Shape phase transition in odd nuclei in a multi-j model: The $U^B(6) \otimes U^F(12)$ case

C. E. Alonso,[1] J. M. Arias,[1] and A. Vitturi[1,2]

[1]*Departamento de Física Atómica, Molecular y Nuclear, Facultad de Física, Universidad de Sevilla, Apartado 1065, E-41080 Sevilla, Spain*

[2]*Dipartimento di Fisica Galileo Galilei and INFN, Via Marzolo 8, I-35131 Padova, Italy*

(Received 2 April 2007; published 22 June 2007)

The phase transition in odd nuclei when the underlying even-even core nuclei experience a transition from spherical to deformed γ-unstable shapes is investigated. The odd particle is assumed to be moving in the three single-particle orbitals $j = 1/2, 3/2$, and $5/2$. At the critical point in the phase transition, an analytic solution to the corresponding Bohr Hamiltonian, called E(5/12), is worked out. Energy spectra and electromagnetic transitions and moments are presented. The same problem is also attacked in the framework of the interacting boson-fermion model (IBFM). Two different Hamiltonians are used. The first one is constructed *ad hoc* so as to mimic the situation in the E(5/12) model. The second one leads to the occurrence of the $O^B(6) \otimes U^F(12)$ symmetry when the boson part approaches the O(6) condition. The entire transition line is studied with this Hamiltonian and, in particular, the critical point. Both IBFM calculations at the critical point are consistent with the E(5/12) results.

DOI: 10.1103/PhysRevC.75.064316 PACS number(s): 21.60.Fw, 21.60.Ev

I. INTRODUCTION

Atomic nuclei have been classified for years in terms of collective deformations that are described by introducing two collective variables [1]. The deformation parameter β measures the axial deviation from sphericity, while the angle variable γ controls the departure from axial symmetry. Three analytical solutions for the collective Bohr Hamiltonian have been known: the vibrational, rotational, and deformed γ-unstable limits [1–3]. These situations correspond, respectively, to the spherical, axially deformed, and deformed γ-independent ground state shapes. Of course, few nuclei present properties that fit exactly one of these limiting situations, and most of them display a transitional behavior. The occurrence of shape transitional behavior characterizes both even-even and odd nuclei. As the nucleon number changes, the structure of the system changes from one character to another, allowing one to talk about a ground state phase transition (nowadays known as a quantum phase transition). The properties of the nucleus at the vicinity of the critical point vary rapidly, and, consequently, the description of critical nuclei has been thought to be very difficult. Recently, Iachello proposed in a series of papers a new class of symmetries that are formulated in terms of the Bohr Hamiltonian and that can be applied to critical point situations [4–6]. In particular, at the critical point from spherical to γ-unstable shapes, called E(5) [4], at the critical point from spherical to axially deformed shapes, called X(5) [5], and at the critical point from axially deformed shapes to triaxial shapes, called Y(5) [6]. Since the introduction of these limits, many theoretical [7–12] and experimental [13–22] studies have been presented in order to look for nuclei that exhibit the properties of criticality and to classify the corresponding phase transitions. Many studies have extended these original models to more complex situations [23–27].

Because of the further complexity embodied in the description of the coupling of the odd particle to the even core, the study of phase transitions in odd nuclei has only recently been attacked. The first case considered by Iachello [7,28] was the case of a $j = 3/2$ fermion coupled to a boson core that undergoes a transition from spherical to γ-unstable situation. At the critical point, an elegant analytic solution, called E(5/4), was obtained starting from the Bohr Hamiltonian. The possible occurrence of this symmetry in ^{135}Ba has been recently suggested [29]. However, the restriction of the fermion space to a single-j orbit makes difficult the identification of a particular nucleus as critical. A recent work [30] considered the richer case of a collective core coupled to a particle moving in the $j = 1/2, 3/2$, and $5/2$ single-particle orbits, with the boson part undergoing a transition from sphericity to deformed γ-instability. For the critical point in this situation, a new analytic solution, called E(5/12), has been presented. Since this situation is likely to occur in several mass regions, E(5/12) provides a better chance of finding an example of criticality in odd-even nuclei. In this paper we extend the previous study on E(5/12) so as to provide the grounds for experimental studies looking for critical behavior in odd-even nuclei. Spectrum, electric quadrupole, and magnetic dipole transitions along with quadrupole and magnetic moments are worked out in the model.

It is worth mentioning that all these studies have been performed in the framework of the collective model. However, a parallel treatment can be done within the interacting boson model (IBM) [31]. Although this model is formulated from the beginning in second quantized form, it is possible to obtain its geometric picture by using coherent states and shape variables [32–34]. Three dynamical IBM symmetries are known: U(5), SU(3), and O(6) corresponding to geometrical spherical, axially deformed, and γ-unstable deformed shapes, respectively [35]. Transitional classes among these limits have been thoroughly studied since the introduction of the model. Even ground state phase transitions were studied soon after [33,36–38], but these studies have been revitalized recently with the above discussed works on quantum phase transitions within the Bohr Hamiltonian framework. Numerical studies for even-even nuclei on the critical point from U(5) to O(6) and from U(5) to SU(3) have been presented, and the analysis of the phase transition characteristics has been the object of many papers [39–61]. A recent review on the topic can be found in Ref. [62]. For the case of odd-even nuclei, the simplest IBM extension is called the interacting boson-fermion model (IBFM) [63]. The IBFM has been used to study the phase transition along the U(5)-O(6) line for the case of a fermion in a single $j = 3/2$ orbital [64,65], which is the situation studied in the E(5/4) model. The choice of the $j = 3/2$ orbital allows one to recover at the γ-soft extreme the $O^B(6) \otimes SU^F(4)$ boson-fermion symmetry. This is because in this case the full boson-fermion quadrupole-quadrupole term in the Hamiltonian can be expressed in terms of the Casimir operators of the general $U^{BF}(5)$, $O^{BF}(5)$, $\text{Spin}^{BF}(6)$, and $\text{Spin}^{BF}(5)$ algebras. The case of the $j = 3/2$ is not, however, unique. It is known in the literature [63,66] that also in the multi-j case with $j = 1/2, 3/2, 5/2$, with a proper choice of the boson-fermion Hamiltonian, one can obtain a transitional Hamiltonian expressed in terms of Casimir operators associated with the $O^{BF}(6)$ algebra. This has allowed one to study within the IBFM a similar situation to that studied in the E(5/12) model. The IBFM study has been shown to be useful to check the E(5/4) [64] and E(5/12) [30] cases. In this paper, we present for $j = 1/2, 3/2, 5/2$ the equivalent E(5/12) results for selected IBFM Hamiltonians. The confrontation of E(5/12) and IBFM at the corresponding critical point will provide useful guidelines for experimental studies.

The structure of the paper is as follows. In Sec. II, the E(5/12) model is presented in detail and the wave functions are used to calculate different spectroscopic properties of experimental interest. In Secs. III and IV, two IBFM model Hamiltonians produced to study the critical point within the IBM framework are presented. We will therefore consider within the IBFM the case of the coupling of an odd particle moving in the $j = 1/2, 3/2$, and 5/2 orbitals to a boson core undergoing a transition from U(5) to O(6) situations. We follow along the transition the evolution of the spectrum and study in detail energies and transitions in correspondence to the critical point. We will show that the results of IBFM and E(5/12) models give comparable behaviors producing robust results indicating criticality in odd-even nuclei.

II. THE BOHR HAMILTONIAN AT THE CRITICAL POINT AND THE E(5/12) MODEL

In this section, we study the coupling of an even core to a particle moving in the 1/2, 3/2, and 5/2 orbitals. We show that it can also display analytic solutions within the Bohr collective model, in analogy to the case of the $j = 3/2$ particle, described by Iachello within the E(5/4) critical point symmetry. We will name this case as E(5/12). But first, we would like to present briefly the E(5) case for even-even nuclei, since it will be an important reference for our study. The E(5) critical point situation corresponds to the solution for the Bohr Hamiltonian

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^4}\sum_\kappa \frac{Q_\kappa^2}{\sin^2\left(\gamma - \frac{2}{3}\pi\kappa\right)}\right] + U(\beta,\gamma), \tag{2.1}$$

in which the potential energy U is γ independent and modeled as an infinite square well in β

$$\begin{aligned} U(\beta) &= 0, \quad \beta < \beta_w, \\ U(\beta) &= \infty, \quad \beta \geqslant \beta_w. \end{aligned} \tag{2.2}$$

This situation is supposed to mimic the critical point from spherical (minimum of the energy surface at $\beta = 0$) to deformed γ unstable (minimum of the energy surface at $\beta \neq 0$ and γ independent). In this situation, analytical solutions are found. Energy levels are directly related to zeros of appropriate Bessel functions [4,7]. In Fig. 1, the spectrum and a few selected $E2$ transition probabilities are displayed. For notation and more details, we refer to Refs. [4,7]. We must mention that in this situation, because of the symmetry of the problem, states are classified by the totally symmetric irreducible representations of the O(5) and O(3) groups, $(\tau, 0)$ and L, respectively. In addition, an extra label ξ is introduced to mark different solutions for the radial (β) equation. Thus, states are labeled by $|\xi, \tau, L\rangle$. The solutions of Eq. (2.1) are

$$\psi(\beta,\gamma,\theta_i) = f_{\xi,\tau}(\beta)\Phi^{\tau\mu LM}(\gamma,\theta_i), \tag{2.3}$$

where the Φ functions were determined in Refs. [3,67]. These functions are eigenstates of any quadrupole Hamiltonian which displays $O(5) \supset O(3)$ symmetry, i.e., for $U(\beta,\gamma) = U(\beta)$. The $f(\beta)$ functions depend on the selection of the β potential. For the infinite square well proposed, the wave functions in β take the simple form

$$f_{\xi,\tau}(\beta) = C_{\xi,\tau}\beta^{-3/2}J_{\tau+3/2}\left(\frac{x_{\xi,\tau}}{\beta_\omega}\beta\right), \tag{2.4}$$

