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# A faithful functor among algebras and graphs

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**Abstract**

The problem of identifying a functor between the categories of algebras and graphs is currently open. Based on a known algorithm that identifies isomorphisms of Latin squares with isomorphism of vertex-colored graphs, we describe here a pair of graphs that enable us to find a faithful functor between finite-dimensional algebras over finite fields and these graphs.

*Key words: graph theory, algebra, finite field, isomorphism, isotopism, invariant.*

## 1 Introduction

Graph Theory has revealed to be an interesting tool to deal with distinct aspects on the study of algebras [3, 4, 7, 9]. Nevertheless, the problem of identifying a functor that relates the category of algebras with that of graphs remains still open. Both categories are referred with respect to their corresponding isomorphisms among algebras and graphs. Based on a proposal of McKay et al. [10] for identifying isomorphisms of Latin squares with isomorphism of vertex-colored graphs, we describe here a pair of graphs that enable us to find a faithful functor between finite-dimensional algebras over finite fields and these graphs. We focus in particular on the distribution of partial-magma algebras into isomorphism classes by means of some isomorphism invariants related to the mentioned graphs.

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# 2 Preliminaries

## 2.1 Isotopisms of algebras

In 1942, Albert [1] introduced the concept of isotopism of algebras as a generalization of that of isomorphism. Specifically, two $n$-dimensional algebras $(A, \cdot)$ and $(A', \circ)$ defined over the same field $\mathbb{K}$ are said to be *isotopic* if there exist three non-singular linear transformations $f$, $g$ and $h$ from $A$ to $A'$ such that

$$f(u) \circ g(v) = h(u \cdot v), \text{ for all } u, v \in A. \tag{1}$$

Hereafter, in order to simplify the notation and whenever no confusion arises, we do not write explicitly the products $\cdot$ and $\circ$. That is, we write the previous identity as $f(u)g(v) = h(uv)$, for all $u, v \in A$. The triple $(f, g, h)$ is an *isotopism* between the algebras $A$ and $A'$.

Let $A$ be an $n$-dimensional algebra over a field $\mathbb{K}$ and let $\{e_1, \ldots, e_n\}$ be a basis of this algebra. The *structure constants* of $A$ are the numbers $c_{ij}^k \in \mathbb{K}$ such that

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k, \text{ for } 1 \leq i, j \leq n. \tag{2}$$

If the structure constants of an algebra are all of them zeros, then this algebra is called *abelian* .

**Lemma 1.** *The $n$-dimensional abelian algebra is not isotopic to any other $n$-dimensional algebra.* □

Let $S$ be a vector subspace of an algebra $A$. The *left* and *right annihilators* of $S$ in $A$ are respectively defined as the sets

$$\mathrm{Ann}_{A^-}(S) = \{u \in A \mid uv = 0, \text{ for all } v \in S\}. \tag{3}$$

$$\mathrm{Ann}_{A^+}(S) = \{u \in A \mid vu = 0, \text{ for all } v \in S\}. \tag{4}$$

The intersection of both sets is called the *annihilator* of $S$ in $A$. It is defined as

$$\mathrm{Ann}_A(S) = \{u \in A \mid uv = vu = 0, \text{ for all } v \in S\}. \tag{5}$$

**Lemma 2.** *Let $(f, g, h)$ be an isotopism between two $n$-dimensional algebras $A$ and $A'$. Let $S$ be a vector subspace of $A$. Then,*

*a)* $f(\mathrm{Ann}_{A^-}(S)) = \mathrm{Ann}_{A'^-}(g(S))$.

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*b)* $g(\mathrm{Ann}_{A^+}(S)) = \mathrm{Ann}_{A'^+}(f(S))$.

*c)* $f(\mathrm{Ann}_{A^-}(S)) \cap g(\mathrm{Ann}_{A^+}(S)) = \mathrm{Ann}_{A'}(f(S) \cap g(S))$. □

**Proposition 1.** *Let $(f, g, h)$ be an isotopism between two $n$-dimensional algebras $A$ and $A'$. Then,*

*a)* $f(\mathrm{Ann}_{A^-}(A)) = \mathrm{Ann}_{A'^-}(A')$.

