Distance domination, guarding and vertex cover for maximal outerplanar graphs

Santiago Canales[1], Gregorio Hernández[*2], Mafalda Martins[‡†3], and Inês Matos[‡3]

[1]Universidad Pontificia Comillas de Madrid, Spain
[2]Universidad Politécnica de Madrid, Spain
[3]Universidade Aveiro & CIDMA, Portugal

Abstract

In this paper we define a distance guarding concept on plane graphs and associate this concept with distance domination and distance vertex cover concepts on triangulation graphs. Furthermore, for any n-vertex maximal outerplanar graph, we provide tight upper bounds for $g_{2d}(n)$ ($2d$-guarding number), $\gamma_{2d}(n)$ ($2d$-domination number) and $\beta_{2d}(n)$ ($2d$-vertex cover number).

Introduction

Domination, covering and guarding are widely studied subjects in graph theory. Given a graph $G = (V, E)$ a *dominating set* is a subset D of V such that every vertex not in D is adjacent to a vertex in D. A subset C of V is a *vertex cover* if each edge of the graph is incident to at least one vertex of the set. Thus, a dominating set guards the *vertices* of a graph while a vertex cover guards its *edges*. In plane graphs these concepts differ from the *guarding set* concept that guards the *faces* of the graph. Let $G = (V, E)$ be a plane graph, a guarding set is a subset S of V such that every face has a vertex in S. There are many papers and books about domination and its many variants in graphs, e.g. [4, 8, 9, 10]. Domination was extended to *distance domination* by Meir and Moon [11]: given a graph G, a subset D of V is said to be a *distance k-dominating set* if for each vertex $u \in V - D$, $dist_G(u, v) \leq k$ for some $v \in D$. The minimum cardinality of a distance k-dominating set is said to be the *distance k-domination number* of G and is denoted by $\gamma_{kd}(G)$. In the case of distance domination, there are also some known results concerning bounds for $\gamma_{kd}(G)$, e.g., [13, 14]. However, if graphs are restricted to triangulations, as far as we know, there are no known bounds for $\gamma_{kd}(G)$. The distance domination was generalized to *broadcast domination* when the power of each vertex may vary [6]. Given a graph $G = (V, E)$, a *broadcast* is a function $f : V \to \mathbb{N}_0$. A broadcast is *dominating* if for every vertex v, there is a vertex u such that $f(u) > 0$ and $d(u, v) \leq f(u)$. A dominating broadcast f is *optimal* if $\sum_{v\in V} f(v)$ is minimum over all choices of broadcast dominating functions for G. The *broadcast domination problem* consists in building this optimal function. Note that, if $f(V) = \{0, k\}$, then the broadcast domination problem is the distance k-dominating problem. If a broadcast f provides coverage to the edges of G instead of covering its vertices, we have a generalization of the vertex cover concept [2]. A broadcast f is *covering* if for every edge $(x, y) \in E$, there is a path P in G that includes the edge (x, y) and one end of P must be a vertex u, where $f(u)$ is at least the length of P. A covering broadcast f is *optimal* if $\sum_{v\in V} f(v)$ is minimum over all choices of broadcast covering functions for G. Note that, if $f(V) = \{0, 1\}$, then the broadcast cover problem coincides with the problem of finding a minimum vertex cover. Regarding the broadcast cover problem when all vertices have the same power (i.e., when $f(V) = \{0, k\}$, for a fixed $k \neq 1$), as far as we know, there are no published results besides [5] where the authors propose a centralized and distributed approximation algorithm to solve it. Concerning the guarding concept there are also known bounds on the guarding number $g(G)$. For example, $g(G) \leq \lfloor \frac{n}{2} \rfloor$, for any n-vertex plane graph [3], and if G is a maximal outerplanar graph this bound is $\lfloor \frac{n}{3} \rfloor$ [7]. Contrary to the notions of domination and vertex cover on plane graphs that were extended to include its distance versions, the guarding concept was not generalized to its distance version.

In this paper we generalize the guarding concept on plane graphs to its *distance guarding* version. We also formalize the broadcast cover problem when all vertices have the same power, which we call *distance*

*Research supported by ESF EUROCORES programme EuroGIGA - ComPoSe IP04 - MICINN Project EUI-EURC-2011-4306.