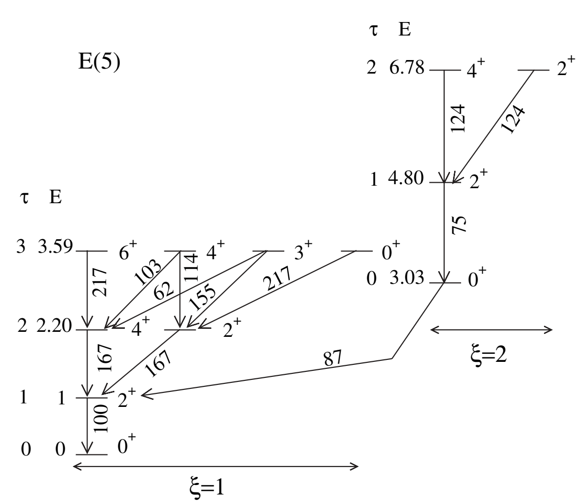

FIG. 1. Energy levels (normalized to the energy of the first excited state) and $B(E2)$ transition probabilities for E(5).

where $\tau = 0, 1, 2, \ldots$ labels the SO(5) group, ξ is an index that enumerates the successive zeros of the Bessel functions, $x_{\xi,\tau}$ is the ξth zero for a given τ, $C_{\xi\tau}$ is a normalization, and β_ω is the range of the square potential. The full states in Eq. (2.3) can then be denoted by

$$|\xi\tau\mu LM\rangle, \tag{2.5}$$

where μ is a label that distinguishes between repeated L for a given value of τ. Once the wave functions are known, the transition probabilities are related to the reduced matrix element of the relevant electromagnetic operator,

$$B(E/M, \lambda) = \frac{1}{2L_i + 1}\langle \xi_f \tau_f \mu_f L_f || T_C^{(\lambda)} || \xi_i \tau_i \mu_i L_i \rangle^2, \tag{2.6}$$

its calculation includes an integral of products of $f(\beta)$ functions on the β variable, plus integrals of products of $\Phi^{\tau\mu LM}(\gamma, \theta_i)$ functions in the γ variable and the Euler angles. Details of these integrations can be found in Ref. [7].

Next we go to our case of interest, in which a single particle that can occupy the $j = 1/2, 3/2$, and $5/2$ orbitals is coupled to an E(5) even-even core. In this situation, one can use the separation of the single-particle angular momentum into a pseudospin and pseudo-orbital angular momentum (0 and 2) and assume no coupling due to the pseudospin. In this situation, an E(5) core is coupled to a two-level system representing the odd particle. The lower level with $L_F = 0$ ($j = 1/2$) has zero energy, and the upper one with $L_F = 2$ ($j = 3/2, 5/2$ degenerate) has some κ' energy. The pseudo-orbital part transforms as the representations $[\tau_F, 0]$ of $O^F(5)$, with τ_F being either 0 or 1. One can use in this case a model Hamiltonian of the form

$$\begin{aligned} H = & -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} \right. \\ & \left. - \frac{1}{4\beta^4}\sum_\kappa \frac{Q_\kappa^2}{\sin^2\left(\gamma - \frac{2}{3}\pi\kappa\right)}\right] + u(\beta) + kg(\beta) \\ & \times [2\hat{\mathcal{L}}_B \circ \hat{\mathcal{L}}_F] + k' g(\beta)\hat{\mathcal{L}}_F^2, \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} u(\beta) &= 0, \quad \beta < \beta_w, \\ u(\beta) &= \infty, \quad \beta \geqslant \beta_w, \\ g(\beta) &= \frac{\hbar^2}{2B\beta^2}. \end{aligned} \tag{2.8}$$

Note that $\hat{\mathcal{L}}_B$ and $\hat{\mathcal{L}}_F$ are the five-dimensional boson and fermion angular momenta, and that the dot $\circ$ indicates the five-dimensional scalar product. Here and in the following we will use the calligraphic letter $\hat{\mathcal{L}}$ to denote the five-dimensional O(5) angular momenta and the italic letter L to denote the usual three-dimensional O(3) angular momenta. It is worth noting that the Hamiltonian (2.7) is a generalization of the one proposed for the E(5/4) case. Since in E(5/12) there are two $O^F(5)$ representations involved, in Eq. (2.7) the last term has been introduced to include the energy difference between $j = 1/2$ and $j = 3/2, 5/2$ single-particle orbitals. The coupling scheme is presented in Fig. 2. In the lowest part of the figure the even-even degrees of freedom, given by the E(5) model, have to be coupled to the odd-particle degrees of freedom. These last ones are separated into a pseudo-orbital part (two-level system) plus a pseudospin part. First, the core E(5), whose states are classified by (ξ, τ_B, L_B) as explained above, is coupled to the pseudo-orbital part of the odd particle, labeled (τ_F, L_F). This gives rise to a

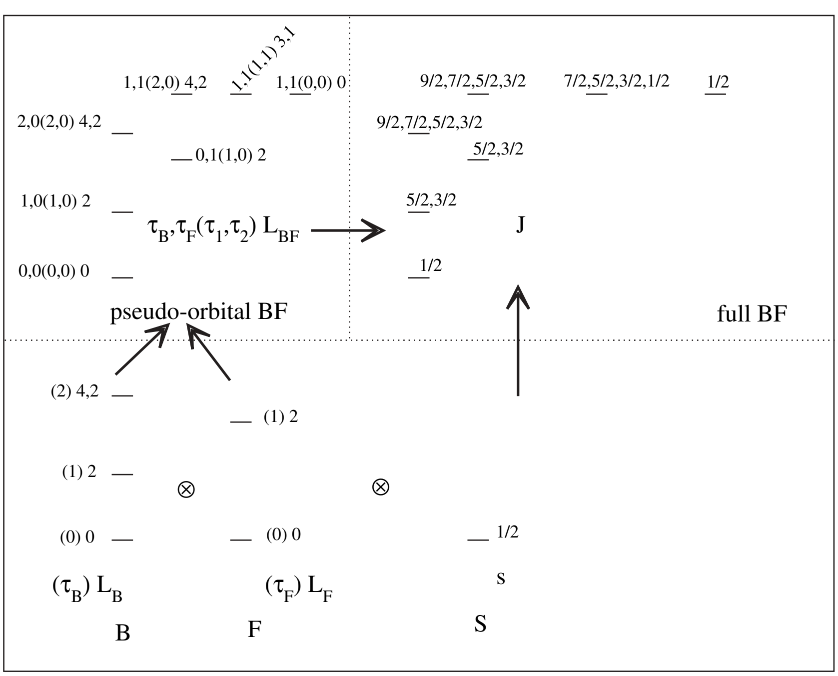

FIG. 2. Coupling scheme for E(5/12) model. Lower part: an even-even E(5) core (represented by the lowest three states within the ground state band, $\xi = 1$) is coupled first to a two-level system that represents the pseudo-orbital angular momentum of the odd particle. The resulting total orbital angular momentum for the odd-even system is finally coupled to the spin part of the odd particle.

boson-fermion pseudo-orbital coupling presented in the upper left panel. The states are labeled τ_B characterizing the $O^B(5)$ group and τ_F characterizing the $O^F(5)$ group. These two representations are coupled to a common $O^{BF}(5)$ group whose irreducible representations are labeled (τ_1, τ_2). If the coupling is with $\tau_F = 0$, $L_F = 0$, all the obtained states are as in the E(5) even-even case. When the coupling is with $\tau_F = 1$, the allowed coupled representations are $(\tau_1, \tau_2) = (\tau_B + 1, 0)$, $(\tau_B, 1)$, and $(\tau_B - 1, 0)$. Once the $O^{BF}(5)$ representations are known, the usual reduction to angular momentum is performed. In the upper left panel of Fig. 2, such a pseudo-orbital coupling is represented for the $\xi = 1$ states in the E(5) part, the left band comes from the coupling of the E(5) states to the $(\tau_F = 0, L_F = 0)$ state and then, as mentioned above, are as in the E(5) model. The other three bands come from the coupling of the E(5) part to the $(\tau_F = 1, L_F = 2)$ state. This coupling for the $\tau_B = 0$ produces just one state $(\tau_1 = 1, \tau_2 = 0)$ at the same energy as the excitation energy of the $\tau_F = 1$ with respect to the $\tau_F = 0$ state for the odd particle. This state is labeled $\tau_B, \tau_F(\tau_1, \tau_2)L_{BF} = 0, 1(1, 0)2$ (L_{BF} is the boson-fermion pseudo-orbital angular momentum). The coupling of the boson $\tau_B = 1$ states to the single-particle pseudo-orbital $\tau_F = 1$ state give three possibilities $(\tau_1, \tau_2) = (2, 0), (1, 1)$, and $(0, 0)$; the corresponding reduction rules give $L_{BF} = 4, 2$, $L_{BF} = 3, 1$, and $L_{BF} = 0$, respectively. This is represented schematically in Fig. 2 (upper left panel). Finally, these states are coupled to the pseudospin part and give the spectrum for the full boson-fermion system, presented in the upper right part of Fig. 2. Each line represents degenerate states with total angular momentum given by J, produced by the coupling of the boson-fermion pseudo-orbital angular momentum L_{BF} to the pseudospin s.

