An algorithm that constructs irreducible triangulations of once-punctured surfaces

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Abstract

A triangulation of a surface is irreducible if there is no edge whose contraction produces another triangulation of the surface. In this work we propose an algorithm that constructs the set of irreducible triangulations of any surface with precisely one boundary component.

Introduction and terminology

We restrict our attention to the following objects:

- $S = S_g$ or N_k is the closed orientable surface S_g of genus g or closed nonorientable surface of nonorientable genus k. In particular, S_0 and S_1 are the sphere and torus, N_1 and N_2 are the projective plane and the Klein bottle (respectively).
- $S - D$ is S minus an open disk D (the hole). This compact surface is called the *once-punctured* surface. We assume that the boundary ∂S of S, which is equal to ∂D, is homeomorphic to a circle.

We use the notation Σ whenever we assume the general case - that is, $\Sigma \in \{S, S - D\}$.

If a (finite, undirected, simple) graph G is 2-cell embedded in Σ, the components of $\Sigma - G$ are called *faces*. A triangulation of Σ with graph G is a 2-cell embedding $T : G \to \Sigma$ in which each face is bounded by a 3-cycle (that is, a cycle of length 3) of G and any two faces are either disjoint, share a single vertex, or share a single edge. We denote by $V = V(T)$, $E = E(T)$, and $F = F(T)$ the sets of the vertices, the edges, and the faces of T, respectively. Equivalently, the triangulation T of a surface can be defined as a hypergraph of rank 3 or a 3-graph, with the vertex set $V(T)$ and the collection $F(T)$ of triplets of $V(T)$ (called 3-edges, or triangles, of T) (see [4]).

By $G(T)$ we denote the graph $(V(T), E(T))$ of triangulation T. Two triangulations T_1 and T_2 are called *isomorphic*, denoted $T_1 \cong T_2$, if there is a bijection $\alpha : V(T_1) \to V(T_2)$ such that $uvw \in F(T_1)$ if and only if $\alpha(u)\alpha(v)\alpha(w) \in F(T_2)$. Throughout this paper we distinguish triangulations only up to isomorphism.

In the case $\Sigma = S - D$ let ∂T, which is equal to ∂D, denote the boundary cycle of T. The vertices and edges of ∂T are called *boundary vertices* and *boundary edges* of T, respectively.

A triangulation of a punctured surface is *irreducible* (term which is more accurately defined in Section 1) if no edge can be shrunk without producing multiple edges or changing the topological type of the surface. The irreducible triangulations of Σ form a basis for the family of all triangulations of Σ, in the sense that any triangulation of Σ can be obtained from a member of the basis by applying the *splitting* operation (introduced in Section 1) a finite number of times. To have such a basis in hand can be very useful in practical application of triangulation generating; the papers [6] and [19] are worth mentioning.

It is known that for every surface Σ the basis of irreducible triangulations is finite (the case of closed surfaces is proved in [2], [13], [12], and [7] and the case of surfaces with boundary is proved in [3]). At present such bases are known only for seven closed surfaces and two once-punctured surfaces: the sphere ([14]), projective plane ([1]), torus ([8]), Klein bottle ([9] and [15]), S_2, N_3, and N_4 ([16, 17]), the disk and Möbius band ([5]).

In this paper we present an algorithm which is designed as an application of some recent advances on the study of irreducible triangulations of once-punctured surface collected in [5]. Specifically, Lemmas 1-3 (in Section 1) are the main supporting theoretical results for the mentioned algorithm. As a particular example, all the non-isomorphic combinatorial types (293 in number) of triangulations of the once-punctured torus are determined.

1 Previous results

Let T be a triangulation of Σ. An unordered pair of distinct adjacent edges vu and vw of T is called

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a *corner* of T at vertex v, denoted by $\langle u, v, w\rangle$ $(=\langle w, v, u\rangle)$. The *splitting* of a corner $\langle u, v, w\rangle$, denoted by $\mathrm{sp}\langle u, v, w\rangle$, is the operation which consists of cutting T open along the edges vu and vw and then closing the resulting hole with two new triangular faces, $v'v''u$ and $v'v''w$, where v' and v'' denote the two images of vertex v appearing as a result of cutting. Under this operation, vertex v is extended to the edge $v'v''$ and the two faces having this edge in common are inserted into the triangulation; therefore the order increases by one and the number of edges increases by three.

If a corner $\langle u, v, w\rangle$ is composed of two edges vu and vw neighboring around vertex v, $\mathrm{sp}\langle u, v, w\rangle$ is equivalent to the stellar subdivision of the face uvw.

Especially in the case $\{\Sigma = S - D, uv \in E(T)$, and $v \in V(T)\}$, the operation $\mathrm{sp}\langle u, v]$ of *splitting a truncated corner* $\langle u, v]$ produces a single triangular face $uv'v''$, where $v'v'' \in E(\partial(\mathrm{sp}\langle u, v](T)))$.