*b)* $g(\mathrm{Ann}_{A^+}(A)) = \mathrm{Ann}_{A'^+}(A')$.

*c)* $f(\mathrm{Ann}_{A^-}(A)) \cap g(\mathrm{Ann}_{A^+}(A)) = \mathrm{Ann}_{A'}(A')$.

*Proof.* The result follows straightforward from Lemma 2 and the regularity of $f$ and $g$. □

Hereafter, given a vector subspace $S$ of an algebra $A$, we define the vector subspace $SA = \{uv \mid u \in S \text{ and } v \in A\}$. The *derived algebra* of the algebra $A$ is then defined as the subalgebra

$$A^2 = AA = \{uv \mid u, v \in A\} \subseteq A. \tag{6}$$

**Lemma 3.** *Let $(f, g, h)$ be an isotopism between both algebras $A$ and $A'$. Then, $h(A^2) = A'^2$ and* $\dim(A^2) = \dim(A'^2)$. □

## 2.2 Partial-magma algebras

A *partial magma* is a finite set endowed with a partial binary operation. Hereafter, we suppose this set to be $[n] = \{1, \ldots, n\}$ and we denote the operation as $\cdot$. In this case, $n$ is the *order* of the partial magma. Two partial magmas $([n], \cdot)$ and $([n], \circ)$ are said to be *isotopic* if there exist three permutations $\alpha$, $\beta$ and $\gamma$ in the symmetric group $S_n$ such that

$$\alpha(i) \circ \beta(j) = \gamma(i \cdot j), \text{ for all } i, j \leq n \text{ such that } i \cdot j \text{ exists.} \tag{7}$$

If $\alpha = \beta = \gamma$, then the partial magmas are said to be *isomorphic*. The triple $(\alpha, \beta, \gamma)$ constitutes an *isotopism* of magmas (an *isomorphism* if $\alpha = \beta = \gamma$).

A *partial quasigroup* is a partial magma $([n], \cdot)$ such that if the equations $ix = j$ and $yi = j$, with $i, j \in [n]$, have solutions for $x$ and $y$ in $[n]$, then these solutions are unique. Every partial quasigroup of order $n$ is the multiplication table of a *partial Latin square* of order $n$, that is, an $n \times n$ array in which each cell is either empty or contains one element chosen from the set $[n]$, such that each symbol occurs at most once in each row and in each column. Every isotopism of a partial quasigroup is uniquely related to a permutation of the rows, columns and symbols of the corresponding partial Latin square. The distribution

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of partial Latin squares into isotopism classes is known for order up to 6 [5, 6]. Finally, if two Latin squares are isotopic after a reordering of the components of all their entries, then they are said to be *paratopic*.

In 1944, Bruck [2] introduced the concept of *quasigroup algebra* as an $n$-dimensional algebra over a base field $\mathbb{K}$ such that there exists a basis $\{e_1, \ldots, e_n\}$ and a quasigroup $([n], \cdot)$ satisfying that $e_i e_j = c_{ij} e_{i \cdot j}$ for each pair of elements $i, j \leq n$ and some non-zero structure constant $c_{ij} \in \mathbb{K} \setminus \{0\}$. The algebra is then said to be *based on* the quasigroup $([n], \cdot)$. If all its structure constants are equal to 1, then this is called a *quasigroup ring*. *Partial-magma algebras* constitute a natural generalization of the concept of quasigroup algebra, once the condition of being based on a quasigroup is replaced by that of being based on a partial magma.

## 2.3 Graph theory

A *graph* is a pair $G = (V, E)$ formed by a set $V$ of points or *vertices* and a set $E$ of lines or *edges* formed by subsets of two vertices of $V$. The *degree* of a vertex $v \in V$ is the number $d(v)$ of edges containing this vertex. A graph is said to be *vertex-colored* if there exists a partition into color sets of its set of vertices. The color of a vertex $v$ is denoted as $\mathrm{color}(v)$. An *isomorphism* between two vertex-colored graphs $G = (V, E)$ and $G' = (V', E')$ is any bijective map $f$ between the set of vertices $V$ and $V'$ that preserves collinearity and such that $\mathrm{color}(f(v)) = \mathrm{color}(v)$, for all $v \in V$.