With this choice of the coupling, the total wave function can be factorized in the form

$$\Psi = F(\beta)[\Phi(\gamma, \theta_i, \eta_{\rm orb}) \otimes \chi(\eta_{\rm spin})]_{J_{BF}}, \tag{2.9}$$

where $\eta_{\rm orb}$ and $\eta_{\rm spin}$ represent the particle coordinates associated with the pseudo-orbital and pseudospin spaces. The wave function Φ is characterized by good quantum numbers $(\tau_B, \tau_F, \tau_1, \tau_2, L_{BF}, M_L)$ and is given by

$$\begin{aligned}
&\Phi_{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF},M_L}(\gamma, \theta_i, \eta_{\rm orb}) \\
&\quad = \sum_{L_B, M_{L_B}, L_F, m_{L_F}} \left\langle \begin{matrix} (\tau_B, 0) & (\tau_F, 0) & | & (\tau_1, \tau_2) \\ L_B & L_F & | & L_{BF} \end{matrix} \right\rangle \\
&\qquad \times \left\langle \begin{matrix} L_B & L_F & | & L_{BF} \\ M_{L_B} & m_{L_F} & | & M_L \end{matrix} \right\rangle \phi_{\tau_B, L_B, M_{L_B}}(\gamma, \theta_i) \\
&\qquad \times X_{\tau_F, L_F, m_{L_F}}(\eta_{\rm orb}),
\end{aligned} \tag{2.10}$$

where the symbols after the summation are isoscalar factors for the O(5)-O(3) and the O(3)-O(2) reductions. The function $X(\eta_{\rm orb})$ describes the five-dimensional pseudo-orbital part, while $\chi(\eta_{\rm spin})$ is the pseudospin wave function, which does not contribute to the energy. The equation for $F(\beta)$ then acquires the familiar form

$$\left[-\frac{\hbar^2}{2B}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\hbar^2}{2B}\frac{\Lambda}{\beta^2} + u(\beta)\right] F(\beta) = EF(\beta), \tag{2.11}$$

with

$$\begin{aligned}
\Lambda &= \tau_B(\tau_B + 3) + k[\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1) \\
&\quad - \tau_B(\tau_B + 3) - \tau_F(\tau_F + 3)] + k'\tau_F(\tau_F + 3).
\end{aligned} \tag{2.12}$$

The solutions of this equation, as in the case of the E(5/4) model, are given in terms of Bessel functions of order $\nu = \sqrt{\Lambda + 9/4}$, in the form

$$\begin{aligned}
&F_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}(\beta) \\
&\quad = c_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}\beta^{-3/2} J_\nu(x_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}\beta/\beta_w),
\end{aligned} \tag{2.13}$$

where $c_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}$ is a normalization constant, and $x_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}$ the ξth zero of $J_\nu(z)$. The corresponding eigenvalues are given as

$$E_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}} = \frac{\hbar^2}{2B}\left(\frac{x_{\xi,\{\tau_B,\tau_F,(\tau_1,\tau_2),L_{BF}\}}}{\beta_w}\right)^2. \tag{2.14}$$

The corresponding spectrum is given in Fig. 3 for the choice of the parameters $k = -1/4$ and $k' = 5/2$. The states are labeled according to the quantum numbers $\xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}$. One should remember that each state actually represents a multiplet of degenerate levels, whose angular momenta can be extracted from Table I.

The parameters k and k' control the order of the bands. In Fig. 4 we present the evolution of the bandheads as a function of the k and k' parameters.

Once the wave functions are known, one can calculate any other spectroscopic observable of interest. In general, the electromagnetic operator for the odd system will have both contributions from the collective core and from the single fermion. We will have to calculate reduced matrix elements

$$\langle \Psi_f || T^{(\lambda} || \Psi_i \rangle, \tag{2.15}$$

where Ψ are given by Eq. (2.9). Since in this formulation the total angular momentum for the fermion is not a good quantum number, we prefer to perform a simple change of coupling transformation given by

$$\begin{aligned}
&|(L_B, L_F), L_{BF}, s; J\rangle \\
&\quad = \sum_j (-1)^{L_B + L_F + 1/2 + J}\sqrt{(2L_{BF} + 1)(2j + 1)} \\
&\qquad \times \left\{ \begin{matrix} L_B & L_F & L_{BF} \\ 1/2 & J & j \end{matrix} \right\} |L_B, (L_F, s), j; J\rangle.
\end{aligned} \tag{2.16}$$

With this it is easy to use standard angular momentum coupling to calculate both collective, $T_C^{(\lambda)}$, and single particle, $T_F^{(\lambda)}$, parts of any electromagnetic operator:

TABLE I. Reduction (branching rules) from the $O^{BF}(5)$ irreducible representations (τ_1, τ_2) (first column) to the $O^{BF}(3)$ irreducible representations L_{BF} (second column). They can be found, for instance, in Ref. [63] (p. 73, Eqs. 3.85 and 3.86). In the last column, the angular momentum content J_{BF} in each $O^{BF}(5)$ representation is presented. It comes from the coupling of L_{BF} to the fermion pseudospin $s = 1/2$. The superscript on a given L_{BF} (or J) means the number of times that L_{BF} (or J) is obtained.

$O^{BF}(5)$ irrep (τ_1, τ_2)	$O^{BF}(3)$ irrep L_{BF}	Total ang. mom. J
(0,0)	0	1/2
(1,0)	2	5/2,3/2
(2,0)	4,2	9/2,7/2,5/2,3/2
(3,0)	6,4,3,0	13/2,11/2,9/2,$(7/2)^2$,5/2,1/2
(4,0)	8,6,5,4,2	17/2,15/2,13/2,$(11/2)^2$,$(9/2)^2$,7/2,5/2,3/2
(1,1)	3,1	7/2,5/2,3/2,1/2
(2,1)	5,4,3,2,1	11/2,$(9/2)^2$,$(7/2)^2$,$(5/2)^2$,$(3/2)^2$,1/2
(3,1)	7,6,5^2,4,3^2,2,1	15/2,$(13/2)^2$,$(11/2)^3$,$(9/2)^3$,$(7/2)^3$,$(5/2)^3$,$(3/2)^2$,1/2

$$\begin{aligned}
&\langle \xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}, 1/2; J || T_C^{(\lambda)} || \xi', \tau_B', \tau_F', (\tau_1', \tau_2'), L_{BF}', 1/2; J' \rangle \\
&\quad = \sum_{L_B, L_F, L_B', L_F'} \left\langle \begin{matrix} (\tau_B, 0) & (\tau_F, 0) & | & (\tau_1, \tau_2) \\ L_B & L_F & | & L_{BF} \end{matrix} \right\rangle \left\langle \begin{matrix} (\tau_B', 0) & (\tau_F', 0) & | & (\tau_1', \tau_2') \\ L_B' & L_F' & | & L_{BF}' \end{matrix} \right\rangle \\
&\qquad \times \sum_{j, j'} (-1)^{L_B' + J + j + \lambda} \sqrt{(2L_{BF}+1)(2L_{BF}'+1)(2J+1)(2J'+1)(2j+1)(2j'+1)} \\
&\qquad \times \begin{Bmatrix} L_B & L_F & L_{BF} \\ 1/2 & J & j \end{Bmatrix} \begin{Bmatrix} L_B' & L_F' & L_{BF}' \\ 1/2 & J' & j' \end{Bmatrix} \begin{Bmatrix} L_B & J & j \\ J' & L_B' & \lambda \end{Bmatrix} \langle \xi, \tau_B, L_B || T_C^{(\lambda)} || \xi', \tau_B', L_B' \rangle \delta_{\tau_F, \tau_F'} \delta_{L_F, L_F'} \delta_{j, j'},
\end{aligned} \tag{2.17}$$

where $\langle \xi, \tau_B, L_B || T_C^{(\lambda)} || \xi', \tau_B', L_B' \rangle$ is the collective part already calculated for the even-even system. For the fermion part

$$\begin{aligned}
&\langle \xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}, 1/2; J || T_F^{(\lambda)} || \xi', \tau_B', \tau_F', (\tau_1', \tau_2'), L_{BF}', 1/2; J' \rangle \\
&\quad = \sum_{L_B, L_F, L_B', L_F'} \left\langle \begin{matrix} (\tau_B, 0) & (\tau_F, 0) & | & (\tau_1, \tau_2) \\ L_B & L_F & | & L_{BF} \end{matrix} \right\rangle \left\langle \begin{matrix} (\tau_B', 0) & (\tau_F', 0) & | & (\tau_1', \tau_2') \\ L_B' & L_F' & | & L_{BF}' \end{matrix} \right\rangle \\
&\qquad \times \sum_{j, j'} (-1)^{L_B' + J' + j' + \lambda} \sqrt{(2L_{BF}+1)(2L_{BF}'+1)(2J+1)(2J'+1)(2j+1)(2j'+1)} \\
&\qquad \times \begin{Bmatrix} L_B & L_F & L_{BF} \\ 1/2 & J & j \end{Bmatrix} \begin{Bmatrix} L_B' & L_F' & L_{BF}' \\ 1/2 & J' & j' \end{Bmatrix} \begin{Bmatrix} j & J & L_B \\ J' & j' & \lambda \end{Bmatrix} \\
&\qquad \times \langle j || T_F^{(\lambda)} || j' \rangle \delta_{\xi, \xi'} \delta_{\tau_B, \tau_B'} \delta_{L_B, L_B'},
\end{aligned} \tag{2.18}$$

where $\langle j || T_F^{(\lambda)} || j' \rangle$ is a single-particle matrix element that is obtained from

$$\langle j || (a_{j_1}^\dagger \times \tilde{a}_{j_2})^{(\lambda)} || j' \rangle = -\sqrt{2\lambda + 1} \delta_{j_2, j'} \delta_{j_1, j}. \tag{2.19}$$

With these general expressions, any multipole transition and moment can be evaluated. In particular, we are studying $E2$ and $M1$ transitions and moments below.