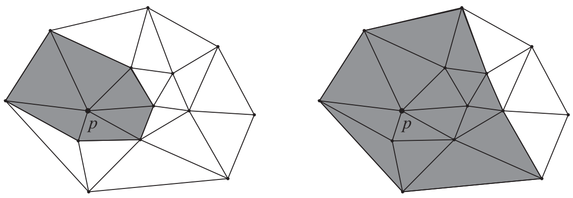
†Email: mafalda.martins@ua.pt. Research also supported by FCT grant SFRH/BPD/66431/2009.

‡Research supported by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness, CIDMA and FCT within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690.

k-vertex cover. Since there are no combinatorial results for the concepts of domination, vertex cover and guarding on its distance versions on triangulation graphs (*triangulations*, for short), we also address this problem. In the next section we describe some of the terminology that will be used throughout this paper.

1 Preliminaries

Given a triangulation of a point set $T = (V, E)$, we say that a bounded face T_i of T (i.e., a triangle) is *kd-visible* from a vertex $p \in V$, if there is a vertex $x \in T_i$ such that $dist_T(x, p) \leq k - 1$. The *kd-visibility region* of a vertex $p \in V$ comprises the triangles of T that are kd-visible from p (see Fig. 1).



(a) (b)

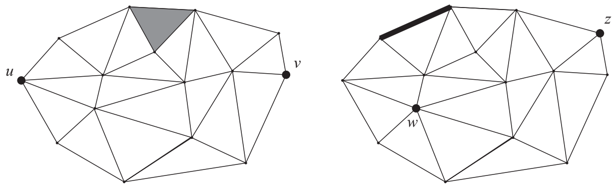
Figure 1: The kd-visible region of p for: (a) $k = 1$; (b) $k = 2$.

A *kd-guarding set* for T is a subset $F \subseteq V$ such that every triangle of T is kd-visible from an element of F. We designate the elements of F by *kd-guards*. The *kd-guarding number* $g_{kd}(T)$ is the number of vertices in a smallest kd-guarding set for T. Note that, to avoid confusion with *multiple guarding* [1] - where the typical notation is k-guarding - we will use kd-guarding, with an extra "*d*". Given a set S and a positive integer n, we define $g_{kd}(S) = max\{g_{kd}(T) : T = (V, E)$ is triangulation with $V = S\}$ and $g_{kd}(n) = max\{g_{kd}(S) : |S| = n\}$. A *kd-vertex cover* for T, or *distance k-vertex cover* for T, is a subset $C \subseteq V$ such that for each edge $e \in E$ there is a path of length at most k, which contains e and a vertex of C. The *kd-vertex cover number* $\beta_{kd}(T)$ is the number of vertices in a smallest kd-vertex cover set for T. Given a set S and a positive integer n, we define $\beta_{kd}(S) = max\{\beta_{kd}(S) : T = (V, E)$ is triangulation with $V = S\}$ and $\beta_{kd}(n) = max\{\beta_{kd}(S) : |S| = n\}$. Finally, as already defined by other authors, a *kd-dominating set* for T, or *distance k-dominating set* for T, is a subset $D \subseteq V$ such that each vertex $u \in V - D$, $dist_T(u, v) \leq k$ for some $v \in D$. The kd-domination number $\gamma_{kd}(T)$ is the number of vertices in a smallest kd-dominating set for T. Given a set S and a positive integer n, we define $\gamma_{kd}(S) = max\{\gamma_{kd}(T) : T = (V, E)$ is triangulation with $V = S\}$ and $\gamma_{kd}(n) = max\{\gamma_{kd}(S) : |S| = n\}$. Our main goal is to obtain bounds on $\gamma_{kd}(T)$, $g_{kd}(T)$ and $\beta_{kd}(T)$. We start by establishing a tight upper bound for $g_{2d}(n)$ for a special class of triangulations, namely the *maximal outerplanar graphs*. A graph is a maximal outerplanar graph if it is a triangulation of a simple polygon without holes [12]. Edges on the exterior face are called *exterior edges*, and *interior edges* otherwise.

In the next section we show that there is a relationship between $2d$-guarding, $2d$-dominating and $2d$-vertex cover sets on triangulations. In sections 3 and 4 we provide upper bounds for $g_{2d}(n)$, $\gamma_{2d}(n)$ and $\beta_{2d}(n)$ on maximal outerplanar graphs and show that these bounds are tight.

2 Relationship between distance vertex cover, guarding and domination on triangulations

We start by showing that the three concepts are different. Fig. 2 depicts $2d$-dominating and $2d$-guarding sets for a given triangulation. Note that in Fig. 2(a) the set $\{u, v\}$ is $2d$-dominating since the remaining vertices are at distance 1 or 2. However, it is not a $2d$-guarding set because the shaded triangle is not guarded, as its vertices are at distance 2 from $\{u, v\}$. In 2(b) $\{w, z\}$ is a $2d$-guarding set, however it is not a $2d$-vertex cover since any path between the bold edge and w or z has length at least 3. Therefore, the bold edge is not covered.