Let $L = (l_{ij})$ be a Latin square of order $n$. McKay et al. [10] defined the vertex-colored graph $G_2(L)$ with $n^2 + 3n$ vertices

$$\{r_i \mid i \leq n\} \cup \{c_i \mid i \leq n\} \cup \{s_i \mid i \leq n\} \cup \{t_{ij} \mid i, j \leq n\},$$

where each of the four subsets (related to the rows $(r_i)$, columns $(c_i)$, symbols $(s_i)$ and cells $(t_{ij})$ of the Latin square $L$) has a different color, and $3n^2$ edges

$$\{r_i e_{ij}, c_j e_{ij}, s_{ij} t_{ij} \mid i, j \leq n\}\}.$$

They also defined the vertex-colored graph $G_1(L)$ from the graph $G_2(L)$ by adding 3 additional vertices $\{R, C, S\}$ and $3n$ additional edges $\{Rr_i, Cc_i, Ss_i \mid i \leq n\}$. Here, there are three colors: one for $\{R, C, S\}$, one for $\{r_i, c_i, s_i \mid i \leq n\}$ and one for the rest of vertices. Finally, they defined the vertex-colored graph $G_3(L)$ from the graph $G_2(L)$ by adding $3n$ additional edges $\{r_i c_i, c_i s_i, r_i s_i \mid i \leq n\}$. Here, the color of the vertices coincides with those of $G_1(L)$. These authors proved (Theorem 6 in [10]) that two Latin squares $L_1$ and $L_2$ of the same order are paratopic (respectively, isotopic or isomorphic) if and only if the graphs $G_1(L_1)$ and $G_1(L_2)$ (respectively, $G_2(L_1)$ and $G_2(L_2)$, and $G_3(L_1)$ and $G_3(L_2)$) are

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isomorphic. Figure 1 shows an example of the three graphs related to the next Latin square of order 2.

$$L = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have used distinct styles (∘, ▲, ▶, ◀ and •) in the vertices of the graphs to represent their colors.

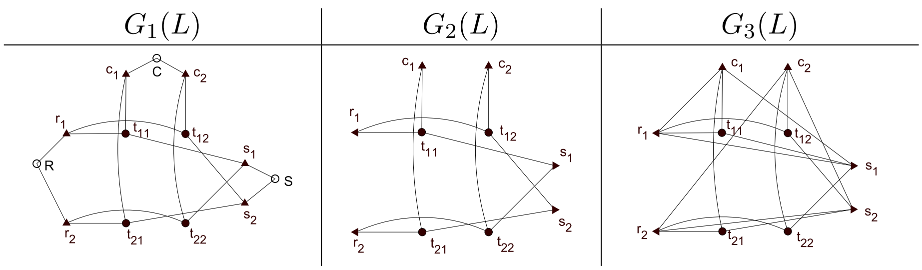


Figure 1: Graphs related to a Latin square of order 2.

## 3 The proposed graph

Based on the proposal of McKay et al. for Latin squares, we describe now a pair of graphs that are uniquely related to a finite-dimensional algebra over a finite field and which enable us to ensure that any two isotopic or isomorphic algebras map to two isomorphic graphs. To this end, let $A$ be an $n$-dimensional algebra over a finite field $\mathbb{K}$. Firstly, we define the vertex-colored graph $G_1(A)$ with four maximal monochromatic subsets

$$\begin{cases} R_A = \{r_u \mid u \in A \setminus \mathrm{Ann}_{A^-}(A)\}, \\ C_A = \{c_u \mid u \in A \setminus \mathrm{Ann}_{A^+}(A)\}, \\ S_A = \{s_u \mid u \in A^2 \setminus \{0\}\}, \\ T_A = \{t_{u,v} \mid u, v \in A, uv \neq 0\}. \end{cases}$$

and edges

$$\{r_u t_{u,v}, c_v t_{u,v}, s_w t_{u,v} \mid u, v, w \in A, uv = w \neq 0\}.$$

From this graph we also define the vertex-colored graph $G_2(A)$ by adding the edges

$$\{r_u c_u, \mid u \in A \setminus \mathrm{Ann}_A(A)\} \cup \{c_u s_u \mid u \in A^2 \setminus \mathrm{Ann}_{A^+}(A)\} \cup \{r_u s_u \mid u \in A^2 \setminus \mathrm{Ann}_{A^-}(A)\}.$$

Figure 2 shows, for instance, the two graphs $G_1$ and $G_2$ that are related to any $n$-dimensional anticommutative algebra over the finite field $\mathbb{F}_2$, with basis $\{e_1, \ldots, e_n\}$, that is described by the non-zero product $e_1 e_2 = e_1$.