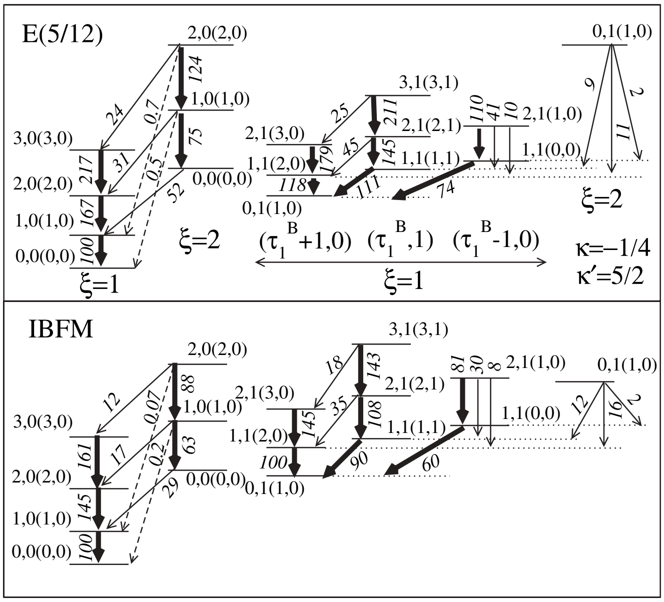

FIG. 3. Upper frame: Energy levels in the odd system at the E(5/12) critical point (normalized to the energy of the first excited state). Each (degenerate) state is characterized by the $\xi, \tau_B, \tau_F, (\tau_1, \tau_2)$ quantum numbers. The values $k = -1/4$ and $k' = 5/2$ were used in the Hamiltonian (2.7). The values of the angular momenta for each degenerate state can be extracted from Table I. Some selected values of $B(E2)$, corresponding to the transitions between the highest spins of initial and final multiplets, are given in the figure [the $B(E2)$ associated with the lowest 5/2-1/2 transition is normalized to 100]. Lower frame: Energy levels in the odd system within the IBFM Hamiltonian (3.6) at the critical point. The number of bosons $N = 7$ has been assumed. The values of k and k' are the same as in the E(5/12) Hamiltonian. For $B(E2)$, see the upper frame.

A. *E*2 transition and moments

Let us first consider the electric quadrupole operator. This can be written as the sum of the collective contribution plus the odd-particle one:

$$T^{(E2)} = T_C^{(E2)} + T_F^{(E2)}. \tag{2.20}$$

For the collective part, usually a quadrupole operator depending linearly on β is used in the collective models

$$\begin{aligned}\left(T_C^{(E2)}\right)_\mu &= t_C^{(2)} \alpha_{2,\mu} = t_C^{(2)} \beta \Big[\mathcal{D}^{(2)}_{\mu 0}(\theta_i) \cos\gamma \\ &\quad + \frac{1}{\sqrt{2}} \big(\mathcal{D}^{(2)}_{\mu 2}(\theta_i) + \mathcal{D}^{(2)}_{\mu -2}(\theta_i) \big) \sin\gamma \Big],\end{aligned} \tag{2.21}$$

where $t_C^{(2)}$ is a scale factor. However, the importance of the next term quadratic in β has been shown. Because of that we will include the next order in the quadrupole coordinates, which is given by [67]

$$\begin{aligned}\left(T_C^{(E2)}\right)_\mu &= t_C^{(2)} \left(\alpha_{2\mu} + \frac{\chi}{2\sqrt{7}} [\alpha_2 \times \alpha_2]^{(2)}_\mu \right) \\ &= t_C^{(2)} \sum_m D^{(2)*}_{\mu m} \bar{Q}^{(2)}_m,\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\bar{Q}^{(2)}_0 &= \beta\cos\gamma - \sqrt{\frac{1}{14}} \chi \beta^2 \cos(2\gamma), \\ \bar{Q}^{(2)}_2 &= Q^{(2)}_{-2} = \frac{1}{\sqrt{2}} \left[\beta \sin\gamma + \sqrt{\frac{1}{14}} \chi \beta^2 \sin(2\gamma) \right], \\ \bar{Q}^{(2)}_1 &= \bar{Q}^{(2)}_{-1} = 0.\end{aligned} \tag{2.23}$$

The $2\sqrt{7}$ denominator in Eq. (2.22) is introduced here for convenience to match the expression based on the interacting boson model (IBM) quadrupole operator. The χ parameter controls the weight of the quadratic relative to the linear term. The $\alpha_{2\mu}$ variable behaves as the $O(5) \supset O(3)$ tensor $(\tau, L) = (1, 2)$, while the second-order term $[\alpha \times \alpha]^{(2)}_\mu$ is associated with the tensor (2, 2) [67]. This means that they fulfill strict selection rules in the $\phi^{\tau\mu LM}(\gamma, \theta_i)$ collective space: $\Delta\tau_B = \pm 1$ for the former and $\Delta\tau_B = 0, \pm 2$ for the latter, with the usual restrictions in angular momentum coupling [31,67]. Thus the first term leads to "allowed" transitions in first order, and the second to nonzero quadrupole moments and to small values for "forbidden" ones, in the E(5) scheme of Ref. [4].

For the single-particle part, in general,

$$T_F^{(E2)} = \sum_{j,j'} t^{(2)}_{j,j'} (a^\dagger_j \times \tilde{a}_{j'})^{(2)}, \tag{2.24}$$

where $t^{(2)}_{j,j'}$ are parameters. To reduce the number of parameters, the following form dictated by the U(6/12) symmetry has been used:

$$\begin{aligned}T_F^{(E2)} &= t_F^{(2)} \Big[-\sqrt{\frac{4}{5}} (a^\dagger_{1/2} \times \tilde{a}_{3/2} + \text{h.c.})^{(2)} \\ &\quad - \sqrt{\frac{6}{5}} (a^\dagger_{1/2} \times \tilde{a}_{5/2} + \text{h.c.})^{(2)} \Big].\end{aligned} \tag{2.25}$$

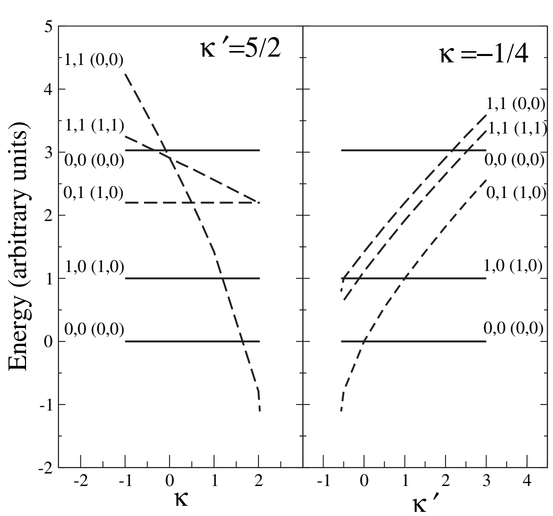

FIG. 4. Behavior of the few lowest energy E(5/12) bandheads as a function of parameters k and k' in the Hamiltonian.

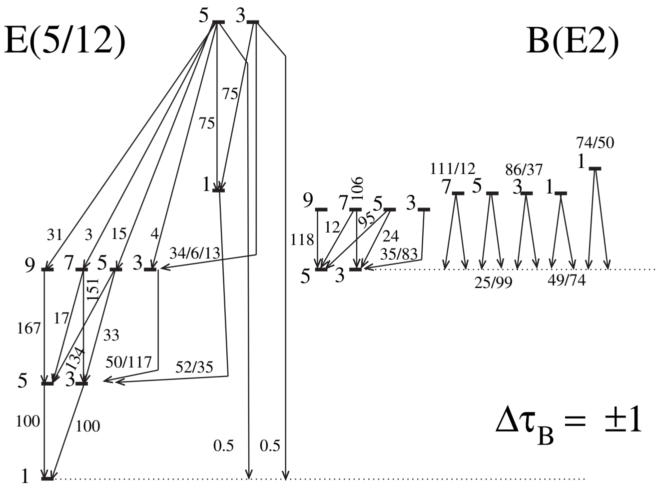

FIG. 5. $B(E2)$ strengths induced by the linear term of the collective $E2$ operator for the E(5/12) model. Spins given are twice the actual value. Transitions obey the selection rule $\Delta\tau_B = \pm1$. The transition $B(E2; 5/2_1 \rightarrow 1/2_1)$ is normalized to 100. To get absolute values, all the strengths have to be multiplied by $(0.07453)(t_C^{(2)})^2\beta_w^2$.

For the odd system, contributions from both collective and single-particle operators are to be calculated. In the upper panel of Fig. 3, some $B(E2)$ values are presented for the collective contribution and transitions between the largest spin in each degenerate multiplet. In Figs. 5 and 6, $B(E2)$ values for $\Delta\tau_B = \pm1$ and $\Delta\tau_B = 0, \pm2$ are presented. In the first case, only the linear term is effective; while in the second case, only the quadratic contribution plays a role.

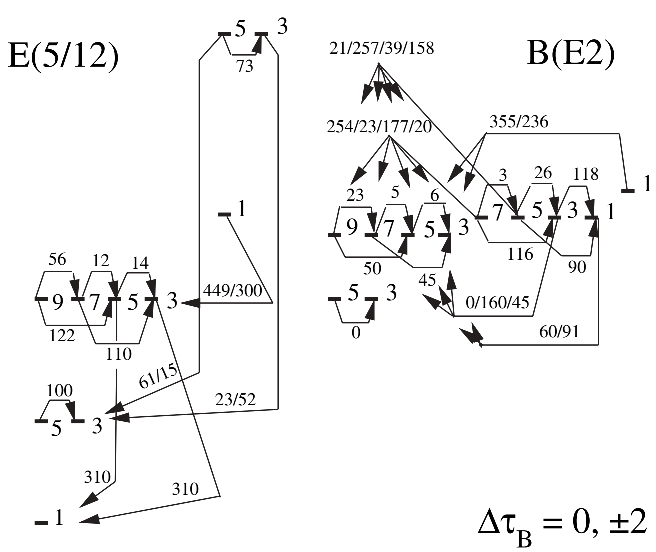

FIG. 6. $B(E2)$ strengths induced by the quadratic term of the collective $E2$ operator for the E(5/12) model. Spins given are twice the actual value. Transitions obey the selection rule $\Delta\tau_B = 0, \pm2$. The transition $B(E2; 5/2_1 \rightarrow 3/2_1)$ is normalized to 100. To get absolute values, all the strengths have to be multiplied by $(0.0007855)\chi^2(t_C^{(2)})^2\beta_w^4$.