Under the inverse operation, *shrinking* the edge $v'v''$, denoted by $\mathrm{sh}\rangle v'v''\langle$, this edge collapses to a single vertex v, the faces $v'v''u$ and $v'v''w$ collapse to the edges vu and vw, respectively. Therefore $\mathrm{sp}\langle u, v, w\rangle \circ sh\rangle v'v''\langle(T) = T$. It should be noticed that in the case $\{\Sigma = S - D,\ v'v'' \in E(\partial T)\}$, there is only one face incident with $v'v''$, and only that single face collapses to an edge under $\mathrm{sh}\rangle v'v''\langle$. Clearly, the operation of splitting doesn't change the topological type of Σ if $\Sigma \in \{S, S - D\}$. We demand that the shrinking operation must preserve the topological type of Σ as well; moreover, multiple edges must not be created in a triangulation. A 3-cycle of T is called *nonfacial* if it doesn't bound a face of T. In the case in which an edge $e \in E(T)$ occurs in some nonfacial 3-cycle, if we still insist on shrinking e, multiple edges would be produced, which would expel $\mathrm{sh}\rangle e\langle(T)$ from the class of triangulations. An edge e is called *shrinkable* or a *cable* if $\mathrm{sh}\rangle e\langle(T)$ is still a triangulation of Σ; otherwise the edge is called *unshrinkable* or a *rod*. The subgraph of $G(T)$ made up of all cables is called the *cable-subgraph of* $G(T)$.

The impediments to edge shrinkability in a triangulation T of a punctured surface $S-D$ are identified in [2, 3, 1, 8]; an edge $e \in E(T)$ is a rod if and only if e satisfies one of the following conditions:

(1) e is in a nonfacial 3-cycle of $G(T)$. In particular, e is a boundary edge in the case in which the boundary cycle is a 3-cycle.

(2) e is a chord of D -that is, the end vertices of e are in $V(\partial D)$ but $e \notin E(\partial D)$.

From now on, we assume that $S \neq S_0$ and make an agreement that by "non-facial 3-cycle" we mean a non-null-homotopic 3-cycle whenever we refer to conditions (1) and (2). Therefore, an edge e is a rod provided e occurs in some non-null-homotopic 3-cycle, and e is a cable otherwise. Especially, the boundary edges of stellar subdivided faces are now regarded as cables unless they occur in some non-null-homotopic 3-cycles. The convenience of this agreement is that once a rod turns into a cable in the course of any splitting sequence, it always remains a cable under forthcoming splittings.

A triangulation is said to be *irreducible* if it is free of cables or equivalently, each edge is a rod. For instance, a single triangle is the only irreducible triangulation of the disk $S_0 - D$.

Let T be an irreducible triangulation of a punctured surface $S - D$ where $S \neq S_0$. Let us close the hole in T by restoring the disk D add a vertex, p, in D, and join p to the vertices in ∂D. We thus obtain a triangulation, T^*, of the closed surface S. In this setting we call D the *patch*, call p the *central vertex of the patch*, and say that T is obtained from the corresponding triangulation T^* of S by the *patch removal.*

Notice that T^* may be an irreducible triangulation of S but not necessarily. Using the assumption that T is irreducible and the fact that each cable of T^* fails to satisfy condition (1) (in the strong non-null-homotopic sense), it can be easily seen that in the case T^* is not irreducible, all cables of T^* have to be entirely in $D \cup \partial D$ and, moreover, there is no cable which is entirely in ∂D whenever the length of the boundary cycle ∂D is greater than or equal to 4. In particular, we observe that each chord of D (if any) is a rod in T because it meets condition (2), and is also a rod in T because it meets condition (1). We now come to a lemma which is to be stated shortly after some necessary definitions.

A vertex of a triangulation R of S is called a *pylonic vertex* if that vertex is incident with all cables of R. A triangulation which has at least one cable and at least one pylonic vertex is called a *pylonic triangulation.*

A triangulation may have a unique cable and therefore two pylonic vertices. However, if the number of cables in a pylonic triangulation R is at least 2, R has exactly one pylonic vertex.

Lemma 1 *Suppose an irreducible triangulation T of a punctured surface $S - D$ $(S \neq S_0)$ is obtained from the corresponding triangulation T^* of S by the patch removal. If T^* has at least two cables, then either the central vertex p of the patch is the only pylonic vertex of T^*, or else the length of ∂D is equal to 3.*

Let $\Xi_n = \Xi_n(S)$ denote the set of triangulations of a fixed closed surface S that can be obtained from an irreducible triangulation of S by a sequence of exactly n repeated splittings.

In the following result, by the "removal of a vertex v" we mean the removal of v together with the interiors of the edges and faces incident with v and by the "removal of a face" we mean the removal of the interior of that face.