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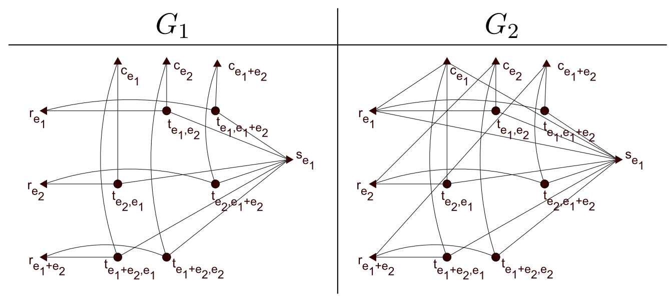


Figure 2: Graphs related to the anticommutative algebra $e_1e_2 = e_1$ over $\mathbb{F}_2$.

**Example 1.** *In order to illustrate in a better way the proposed graphs, we describe how to construct step by step the graph $G_2$ related to the 3-dimensional anti-commutative algebra $A$ over the finite field $\mathbb{F}_2$, with basis $\{e_1, e_2, e_3\}$, that is linearly defined from the non-zero products $e_1e_3 = e_2$ and $e_2e_3 = e_1$. In order to make easier this construction, we place the vertices of the graph in rows and columns as if they were the elements of a matrix. Each one of these vertices can, therefore, be described by its position $(i, j)$ inside this matrix.*

*Step 1. The underlying set of vectors of our algebra is $\{e_1, e_2, e_3, e_1+e_2, e_1+e_3, e_2+e_3, e_1+e_2+e_3\}$. Since $\mathrm{Ann}_A(A) = \emptyset$, these seven vectors can appear as left or right factors of a non-zero product in $A$. We start, therefore, the construction of the graph $G_2(A)$ by drawing seven vertices labeled as $r_u$ in a column at the left of the graph and other seven vertices labeled as $c_u$ in a row on the top of the graph. Each $u$ denotes here a vector of the algebra (see Figure 3 (left)).*

*Step 2. In the body of the graph (the empty zone among the two sets of vertices that have been drawn until now) we draw now those vertices corresponding to non-null brackets. Since we have at most seven times seven brackets, we add at most 49 new vertices labeled as $t_{u,v}$. These vertices are distributed in matrix form according to the left and right factors that determine the corresponding product (see Figure 3 (center)).*

*Step 3. Now, since $A^2 \setminus \{0\} = \{e_1, e_2, e_1 + e_2\}$, we draw three new vertices labeled as $s_u$ in a column at the right of the graph (see Figure 3 (right)).*

*Step 4. We deal now with the construction of the edges. Firstly, we join each vertex $(i, 1)$ at the left of the graph with the vertices $(i, j)$ for $2 \leq i, j \leq 7$. These correspond to the edges $r_u t_{u,v}$ of the description (see Figure 4 (left)).*

*Step 5. After that, we join each vertex $(1, j)$ on the top of the graph with the vertices $(i, j)$ for $2 \leq i, j \leq 7$. These correspond to the edges $c_u t_{u,v}$ of the description (see Figure 4 (left in the center)).*

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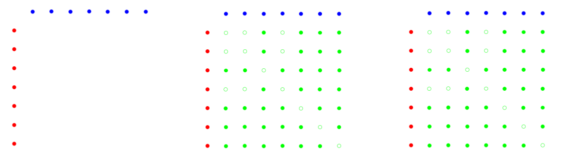

Figure 3: Steps 1–3 in the construction of the graph $G_2(A)$.