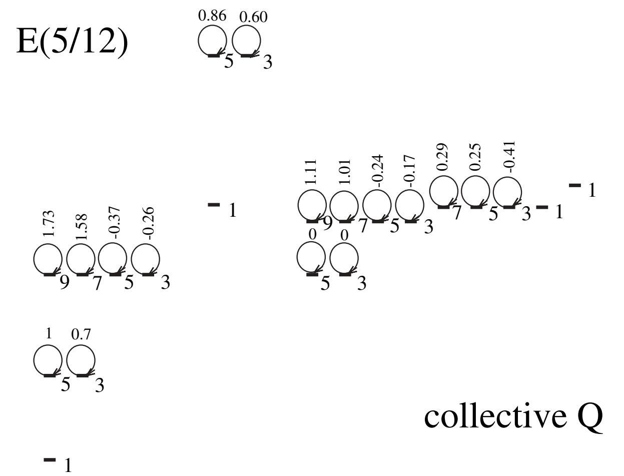

FIG. 7. Quadrupole moments induced by the collective quadratic term in $T^{(E2)}$ for the E(5/12) model. Spins given are twice the actual value. The quadrupole moment of the state $5/2_1$ is normalized to 1. To get absolute values, all the quadrupole moments have to be multiplied by $(0.12229)\chi t_C^{(2)}\beta_w^2$.

Collective quadrupole moments induced by the quadratic term in the $E2$ operator can be calculated as

$$
\begin{aligned}
&eQ(\xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}, 1/2; J) \\
&\quad = \sqrt{\frac{16\pi}{5}}\sqrt{\frac{J(2J-1)}{(J+1)(2J+1)(2J+3)}} \\
&\qquad \times \langle \xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}, 1/2; J||T_C^{(E2)} + T_F^{(E2)}|| \\
&\qquad \times \xi, \tau_B, \tau_F, (\tau_1, \tau_2), L_{BF}, 1/2; J\rangle. \qquad (2.26)
\end{aligned}
$$

In Fig. 7, the collective quadrupole moments induced by the quadratic term in $T_C^{(E2)}$ are presented normalized to the quadrupole moment of the $5/2_1$ state.

The fermionic quadrupole operator (2.25) also contributes to some transitions. However, the structure (2.25) provides a selection rule $\Delta\tau_F = \pm1$, and consequently there are no contributions of the fermion part to the quadrupole moments. For the same reason, there are no transitions where interference between the collective and the fermionic parts exists (for the collective part, $\Delta\tau_F = 0$). In addition, the fermionic $E2$ operator provides a selection rule $\Delta\tau_B = 0$. In Fig. 8, a few fermionic $E2$ transitions induced by the operator (2.25) are presented.

B. $M1$ transition and moments

The magnetic dipole operator can be written as the sum of the collective contribution plus the odd-particle one:

$$T^{(M1)} = T_C^{(M1)} + T_F^{(M1)}. \qquad (2.27)$$

The collective contribution is

$$T_C^{(M1)} = t_{C1}^{(1)}\frac{\hat{L}_B}{\sqrt{10}} + t_{C2}^{(1)}\left(\alpha_2 \times \frac{\hat{L}_B}{\sqrt{10}}\right)^{(1)} + \ldots. \qquad (2.28)$$

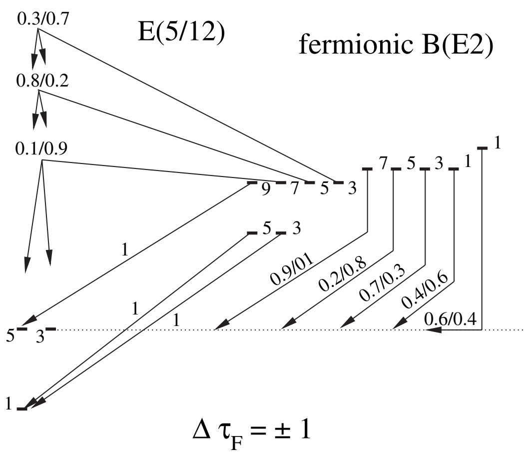

FIG. 8. $B(E2)$ strengths induced by the fermionic operator (2.25) in $T^{(E2)}$ for the E(5/12) model. Spins given are twice the actual value. The strength from the state $[\xi = 1, \tau_B = 0, \tau_F = 1, (\tau_1 = 1, \tau_2 = 0)L_{BF} = 2, J = 5/2]$ to the ground state is 1. To get absolute values, all strengths have to be multiplied by $(t_F^{(2)})^2$.

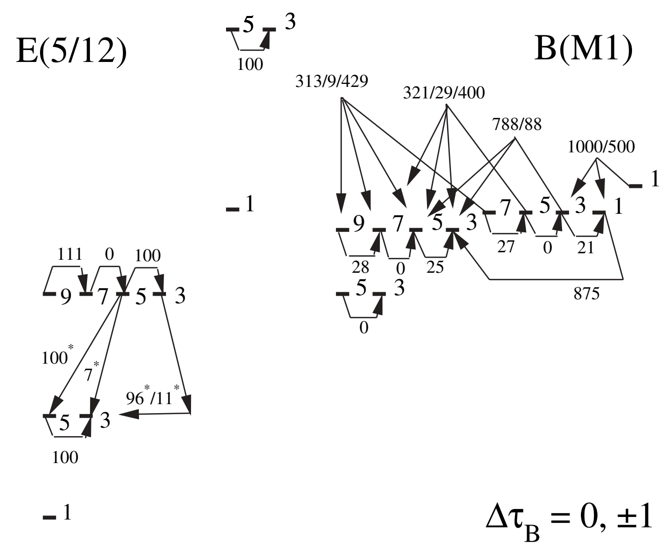

FIG. 9. B(M1) transition strengths induced by the collective $M1$ operator (2.28) for the E(5/12) model. Spins given are twice the actual value. Both $\Delta\tau_B = 0$ and $\Delta\tau_B = \pm 1$ transitions are included. The $\Delta\tau_B = 0$ transitions are normalized to the $B(M1; 5/2_1 \to 3/2_1) = 100$. In order to get absolute values $B(M1)$ with $\Delta\tau_B = 0$ have to be multiplied by $(0.04)(t_{C1}^{(1)})^2$. The $\Delta\tau_B = \pm 1$ are marked with an $*$ and are normalized to the $B(M1; 5/2_{2_g} \to 5/2_1) = 100$ ($5/2_{2_g}$ is the second 5/2 within the ground state band). To get absolute values, $B(M1)$ with $\Delta\tau_B = \pm 1$ have to be multiplied by $(0.02445)(t_{C2}^{(1)})^2$.

The first term is proportional to the collective angular momentum $\hat{L}_B$ and is diagonal for the even-even core but not for the odd system. The transition strengths and dipole moments arising from this term are presented in Figs. 9 and 10.

The fermionic term is proportional to the fermion angular momentum j. For the case we are studying, the appropriate operator is

$$T_F^{(M1)} = -t_F^{(1)}\left[\sqrt{\frac{1}{2}}(a_{1/2}^{\dagger} \times \tilde{a}_{1/2})^{(1)} + \sqrt{5}(a_{3/2}^{\dagger} \times \tilde{a}_{3/2})^{(1)} + \sqrt{\frac{35}{2}}(a_{5/2}^{\dagger} \times \tilde{a}_{5/2})^{(1)}\right]. \tag{2.29}$$

The fermionic magnetic dipole operator (2.29) also contributes to some transitions with the selection rule $\Delta\tau_B = 0$. Since this selection rule is identical for this part and the collective part of the dipole operator, there will be interference between both terms. Table II presents combined strengths for selected transitions in which interference between collective and single-particle terms are important.

Figures 3–10 and Table II give detailed spectroscopic information for the E(5/12) model. In the next two sections, calculations at the critical point in the transition from spherical

TABLE II. $T^{(M1)}$ reduced matrix elements for $B(M1)$ and magnetic moments involving interference effects between collective and single-particle contributions for some selected E(5/12) states.