(a) (b)

Figure 2: (a) $2d$-dominating set for a triangulation T; (b) $2d$-guarding set for T.

Now we are going to establish a relation between $g_{2d}(T)$, $\gamma_{2d}(T)$ and $\beta_{2d}(T)$. The following results can be easily generalized to $g_{kd}(T)$, $\gamma_{kd}(T)$ and $\beta_{kd}(T)$.

Lemma 1 *If C is a $2d$-vertex cover of a triangulation T, then C is a $2d$-guarding set and a $2d$-dominating set of T.*

Lemma 2 *If F is a $2d$-guarding set of a triangulation T, then F is a $2d$-dominating set of T.*

The previous lemmas prove the following result.

Theorem 3 *Given a triangulation T the minimum cardinality $g_{2d}(T)$ of any $2d$-guarding set verifies $\gamma_{2d}(T) \leq g_{2d}(T) \leq \beta_{2d}(T)$.*

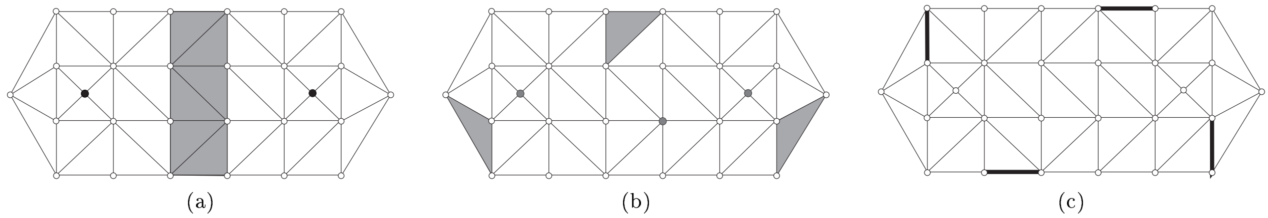


Figure 3: (a) A $2d$-dominating set for a triangulation T (black vertices); (b) a $2d$-guarding set for T (gray vertices); (c) each of the bold edges needs a different vertex to be $2d$-covered.

Note that the inequalities above can be strict, as we can see in the triangulation T presented in Fig. 3, where $\gamma_{2d}(T) = 2$, $g_{2d}(T) = 3$, and $\beta_{2d}(T) \geq 4$.

3 $2d$-domination and $2d$-guarding of maximal outerplanar graphs

In this section we establish upper bounds for $g_{2d}(n)$ and $\gamma_{2d}(n)$ on maximal outerplanar graphs. In order to do this, and following the ideas of O'Rourke [12], we first need to introduce some lemmas.

Lemma 4 *Suppose that $f(m)$ guards are always sufficient to $2d$-guard any outerplanar maximal graph G with m vertices. Then if G has two guards placed at any two adjacent vertices or one guard placed at any one of its vertices, then $f(m-2)$ or $f(m-1)$ additional guards are sufficient to $2d$-guard G, respectively.*

Lemma 5 *Let G be an outerplanar maximal graph with $n \geq 2k$ vertices. There is an interior edge e of G that partitions G into two pieces, one of which contains $m = k, k+1, \ldots, 2k-1$ or $2k-2$ exterior edges of G.*

Theorem 6 *Every n-vertex maximal outerplanar graph, with $n \geq 5$, can be $2d$-guarded by $\lfloor \frac{n}{5} \rfloor$ $2d$-guards. That is, $g_{2d}(n) \leq \lfloor \frac{n}{5} \rfloor$ for all $n \geq 5$.*

Proof. For $5 \leq n \leq 11$, the truth of the theorem can be easily established. It should be noted that for $n = 5$ the $2d$-guard can be placed randomly and for $n = 6$ it can be placed at any vertex of degree greater than 2. Assume that $n \geq 12$ and that the theorem holds for all $n' < n$. Lemma 5 guarantees the existence of an interior edge e that divides G into two maximal outerplanar graphs G_1 and G_2, such that G_1 has m exterior edges of G with $6 \leq m \leq 10$. The vertices of G are labeled with $0, 1, \ldots, n-1$ such that e is $(0, m)$. Each value of m is considered separately. We present the cases $m = 6$ and $m = 9$.

