crossing-critical, it is easy to see that for each edge e there exists an optimal drawing of (G'_n, w'_n) where e is involved in a crossing. $\square$

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