Initial $\lvert i\rangle$ $\lvert\xi, \tau_B, \tau_F(\tau_1, \tau_2)L_{BF}, J\rangle$	Final $\lvert f\rangle$ $\lvert\xi, \tau_B, \tau_F(\tau_1, \tau_2)L_{BF}, J\rangle$	Reduced matrix element $\langle f\lvert\lvert T^{(M1)}\rvert\rvert i\rangle$
1,1,0 (1,0)2,5/2	1,1,0 (1,0)2,3/2	$2\sqrt{\frac{3}{5}}t_F^{(1)} - 0.4899t_{C1}^{(1)}$
1,1,0 (1,0)2,5/2	1,1,0 (1,0)2,5/2	$\sqrt{\frac{21}{10}}t_F^{(1)} + 1.833t_{C1}^{(1)}$
1,1,0 (1,0)2,3/2	1,1,0 (1,0)2,3/2	$-\sqrt{\frac{3}{5}}t_F^{(1)} + 1.4697t_{C1}^{(1)}$
1,2,0 (2,0)4,9/2	1,2,0 (2,0)4,7/2	$2\frac{\sqrt{10}}{3}t_F^{(1)} - 0.6667t_{C1}^{(1)}$
1,2,0 (2,0)2,5/2	1,2,0 (2,0)2,3/2	$2\sqrt{\frac{3}{5}}t_F^{(1)} - 0.4899t_{C1}^{(1)}$
1,2,0 (2,0)4,9/2	1,2,0 (2,0)4,9/2	$\sqrt{\frac{55}{18}}t_F^{(1)} + 4.4222t_{C1}^{(1)}$
1,2,0 (2,0)4,7/2	1,2,0 (2,0)4,7/2	$-\frac{\sqrt{14}}{3}t_F^{(1)} + 3.9441t_{C1}^{(1)}$
1,2,0 (2,0)2,5/2	1,2,0 (2,0)2,5/2	$\sqrt{\frac{21}{10}}t_F^{(1)} + 1.8330t_{C1}^{(1)}$
1,2,0 (2,0)2,3/2	1,2,0 (2,0)2,3/2	$-\sqrt{\frac{3}{5}}t_F^{(1)} + 1.4697t_{C1}^{(1)}$

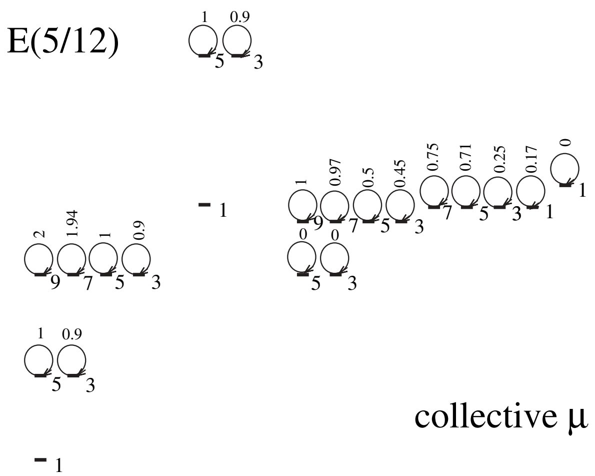

FIG. 10. Magnetic moments induced by the leading collective term in $T^{(M1)}$ for the E(5/12) model. Spins given are twice the actual value. The magnetic moment of the state $5/2_1$ is normalized to 1. To get absolute values, all the magnetic moments have to be multiplied by $(1.29442)t_{C1}^{(1)}$.

to deformed γ-unstable shapes within the IBFM framework are presented. Two different IBFM Hamiltonians for describing the critical point are used in order to see whether the signs of criticality produced by E(5/12) are robust.

Before ending this section, we would like to note that our E(5/12) model yields an infinite number of levels, it corresponds to solving the Bohr equation for a γ-independent infinite square well in β potential. This will not be the case for the IBFM calculations, which are associated with a finite number of bosons and consequently characterized by a finite number of levels. In this latter case, in addition, the spectrum itself will change according to the chosen number of bosons, in particular, its higher part. The comparison between the two approaches will therefore necessarily be significant only for the lower part of the spectrum as in the previous E(5/4) model for odd-even nuclei, in the E(5), X(5) models for even-even nuclei concerning critical points, or, in general, when comparing the collective model with the IBM.

III. AN INTERACTING BOSON-FERMION MODEL HAMILTONIAN THAT MIMICS E(5/12) AT THE CRITICAL POINT

For the odd-even system, the IBFM Hamiltonian is written as

$$H = H_B + H_F + V_{BF}, \tag{3.1}$$

where H_B is the bosonic part, H_F the fermionic one, and the term V_{BF} couples the boson and fermion degrees of freedom. For the bosonic part, we will use a parametrized Hamiltonian that produces a transition between spherical and γ-unstable shapes of the form

$$H_B = x\hat{n}_d - \frac{1-x}{N}\hat{Q}_B \cdot \hat{Q}_B, \tag{3.2}$$

where the operators in the Hamiltonian are given by

$$\hat{n}_d = \sum_\mu d^\dagger_\mu d_\mu, \tag{3.3}$$

$$\hat{Q}_B = (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}, \tag{3.4}$$

and N is the total number of bosons. This Hamiltonian can be recast into the form

$$H_B = xC_1(\mathrm{U}^B(5)) - \frac{1-x}{2N}\left[C_2(\mathrm{O}^B(6)) - C_2(\mathrm{O}^B(5))\right], \tag{3.5}$$

where $C_1(\mathrm{U}^B(5))$ is the linear Casimir operator of the $\mathrm{U}^B(5)$ group, while $C_2(\mathrm{O}^B(6))$ and $C_2(\mathrm{O}^B(5))$ are the quadratic Casimir operators of the $\mathrm{O}^B(6)$ and $\mathrm{O}^B(5)$ groups. We recall that this Hamiltonian produces, when varying the parameter x from 1 to 0, a phase transition between the two extreme situations characteristic of U(5) and O(6) symmetries, at $x_c = \frac{4N-8}{5N-8}$. Note that for any value of x this Hamiltonian maintains the degeneracies typical of the O(5) symmetry. Consistent with this, within the IBM coherent state formalism [32–34], this Hamiltonian always produces an energy surface which is independent of the γ degree of freedom. In the β variable, the energy surfaces display a spherical minimum in $\beta = 0$ for x larger than the critical value, while having a deformed minimum for values of x smaller than the critical values [4,41,58]. It is worth mentioning that when considering the combined boson-fermion system, the energy surface has contributions from both bosons and fermions. The contribution from the fermion is of the order $1/N$ compared with the one from the bosons. Thus, usually it is accepted that the energy surface for the combined system is governed by the bosons with slight contributions from the fermionic degrees of freedom. However, close to the critical point, the boson energy surface is rather flat, and the contribution from the odd fermion and its coupling to the boson core will play a fundamental role in determining the minimum in the potential energy surface. This is an important topic that remains to be investigated in detail. For the purposes of the present paper, i.e., checking the analytical results obtained for the E(5/12) model, we will accept that the critical point of the combined system is just the critical point of the boson part and the fermion is coupled to it without changing the equilibrium configuration.

Depending on the selection of $H_F + V_{BF}$ in Eq. (3.1), different model Hamiltonians to study the critical point are produced. In this section $H_F + V_{BF}$ is selected so as to have a IBFM Hamiltonian as close as possible to the E(5/12) one. Thus, our proposal is

$$H_F + V_{BF} = \frac{k}{2N}[C_2(\mathrm{O}^{BF}(5)) - C_2(\mathrm{O}^B(5)) - C_2(\mathrm{O}^F(5))] + \frac{k'}{2N}C_2(\mathrm{O}^F(5)), \tag{3.6}$$

where $C_2(\mathrm{O}^F(5))$, $C_2(\mathrm{O}^{BF}(5))$ are the quadratic Casimir operators of the $\mathrm{O}^F(5)$ and $\mathrm{O}^{BF}(5)$ algebras, respectively. As mentioned above this Hamiltonian is designed to mimic as much as possible the corresponding Hamiltonian in E(5/12), since the terms $2\hat{\mathcal{L}}_B \circ \hat{\mathcal{L}}_F$ and $\hat{\mathcal{L}}_F^2$ in Eq. (2.7) can be written in

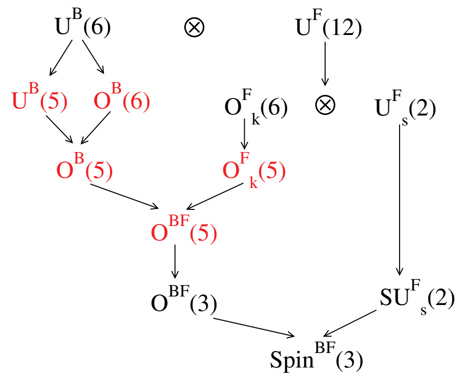

FIG. 11. (Color online) Lattice of algebras involved in the IBFM Hamiltonian (3.6).

terms of Casimir operators as $2\hat{\mathcal{L}}_B \circ \hat{\mathcal{L}}_F = (1/2)[C_2(\mathrm{O}^{BF}(5)) - C_2(\mathrm{O}^B(5)) - C_2(\mathrm{O}^F(5))]$ and $\hat{\mathcal{L}}_F^2 = (1/2)C_2(\mathrm{O}^F(5))$. This boson-fermion Hamiltonian preserves by construction the τ_B and τ_F quantum numbers, and all states are therefore characterized by the same quantum numbers and degeneracies as in the E(5/12) case. For a better clarification, we show in Fig. 11 the lattice of algebras relevant to our problem.

The corresponding spectrum is given in the lower frame of Fig. 3. A number $N = 7$ of bosons is assumed as a typical value. Other choices of N do not alter the qualitative behavior of our results. With the choice $N = 7$ the critical point occurs at $x_{\mathrm{crit}} = \frac{4N-8}{5N-8} = \frac{20}{27} \approx 0.74$. In the Hamiltonian, the values for k and k' have been assumed to be equal to the corresponding values in the E(5/12) case presented in the upper frame.

The spectrum has a structure that clearly resembles the one obtained within the E(5/12) model (upper frame). Energies of selected states in the two models are compared in Table III. As noted before, in the E(5/12) model all states with $\tau_F = 0$ (i.e., with the odd particle in the $j = 1/2$ state) have precisely the same energies as the states in the even E(5) model. Similarly, in the IBFM these states have the same energy as the corresponding states in the boson Hamiltonian (3.2). The agreement for the energies of these states in E(5/12) and IBFM is therefore that already discussed in Ref. [41] for the bosonic part. Excited bands with $\tau_F = 1$ (i.e., with the odd particle

TABLE III. Energies of selected states in E(5/12) and IBFM (normalized to the first excited multiplet). The values $k = -1/4$ and $k' = 5/2$ have been used in both Hamiltonians (2.7) and (3.1)–(3.6). In the IBFM calculation, the number of bosons $N = 7$ has been used.

Level $\xi, \tau_1^B, \tau_1^F(\tau_1, \tau_2)$	Energy E(5/12)	Energy IBFM
1,1,0 (1,0)	1.00	1.00
1,2,0 (2,0)	2.20	2.30
2,0,0 (0,0)	3.03	2.86
1,0,1 (1,0)	2.20	2.69
1,1,1 (1,1)	3.00	3.82

in the $j = 3/2, 5/2$ states), on the other hand, seem to be somewhat compressed in E(5/12) with respect to the IBFM.