Case $m = 6$. G_1 has $m + 1 = 7$ exterior edges, thus G_1 can be $2d$-guarded with one guard. G_2 has $n - 5$ exterior edges including e, and by induction hypothesis, it can be $2d$-guarded with $\lfloor \frac{n-5}{5} \rfloor = \lfloor \frac{n}{5} \rfloor - 1$ guards. Thus G_1 and G_2 together can be $2d$-guarded by $\lfloor \frac{n}{5} \rfloor$ guards.

Case $m = 9$. The presence of any of the internal edges (0,8), (0,7), (0,6), (9,1), (9,2) and (9,3) would violate the minimality of m. Thus, the triangle T in G_1 that is bounded by e is either (0,5,9) or (0,9,4). Since these are equivalent cases, suppose that T is (0,5,9) (see Fig. 4(a)).

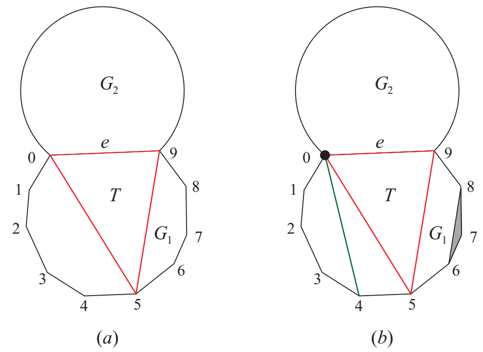


Figure 4: (a) The triangle T is (0,5,9); (b) the internal edge (0,4) and the triangle (6,7,8) are present.

The pentagon (5,6,7,8,9) can be $2d$-guarded by placing one guard at a chosen vertex. However, to $2d$-guard the hexagon (0,1,2,3,4,5) we cannot place a guard randomly.We will consider two separate cases: (i) The internal edge (0,4) is not present: if a guard is placed at vertex 5, then the hexagon (0,1,2,3,4,5) is $2d$-guarded, thus G_1 is $2d$-guarded. Since G_2 has $n-8$ edges it can be $2d$-guarded by $\lfloor \frac{n-8}{5} \rfloor \leq \lfloor \frac{n}{5} \rfloor - 1$ guards applying the induction hypothesis. This yields a $2d$-guarding of G by $\lfloor \frac{n}{5} \rfloor$ guards; (ii) The internal edge (0,4) is present: if a guard is placed at vertex 0, then G_1 is $2d$-guarded unless the triangle (6,7,8) is present in the triangulation (see Fig. 4(b)). In any case, two $2d$-guards placed at vertices 0 and 9 guard G_1. G_2 has $n-8$ exterior edges, including e. By lemma 4 the two guards placed at vertices 0 and 9 allow the remainder of G_2 to be guarded by $f(n-8-2) = f(n-10)$ additional $2d$-guards. Recall that $f(n')$ is the number of $2d$-guards that are always sufficient to guard a maximal outerplanar graph with

n' vertices. By the induction hypothesis $f(n') = \lfloor \frac{n'}{5} \rfloor$. Thus, $\lfloor \frac{n-10}{5} \rfloor = \lfloor \frac{n}{5} \rfloor - 2$ guards suffices to $2d$-guard G_2. Together with the guards placed at vertices 0 and 9 that $2d$-guard G_1, all G is guarded by $\lfloor \frac{n}{5} \rfloor$ $2d$-guards.

□

To prove that this upper bound is tight we need to construct a maximal outerplanar graph G of order n such that $g_{2d}(G) = \lfloor \frac{n}{5} \rfloor$. Fig. 5 shows a maximal outerplanar graph G for which $\gamma_{2d}(G) = \frac{n}{5}$, since the black vertices can only be $2d$-dominated by different vertices. Thus, $\gamma_{2d}(n) \geq \frac{n}{5}$. Note that this example can be generalized to kd-domination to obtain $\gamma_{kd}(n) \geq \frac{n}{(2k+1)}$.

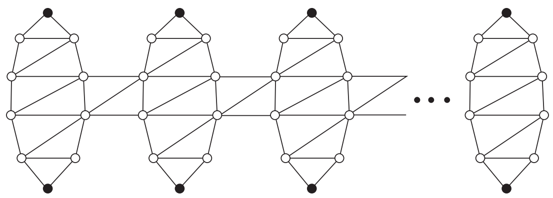


Figure 5: A maximal outerplanar graph G for which $\gamma_{2d}(G) = \frac{n}{5}$.