*Step 6.* *Next, we join each vertex $(i,1)$ with the vertex $(1,i)$ for $2 \leq i \leq 7$. These correspond to the edges $r_u c_u$ of the description. (see Figure 4 (center)).*

*Step 7.* *Now, we join each of the vertices $t_{u,v}$ with the corresponding vertex constructed in Step 3. These correspond to the edges $s_w t_{u,v}$ of the description (see Figure 4 (right in the center)).*

*Step 8.* *Finally, whenever is possible, we join the vertices $(1,j)$ and $(i,1)$ with the corresponding vertices constructed in Step 3, for $2 \leq i,j \leq 7$. These correspond, respectively, to the edges $r_u s_u$ and $c_u s_u$ of the description (see Figure 4 (right)).* $\lhd$

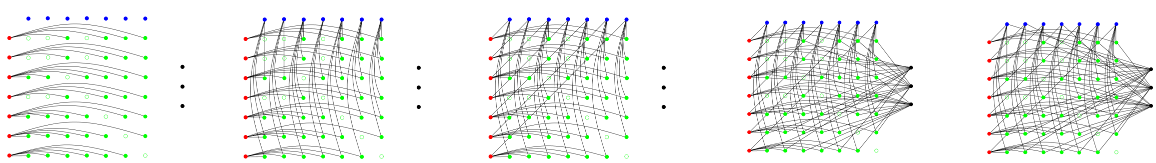

Figure 4: Steps 4–8 in the construction of the graph $G_2(A)$.

**Lemma 4.** *Let $A$ be an $n$-dimensional algebra over a finite field $\mathbb{K}$. Then,*

*a) If the algebra $A$ is abelian, then both graphs $G_1(A)$ and $G_2(A)$ are empty.*

*b) The graph $G_1(A)$ does not contain triangles.*

*c) In both graphs $G_1(A)$ and $G_2(A)$,*

- *The number of vertices is*

$$|A \setminus \mathrm{Ann}_{A^-}(A)| + |A \setminus \mathrm{Ann}_{A^+}(A)| + |A^2| + |\{(u,v) \in A \times A \mid uv \neq 0\}| - 1.$$

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- $d(t_{u,v}) = 3$, *for all* $u, v \in A$ *such that* $uv \neq 0$.

*d) In the graph* $G_1(A)$,

- $d(r_u) = |A \setminus \mathrm{Ann}_{A^+}(\{u\})|$, *for all* $u \in A \setminus \mathrm{Ann}_{A^-}(A)$.
- $d(c_u) = |A \setminus \mathrm{Ann}_{A^-}(\{u\})|$, *for all* $u \in A \setminus \mathrm{Ann}_{A^+}(A)$.
- $d(s_u) = \sum_{v \in A} |\mathrm{ad}_v^{-1}(u)|$, *for all* $u \in A^2 \setminus \{0\}$. *Here,* ad *denotes the adjoint action.*

*e) In the graph* $G_2(A)$,

- $d(r_u) = |A \setminus \mathrm{Ann}_{A^+}(\{u\})| + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^-}(A)}(u) + \mathbf{1}_{A^2}(u)$, *for all* $u \in A \setminus \mathrm{Ann}_{A^-}(\{u\})$.
- $d(c_u) = |A \setminus \mathrm{Ann}_{A^-}(\{u\})| + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^+}(A)}(u) + \mathbf{1}_{A^2}(u)$, *for all* $u \in A \setminus \mathrm{Ann}_{A^+}(\{u\})$.
- $d(s_u) = \mathbf{1}_{A \setminus \mathrm{Ann}_{A^-}(A)}(u) + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^+}(A)}(u) + \sum_{v \in A} |\mathrm{ad}_v^{-1}(u)|$, *for all* $u \in A^2 \setminus \{0\}$.