The same kind of agreement is also present in the electromagnetic transitions. Consistent with the choice presented in the E(5/12), we assume in the IBFM as $E2$ transition operator just the sum of the boson (collective) plus the fermion (single particle) quadrupole operators, i.e., $\hat{T}^{E2} = \hat{Q}_B + \hat{q}_F$, where $\hat{q}_F$ is given by Eq. (2.25). The order of magnitude of the different $B(E2)$ values are reflected in different thicknesses and styles of the arrows shown in Fig. 3. Given the choice of the IBFM Hamiltonian (3.6), the same selection rules valid in the E(5/12) model also apply to the $(\tau_B, \tau_F, \tau_1, \tau_2)$ quantum numbers of initial and final states in the IBFM approach. The $B(E2)$ values of selected transitions are explicitly given in the figure. All $B(E2)$ values are normalized to 100 for the first 5/2-1/2 transition. Similarly to what happens for the energies of the levels, the overall behavior is very similar to that of the E(5/12).

In Figs. 12, 13, and 14 the IBFM results for collective $E2$ transitions and moments are presented and should be compared to Figs. 5, 6, and 7 with the E(5/12) results. All of them are remarkably similar. For the fermionic contribution to the $E2$ transitions, the IBFM and E(5/12) results are identical. Also for $M1$ transitions and moments, both collective and fermionic contributions are identical in the E(5/12) and in the IBFM calculations presented in this section. For the collective part, this is because only the first term in Eq. (2.28) has been used. This term is just proportional to the boson orbital angular momentum. In addition, the states included in Figs. 12–14 have isoscalar factors equal to 1. Thus, for them, the boson orbital angular momentum $\hat{L}_B$ is a good quantum number.

The conclusion of this section is that the IBFM Hamiltonian (3.1) taking Eq. (3.2) for the boson part and Eq. (3.6) for the fermion part and the boson-fermion interaction produces results that are strongly similar to those from the E(5/12) model.

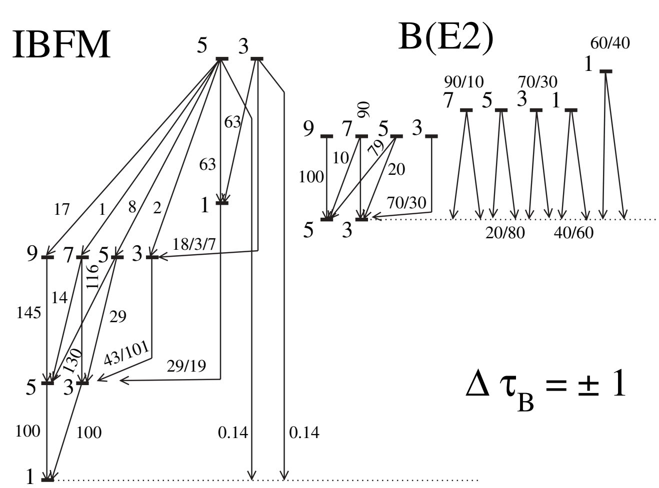

FIG. 12. $B(E2)$ strengths induced by the linear term of the collective $E2$ operator for the IBFM model. Spins given are twice the actual value. Transitions obey the selection rule $\Delta\tau_B = \pm 1$. The transition $B(E2; 5/2_1 \rightarrow 1/2_1)$ is normalized to 100.

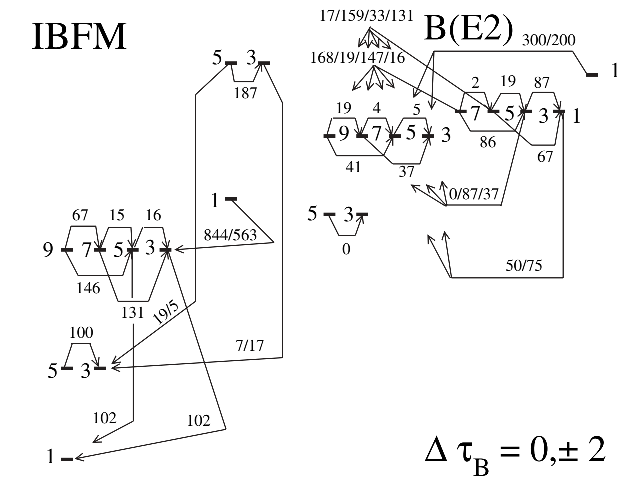

FIG. 13. $B(E2)$ strengths induced by the quadratic term of the collective $E2$ operator for the IBFM model. Spins given are twice the actual value. Transitions obey the selection rule $\Delta\tau_B = 0, \pm 2$. The transition $B(E2; 5/2_1 \rightarrow 3/2_1)$ is normalized to 100.

IV. THE INTERACTING BOSON-FERMION MODEL ALONG THE U(5)-O(6) LINE WITH COUPLING TO A $j = 1/2, 3/2, 5/2$ PARTICLE

In the preceding section, we introduced an IBFM Hamiltonian designed particularly to match the E(5/12) Hamiltonian. In this section, we will present calculations with the Hamiltonian used in many previous works for studying the U(5)-O(6) transition in even-even nuclei (3.2). Then, we will select the critical point in this transition and couple the odd fermion with allowed $j = 1/2, 3/2, 5/2$ to it using a more realistic quadrupole-quadrupole boson-fermion interaction. This is an alternative way of checking the E(5/12) results.

In this case, for the fermion and boson-fermion parts in Eq. (3.1), assuming a particle moving in the single-particle shells $j = 1/2, 3/2$ and $5/2$, we take the form

$$H_F + V_{BF} = \sum_j \epsilon_j a_j^\dagger \cdot \tilde{a}_j - 2\frac{1-x}{N}\hat{Q}_B \cdot \hat{q}_F. \qquad (4.1)$$

The fermion quadrupole operator $\hat{q}_F$ is defined as in Eq. (2.25), $\hat{q}_F = T_F^{(E2)}$, while the single-particle energies are $\epsilon_{1/2} = 0, \epsilon_{3/2} = \epsilon_{5/2} = x + (4-4x)/N$. With these choices, the total Hamiltonian (3.1) can be put into a form formally equivalent to that given in Eq. (3.5) for the boson part, namely,

$$H_{BF} = xC_1(\mathrm{U}^{BF}(5)) - \frac{1-x}{2N}[C_2(\mathrm{O}^{BF}(6)) - C_2(\mathrm{O}^{BF}(5))]. \qquad (4.2)$$

This choice of the fermion space is such that one can usefully visualize the three angular momenta as arising from the coupling of a pseudospin 1/2 with a pseudo-orbital angular momentum 0 or 2. One recovers, in the extreme cases, the Bose-Fermi symmetry [63] associated with $\mathrm{O}^{BF}(6) \otimes U_s^F(2)$ (for $x = 0$) and the vibrational $\mathrm{U}^{BF}(5) \otimes U_s^F(2)$ (for $x = 1$). The factorization of the pseudospin part gives rise to repeated level couplets. But in addition, we have degeneracies coming from the bosonic plus fermionic orbital parts. In fact, in analogy with the overall $\mathrm{O}^B(5)$ structure in the even case, this selection guarantees the preservation of all the degeneracies associated with the $\mathrm{O}^{BF}(5)$ symmetry for any value of x. Our states are therefore characterized (leaving aside the noninfluent coupling to the intrinsic pseudospin 1/2, leading to the total J_{BF}) by the quantum numbers $[N_1, N_2]$ [from the $\mathrm{U}^{BF}(6)$ algebra], (τ_1, τ_2) [from $\mathrm{O}^{BF}(5)$], and L_{BF} [from $\mathrm{O}^{BF}(3)$ [63]. These states are in general a superposition of states of the uncoupled boson-fermion basis, i.e., states of the form $|\tau_B, L_B\rangle \otimes |\tau_F, L_F\rangle$. Since we have only one active fermion, we necessarily have $\tau_F = 0$ and $L_F = 0$ if the fermion is in the $j = 1/2$ shell, while $\tau_F = 1$ and $L_F = 2$ if the fermion is in the 3/2-5/2 shell. Only, in the limiting case of the $\mathrm{O}^{BF}(6)$ symmetry, the states are also characterized by the additional $\mathrm{O}^{BF}(6)$ quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$.

The evolution of the spectrum with the control parameter x is shown in Fig 15. A number of bosons $N = 7$ has been assumed. Energies are normalized to the energy difference between the first excited state (which is always a 3/2-5/2 doublet) and the $J_{BF} = 1/2$ ground state. Each state is characterized by the (τ_1, τ_2) quantum numbers. Most of the states are degenerate. The corresponding allowed angular momenta for each state (τ_1, τ_2) are given in Table I. We give, for each state, the asymptotic quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$ at the $\mathrm{O}^{BF}(6)$ extreme.

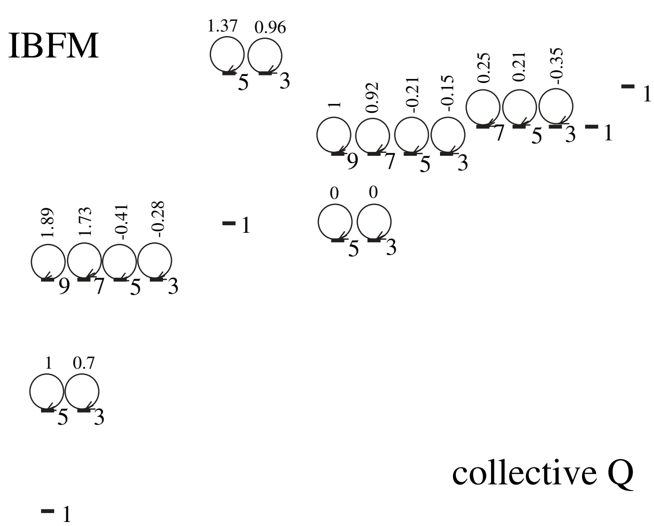

FIG. 14. Quadrupole moments induced by the quadratic term of the collective $E2$ operator for the IBFM model. Spins given are twice the actual value. The quadrupole moment of the $5/2_1$ is normalized to 1.