Lemma 2 *Each irreducible triangulation T of $S - D$ ($S \neq S_0$) can be obtained either*

(i) by removing a vertex from a triangulation in $\Xi_0 = \Xi_0(S)$, or

(ii) by removing a pylonic vertex from a triangulation in $\Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_K$, where the constant K is provided by [3] (whenever a pylonic triangulation occurs), or

(iii) by removing either of the two faces containing a cable in their boundary 3-cycles provided that cable is unique in a triangulation in Ξ_1 (whenever such a situation occurs), or

(iv) by removing the face containing two, or three, cables in its boundary 3-cycle provided those two, or three, cables collectively form the whole cable-subgraph in a triangulation in $\Xi_1 \cup \Xi_2$ (if such a situation occurs).

Lemma 3 *If a triangulation of S has at least two cables but has no pylonic vertex, then no pylonic vertex can be created under further splitting of the triangulation.*

2 Sketch of the algorithm

In this section triangulations are considered to be hypergraphs of rank 3 or 3-graphs. Let T be a 3-graph with $V = V(T)$ and $F = F(T)$ the sets of the vertices and the triangles of T. This 3-graph can be uniquely represented as a bipartite graph $B_T = (V(B_T), E(B_T))$ in the following way: $V(B_T) = V(T) \cup F(T)$, $uv \in E(B_T)$ if and only if the vertex u lies in the triangle v in T.

The algorithm *input* is the set $\mathcal{I}$ of irreducible triangulations of a closed surface $S \neq S_0$. The *output* of the algorithm is the set of all non-isomorphic combinatorial types of irreducible triangulations of the once-punctured surface $S - D$.

The first step is the generation of the set $\Xi_1 \cup \Xi_2$ from the set $\mathcal{I}$. Next, every 3-graph $T \in \Xi_1 \cup \Xi_2$ is then represented by their corresponding bipartite graph B_T. This has been implemented with the computational package *Mathematica* ([18]).

The second step consists of discarding all duplicate bipartite graphs and then, all duplicate triangulations in $\Xi_1 \cup \Xi_2$ will be discarded. That is, the obtention of all non-isomorphic triangulations, denoted $\widetilde{\Xi_1} \cup \widetilde{\Xi_2}$ respectively, which is implemented with the computing packages *Nauty* (and *gtools*, [10], [11]).

Next, all pylonic vertices in $\widetilde{\Xi_1} \cup \widetilde{\Xi_2}$ are detected and operations *(i)–(iv)* described in Lemma 2 are applied to obtain irreducible triangulations (using *Mathematica*).

If Ξ_2 has no pylonic triangulation, immediately proceed to the final step: Discard all duplicate triangulations by using *Nauty* and *gtools*. Otherwise generate Ξ_3 and apply the preceding steps to Ξ_3. Repeat this procedure to process $\Xi_4, \Xi_5, \ldots$ until no pylonic triangulation is left in the current Ξ_n; then the process terminates and a required output is produced.

The validity of this procedure is justified by Lemmas 1 - 3 along with the results of [3]. In particular, the finiteness of the procedure is deduced from the upper bound [3] on the number of vertices in an irreducible triangulation of $S - D$. In particular, that upper bound implies (along with Lemma 3) that Ξ_n does not have a pylonic triangulation for any $n \geq K + 1$, where $K = 945$ for S_1 and $K = 376$ for N_1. In reality K is much less than these values. By computer verification (and also by hand) we have checked that in fact $K = 1$ for S_1, and $K = 2$ for N_1.

Let us now mention two examples.

Firstly, this algorithm has been implemented for the set of two irreducible triangulations of N_1 ([1]). The algorithm gives a set of 6 irreducible triangulations of the Möbius band, $N_1 - D$, which is precisely the same as that obtained by some of the authors of this work in [5], although by using no computational package.

Secondly, we introduce the details of the torus case, S_1.

Example: the once-punctured torus

Input: The set of 21 irreducible triangulations of S_1 (as they are labelled in [8]).

- $\Xi_1(S_1)$ has 433 non-isomorphic triangulations: 232 of them have no pylonic vertex, 193 have an only pylonic vertex and 8 have two pylonic vertices.
- $\Xi_2(S_1)$ has 11612 non-isomorphic triangulations, none of them is a pylonic triangulation.
- Operations described in Lemma 2 provide:
  - (i) 184 triangulations; only 80 of them are non-isomorphic.
  - (ii) 209 triangulations; only 203 of them are non-isomorphic.
  - (iii) 16 triangulations, 10 of them are non-isomorphic.
  - (iv) 0 triangulations.

Output: 293 non-isomorphic combinatorial types of irreducible triangulations of the once-punctured torus $S_1 - D$.

3 Final conclusion

It is clear that this algorithm can be implemented for any closed surface whenever its basis of irreducible triangulations is known. In a future contribution we hope to present the set of irreducible triangulations of the once-punctured Klein bottle, $N_2 - D$.

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