*Here,* $\mathbf{1}$ *denotes the characteristic function.*

**Proposition 2.** *Let* $A$ *be an* $n$*-dimensional algebra over a finite field* $\mathbb{K}$. *Then,*

*a) The number of edges of its related graph* $G_1(A)$ *is*

$$\sum_{u \notin \mathrm{Ann}_{A^-}(A)} (|A \setminus \mathrm{Ann}_{A^+}(\{u\})| + \sum_{v \in A^2 \setminus \{0\}} |\mathrm{ad}_u^{-1}(v)|) + \sum_{u \notin \mathrm{Ann}_{A^+}(A)} |A \setminus \mathrm{Ann}_{A^-}(\{u\})|.$$

*b) The number of edges of its related graph* $G_2(A)$ *coincides with those of* $G_1(A)$ *plus*

$$|A \setminus \mathrm{Ann}_A(A)| + |A^2 \setminus \mathrm{Ann}_{A^-}(A)| + |A^2 \setminus \mathrm{Ann}_{A^+}(A)|.$$

*Proof.* The result follows straightforward from the First Theorem of Graph Theory [8] and assertions (c–e) in Lemma 4. □

**Theorem 1.** *Let* $A$ *and* $A'$ *be two* $n$*-dimensional algebras over a finite field* $\mathbb{K}$. *Then,*

*a) If both algebras are isotopic, then their corresponding graphs* $G_1(A)$ *and* $G_1(A')$ *are isomorphic. Reciprocally, if the graphs* $G_1(A)$ *and* $G_1(A')$ *are isomorphic, then there exist three bijective maps* $f$, $g$ *and* $h$ *between* $A$ *and* $A'$ *such that* $f(u)g(v) = h(uv)$.

*b) If both algebras are isomorphic, then their corresponding graphs* $G_2(A)$ *and* $G_2(A')$ *are also isomorphic. Reciprocally, if the graphs* $G_2(A)$ *and* $G_2(A')$ *are isomorphic, then there exists a multiplicative bijective map between the algebras* $A$ *and* $A'$, *that is, a bijective map* $f : A \to A'$ *so that* $f(u)f(v) = f(uv)$, *for all* $u, v \in A$.

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*Proof.* Let $(f, g, h)$ be an isotopism between the algebras $A$ and $A'$. We define the map $\alpha$ between $G_1(A)$ and $G_1(A')$ such that

$$\begin{cases} \alpha(r_u) = r_{f(u)}, \text{ for all } u \in A \setminus \mathrm{Ann}_{A^-}(A), \\ \alpha(c_u) = c_{g(u)}, \text{ for all } u \in A \setminus \mathrm{Ann}_{A^+}(A), \\ \alpha(s_u) = s_{h(u)}, \text{ for all } u \in A^2 \setminus \{0\}, \\ \alpha(t_{u,v}) = t_{f(u),g(v)}, \text{ for all } u, v \in A \text{ such that } uv \neq 0. \end{cases}$$

The description of both graphs $G_1(A)$ and $G_1(A')$ together with Proposition 1, Lemma 3 and the regularity of $f$, $g$ and $h$ involve $\alpha$ to be an isomorphism between these two vertex-colored graphs, that is, $\alpha$ is a well-defined bijection between the vertices of $G_1(A)$ and $G_1(A')$ that preserves collinearity and the color of the vertices. The same map $\alpha$ constitutes an isomorphism between the graphs $G_2(A)$ and $G_2(A')$ in case of being $f = g = h$, that is, if the algebras $A$ and $A'$ are isomorphic.

Reciprocally, let $\alpha$ be an isomorphism between the graphs $G_1(A)$ and $G_1(A')$. Collinearity involves this isomorphism to be uniquely determined by its restriction to $R_A \cup C_A \cup S_A$. Specifically, the image of each vertex $t_{u,v} \in T_A$ by means of $\alpha$ is uniquely determined by the corresponding images of $r_u$, $c_v$ and $s_{uv}$. Let $\beta$ and $\beta'$ be the respective bases of the algebras $A$ and $A'$ and let $\pi : A \to A'$ be the natural map that preserves the components of each vector with respect to the mentioned bases. That is, $\pi((u_1, \ldots, u_n)_\beta) = (u_1, \ldots, u_n)_{\beta'}$, for all $u_1, \ldots, u_n \in \mathbb{K}$. Let us define three maps $f$, $g$ and $h$ from $A$ to $A'$ such that

$$f(u) = \begin{cases} \pi(u), \text{ for all } u \in \mathrm{Ann}_{A^-}(A), \\ v, \text{ otherwise, where } v \in A \text{ is such that } \alpha(r_u) = r_v. \end{cases}$$

$$g(u) = \begin{cases} \pi(u), \text{ for all } u \in \mathrm{Ann}_{A^+}(A), \\ v, \text{ otherwise, where } v \in A \text{ is such that } \alpha(c_u) = c_v. \end{cases}$$

$$h(u) = \begin{cases} \pi(u), \text{ for all } u \in (A \setminus A^2) \cup \{0\}, \\ v, \text{ otherwise, where } v \in A \text{ is such that } \alpha(s_u) = s_v. \end{cases}$$