According to theorem 3, $\gamma_{2d}(G) \leq g_{2d}(G)$, so $\lfloor \frac{n}{5} \rfloor \leq g_{2d}(G)$. In conclusion, $\lfloor \frac{n}{5} \rfloor$ $2d$-guards are occasionally necessary and always sufficient to guard a n-vertex maximal outerplanar graph G. On the other hand, we can also establish that $\gamma_{2d} = \lfloor \frac{n}{5} \rfloor$, since $\lfloor \frac{n}{5} \rfloor \leq \gamma_{2d}(n)$ and $\gamma_{2d}(n) \leq g_{2d}(n)$, for all n. Thus, it follows:

Theorem 7 *Every n-vertex maximal outerplanar graph with $n \geq 5$ can be $2d$-guarded (and $2d$-dominated) by $\lfloor \frac{n}{5} \rfloor$ guards. This bound is tight.*

4 $2d$-covering of maximal outerplanar graphs

In this section we determine an upper bound for $2d$-vertex cover on maximal outerplanar graphs which is also tight.

Lemma 8 *Suppose that $f(m)$ vertices are always sufficient to $2d$-cover any outerplanar maximal graph G with m vertices. If we randomly choose a vertex of G to be a $2d$-covering vertex, then $f(m-1)$ additional vertices are sufficient $2d$-cover G.*

Theorem 9 *Every n-vertex maximal outerplanar graph, with $n \geq 4$, can be $2d$-covered with $\lfloor \frac{n}{4} \rfloor$ vertices. That is, $\beta_{2d}(n) \leq \lfloor \frac{n}{4} \rfloor$ for all $n \geq 4$.*

Now, we will prove that this upper bound is tight. The bold edges of the maximal outerplanar graph illustrated in Fig. 6 can only be $2d$-covered from different vertices, and therefore $\beta_{2d}(n) \geq \frac{n}{4}$. To conclude:

Theorem 10 *Every n-vertex maximal outerplanar graph with $n \geq 5$ can be $2d$-covered by $\lfloor \frac{n}{4} \rfloor$ vertices. This bound is tight.*

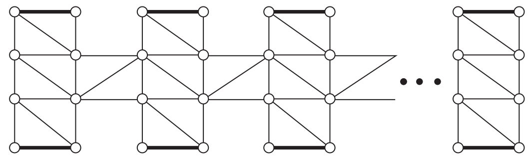


Figure 6: A maximal outerplanar graph G for which $\beta_{2d}(G) = \frac{n}{4}$.

References

[1] P. Belleville, P. Bose, J. Czyzowicz, J. Urrutia, J. Zaks, K-vertex guarding simple polygons, *Comput. Geom. Theory Appl.* , **42(4)** (2009), 352–361.

[2] J.R.S. Blair and S.B. Horton, Broadcast covers in graphs, *Congr. Num.* **173** (2005), 109–115.

[3] P. Bose, T. Shermer, G. Toussaint, and B. Zhu, Guarding Polyhedral Terrains, *Computational Geometry: Theory and Applications*, **6(3)**(1997), 173–185.

[4] C.N. Campos and Y. Wakabayashi, On dominating sets of maximal outerplanar graphs, *Discrete Appl.Math*, **161(3)** (2013), 330–335.

[5] Q. Chen, L. Zhao, Approximation algorithms for the L-distance vertex cover problem, *Proc. ICTMF* (2012), Bali, Indonesia, 100–104.

[6] D. Erwin, Dominating broadcasts in graphs, *Bull. Inst. Combin. Appl.* **42** (2004), 89–105.

[7] S. Fisk. A short proof of Chvatal's watchman theorem, *J. Combin. Theory Ser. B*, **(24)** (1978), 374.

[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, M. Dekker, 1998.

[9] E. King, J. Pelsmajer, Dominating sets in plane triangulations, *Discret Math.* **310(17-18)** (2010), 2221–2230.

[10] L.R. Matheson, R.E. Tarjan, Dominating sets in planar graphs, *Eur. J. Combin.* **17(6)** (1996), 565–568.

[11] A. Meir, J.W. Moon, Relations between packing and covering number of a tree, *Pacific J. Math* 61(1) (1975), 225–233.

[12] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, Inc., New York, USA, 1987.

[13] F. Tian, J. Xu, Bounds for distance domination numbers of graphs, *Journal of University of Science and Technology of China* **34(5)** (2004), 529–534.

[14] F. Tian, J. Xu, A note on distance domination numbers of graphs, *Aust. J. of Combin* **43** (2009), 181–190.