From Proposition 1 and Lemma 3, these three maps are bijective. Let $u, v \in A$. If $u \in \mathrm{Ann}_{A^-}(A)$ or $v \in \mathrm{Ann}_{A^+}(A)$, then there does not exist the vertex $t_{u,v}$ in the graph $G_1(A)$. Since $\alpha$ preserves collinearity, there does not exist the vertex $t_{f(u),g(v)}$ in the graph $G_1(A')$, which means that $f(u) \in \mathrm{Ann}_{A'^-}(A')$ or $g(v) \in \mathrm{Ann}_{A'^+}(A')$. In any case, we have that $f(u)g(v) = 0 = h(uv)$. Finally, if $u \notin \mathrm{Ann}_{A^-}(A)$ and $v \notin \mathrm{Ann}_{A^+}(A)$, then the vertex $t_{u,v}$ connects the vertices $r_u$, $c_v$ and $s_{uv}$ in the graph $G_1(A)$. Now, the isomorphism $\alpha$ maps this vertex $t_{u,v}$ in $G_1(A)$ to a vertex $t_{u',v'}$ in $G_2(A)$ that is connected to the vertices $r_{u'}$, $c_{v'}$ and $s_{u'v'}$. Again, since $\alpha$ preserves collinearity, it is $f(u) = u'$, $g(v) = v'$ and, finally, $h(uv) = f(u)g(v)$.

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In case of being $\alpha$ an isomorphism between the graphs $G_2(A)$ and $G_2(A')$ it is enough to consider $f = g = h$ in the previous description. This is well-defined because of the new edges that are included in the graphs $G_1(A)$ and $G_1(A')$ in order to define, respectively, the graphs $G_2(A)$ and $G_2(A')$. These edges involve the multiplicative character of the bijective map $f$, that is, $f(u)g(v) = h(uv)$, for all $u, v \in A$. □

Theorem 1 enables us to determine non-isotopic and non-isomorphic algebras from their corresponding non-isomorphic graphs. To this end, it is interesting to compute some isomorphism invariants of the corresponding graphs $G_1$ and $G_2$. In this regard, Table 1 shows, for instance, some graph invariants of the graph $G_1$ related to each one of the possible isomorphism classes of 3-dimensional Lie algebras over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$. All of them constitute partial-magma algebras.

| | $\mathbb{F}_2$ | | | $\mathbb{F}_3$ | | |
|---|---|---|---|---|---|---|
| Lie partial-magma algebra | Vertices | Edges | Triangles | Vertices | Edges | Triangles |
| Abelian | 0 | 0 | 0 | 0 | 0 | 0 |
| $e_1e_2 = e_3$ | 37 | 72 | 0 | 482 | 1296 | 0 |
| $e_1e_2 = e_2$ | 37 | 72 | 0 | 482 | 1296 | 0 |
| $e_1e_2 = e_3, e_1e_3 = -e_2$ | - | - | - | 636 | 1728 | 0 |
| $e_1e_2 = e_3, e_1e_3 = e_2$ | 53 | 108 | 0 | 636 | 1728 | 0 |
| $e_1e_2 = e_2, e_1e_3 = e_3$ | 53 | 108 | 0 | 636 | 1728 | 0 |
| $e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = -e_1$ | 63 | 126 | 0 | - | - | - |
| $e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = 2e_1$ | - | - | - | 702 | 1872 | 0 |

Table 1: Graph invariants for the graph $G_1$ related to each isomorphism class of 3-dimensional Lie partial-magma algebras over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.

Thus, for instance, it is known that the $n$-dimensional anticommutative algebra over the finite field $\mathbb{F}_2$, with $n \geq 3$, described by the product $e_1e_2 = e_3$ is not isomorphic to the $n$-dimensional anticommutative algebra over $\mathbb{F}_2$ described by the product $e_1e_2 = e_1$. This follows straightforward from the fact that the corresponding graph $G_2$ related to the former coincides with that associated with the latter, which is shown in Figure 2 (right), up to the vertex $s_{e_1}$, which becomes $s_{e_3}$, and the two edges $r_{e_1}s_{e_1}$ and $c_{e_1}s_{e_1}$, which disappear. Both graphs are, therefore, non-isomorphic and hence, the algebras are neither isomorphic. It is straightforward verified that both algebras are, however, isotopic. Besides, even if this does not constitute a necessary condition, their corresponding graphs $G_1$ are isomorphic. That graph shown in Figure 2 (left) is indeed the graph $G_1$ corresponding to the anticommutative algebra described by the product $e_1e_2 = e_1$.

We finish this paper with an illustrative example that focuses on those graphs $G_1$ and $G_2$ related to the set of non-abelian partial-quasigroup rings over a finite field that are based on the known distribution of partial Latin squares of order 2 into isotopism classes.

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Particularly, Table 2 shows the graph invariants related to the finite field $\mathbb{F}_2$. Partial Latin squares are written row after row in a single line, with empty cells represented by zeros. For each isotopism class we indicate the sequence with the number of vertices of each color, the number of edges and that of triangles of the corresponding graphs $G_1$ and $G_2$.

| | $G_1$ | | | $G_2$ | | |
|---|---|---|---|---|---|---|
| Partial Latin square | Vertices | Edges | Triangles | Vertices | Edges | Triangles |
| 10 00 | (2,2,1,4) | 12 | 0 | (2,2,1,4) | 16 | 7 |
| 10 01 | (3,3,1,6) | 18 | 0 | (3,3,1,6) | 23 | 7 |
| 10 02 | (3,3,3,7) | 21 | 0 | (3,3,3,7) | 30 | 16 |
| 10 20 | (3,2,3,6) | 18 | 0 | (3,2,3,6) | 25 | 12 |
| 12 00 | (2,3,3,6) | 18 | 0 | (2,3,3,6) | 25 | 12 |
| 12 20 | (3,3,3,8) | 24 | 0 | (3,3,3,8) | 33 | 13 |
| 12 21 | (3,3,3,8) | 24 | 0 | (3,3,3,8) | 33 | 13 |

Table 2: Graph invariants for the graphs $G_1$ and $G_2$ related to 2-dimensional non-abelian partial-quasigroup rings over the finite field $\mathbb{F}_2$.

**Theorem 2.** *The set of* 2*-dimensional non-abelian partial-quasigroup rings is distributed into six isotopism classes.*

*Proof.* A computational case study enables us to ensure the result. In particular, if the characteristic of the base field is distinct of two, then the six isotopism classes under consideration are those related to the next partial Latin squares of order 2

| | |
|---|---|
| 1 | |
| | |

| | |
|---|---|
| 1 | |
| | 1 |

| | |
|---|---|
| 1 | |
| 2 | |

| | |
|---|---|
| 1 | 2 |
| | |

| | |
|---|---|
| 1 | 2 |
| 2 | |

| | |
|---|---|
| 1 | 2 |
| 2 | 1 |

Otherwise, if the characteristic of the base field is two, then the isotopism classes related to the last two partial Latin squares coincide. In this case, the next partial Latin square corresponds to the sixth isotopism class

| | |
|---|---|
| 1 | |
| | 2 |

If the characteristic of the base field is distinct of two, the partial-quasigroup ring related to this partial Latin square is isotopic to that related to the unique Latin square of the previous list. □

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## 4 Conclusion and further studies

We have described in this paper a pair of graphs that enable us to define faithful functors between finite-dimensional algebras over finite fields and these graphs. The computation of isomorphism invariants of these graphs plays a remarkable role in the distribution of distinct families of algebras into isotopism and isomorphism classes. Some preliminary results have been exposed in this regard, particularly on the distribution of partial-quasigroup rings over finite fields. Based on the known classification of partial Latin squares into isotopism classes, further work is required to determine completely this distribution.

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