A. The interacting boson-fermion model at the critical point

In Fig. 15, the vertical line at $(1-x) = \frac{N}{5N-8} = \frac{7}{27} \approx 0.26$ gives the position of the critical point. The corresponding spectrum is expanded and compared with E(5/12) in Fig. 16. Each IBFM state is characterized by the $[N_1, N_2](\tau_1, \tau_2)$ quantum numbers and arranged in bands according to the the asymptotic quantum number $(\sigma_1, \sigma_2, \sigma_3)$.

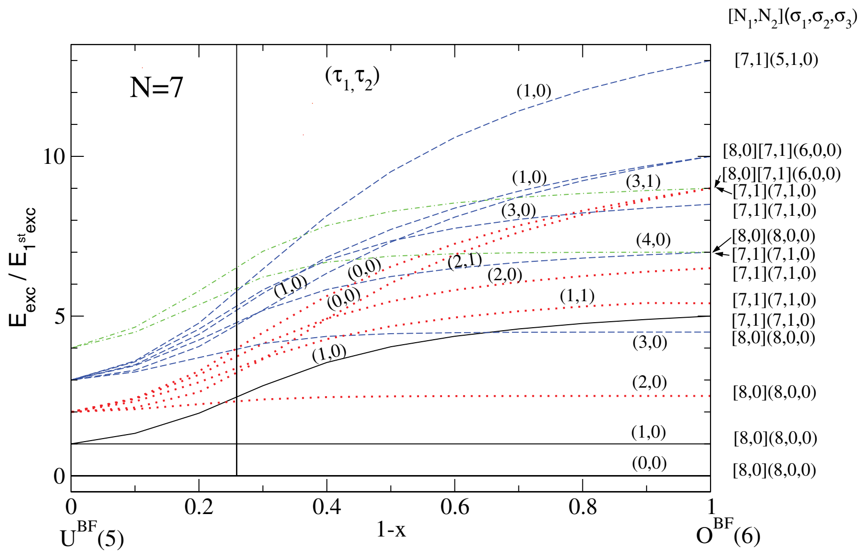

FIG. 15. (Color online) Energy levels (normalized to the energy of the first excited state) as a function of the control parameter x in the Hamiltonian. The number $N = 7$ of bosons has been assumed. Each state is characterized by the (τ_1, τ_2) quantum numbers. The corresponding allowed angular momenta for each state (τ_1, τ_2) are listed in Table I. We also give, for each state, the asymptotic quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$ at the $O^{BF}(6)$ extreme.

The values of the angular momenta for each degenerate state can be extracted from Table I.

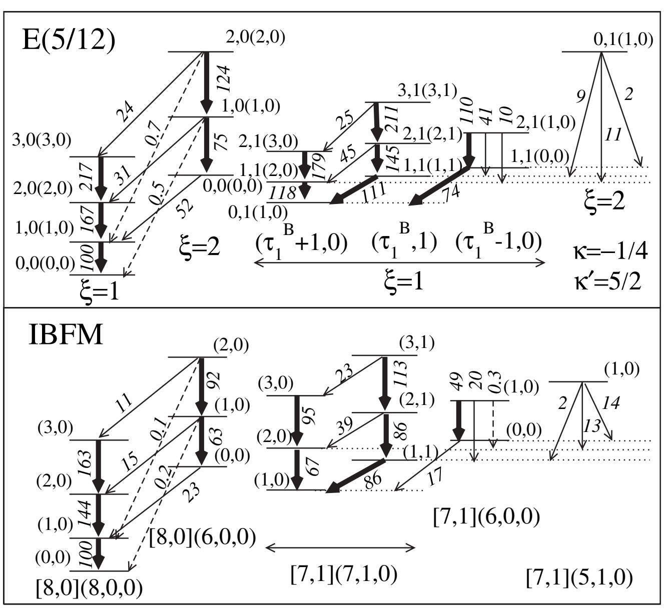

FIG. 16. Upper panel: E(5/12) results as explained in Fig. 3. Lower panel: Energy levels in the odd system at the critical U(5)-O(6) point with the Hamiltonian used in Sec. IV. The number $N = 7$ of bosons has been assumed. Each state is characterized by the $[N_1, N_2](\tau_1, \tau_2)$ quantum numbers and arranged in bands according to the asymptotic quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$. Thus, numbers below each band are $[N_1, N_2](\sigma_1, \sigma_2, \sigma_3)$. The labels (τ_1, τ_2) are beside each level. The corresponding allowed angular momenta for each state (τ_1, τ_2) are listed in Table I. Some selected values of $B(E2)$, corresponding to the transitions between the highest spins of initial and final multiplets, are given in the figure. Wide solid arrows indicate strong $B(E2)$ matrix elements [$B(E2) > 55$, normalizing to 100 the $B(E2)$ associated with the lowest 5/2-1/2 transition], thin solid arrows weaker [$55 > B(E2) > 1$], and dashed arrows very weak (although not vanishing) transitions [$B(E2) < 1$].

The members of the ($J_{BF} = 1/2$) ground state band $[N_1 = N + 1, 0]$, characterized by O(5) quantum numbers $(\tau_1, 0)$, are mainly originated from the coupling of the $\tau_B = \tau_1$ ground band in the even system with the $\tau_F = 0$ fermion state, i.e., with the odd particle in the $j = 1/2$ orbital. As examples, this component amounts to about 90%, 80%, and 70% for the (0,0), (1,0), and (2,0) states. In a similar way, the excited band constructed over the excited (0,0), $J_{BF} = 1/2$ state comes in first approximation from the coupling of the $\tau_B = 0$ boson excited band with the $j = 1/2$ particle. This component is again predominant, of the order of 73%. All these bands belonging to the $[N_1 = N + 1, 0]$ representation are therefore somehow similar (both in energies and electromagnetic transitions) to the bands in the even core at the critical point, with the weak and not influent coupling of the $j = 1/2$ particle (with its pseudo-orbital angular momentum $L_F = 0$).

The $[N_1 = N, N_2 = 1]$ band ([7,1] in our specific case) based on the 3/2-5/2 doublet is instead approximately described as arising from the coupling of the $\tau_B = \tau_1$ ground band in the even system with the $\tau_F = 1$ fermion state, i.e., corresponds to having the odd particle moving in either the $j = 3/2$ or 5/2 orbital. This component amounts to, for example, 86% and 73% for the (1,0) and (2,0) multiplets, being 100% for the states $(1, 1), (2, 1), \ldots, (\tau_1, 1)$, whose structure remains unchanged for all values of x. Finally the percentage in the state (0,0) is 71%. Note that states belonging to the $[N_1 = N, N_2 = 1]$ representation are not present in the even case and are therefore genuine outcomes of the boson-fermion coupling.

The E(5/12) spectrum has a structure that clearly resembles the one obtained within the IBFM approach (Fig. 16). The states are labeled according to the quantum numbers $\xi, \tau_B, \tau_F, (\tau_1, \tau_2)$, but are easily put in correspondence with the IBFM states. One should recall that each state actually represents a multiplet of degenerate levels, whose angular momenta can be extracted from Table I. Energies of selected states in the two models are compared in Table IV. Note that all states with $\tau_F = 0$ have energies that are precisely the same

TABLE IV. Energies of selected states in the IBFM and E(5/12). The quantum numbers characterizing the corresponding states in the two models are separately given. The values $k = -1/4$, $k' = 5/2$ have been used in the Hamiltonian (2.7). In the IBFM calculation, with the Hamiltonian (4.2) at the critical point [$x = (4N - 8)/(5N - 8)$], the number $N = 7$ of bosons has been used.

IBFM		E(5/12)	
Level $[N_1, N_2], (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2)$	Energy	Level $\xi, \tau_B, \tau_F(\tau_1, \tau_2)$	Energy
[8,0] (8,0,0) (1,0)	1.00	1,1,0 (1,0)	1.00
[8,0] (8,0,0) (2,0)	2.34	1,2,0 (2,0)	2.20
[8,0] (6,0,0) (0,0)	3.17	2,0,0 (0,0)	3.03
[7,1] (7,1,0) (1,0)	2.47	1,0,1 (1,0)	2.20
[7,1] (7,1,0) (1,1)	3.38	1,1,1 (1,1)	3.00

as those of the states in the even E(5) model and close to those of the IBFM critical point. Excited bands with $\tau_F = 1$, on the other hand, seem to be somewhat compressed with respect to the other model. Note that, as in the E(5/4) case, the counterpart for having an easy analytic solution to the problem is the fact that one deals here with eigenstates with good values of τ_B and τ_F, while in the more realistic IBFM case presented in this section, we have seen that this is only approximately true.

The nature of the different states is also reflected in the electromagnetic transitions. Consistent with the choice in the Hamiltonian, we assume as total quadrupole operator the sum of the boson and fermion quadrupole operators, as before. The order of magnitude of the different matrix elements is reflected in different thicknesses and styles of the arrows shown in Fig. 16. Wide solid arrows indicate strong $B(E2)$, thin solid arrows weaker transitions, and dashed arrows very weak (although not vanishing) transitions. Remember that most levels are multiple degenerate, so each arrow is in fact representative of the average value of a bunch of transitions. Given the $O^{BF}(5)$ nature of our system, selection rules apply to the (τ_1, τ_2) quantum numbers of initial and final states. Allowed transitions satisfy either $|\Delta\tau_1| = 1, \Delta\tau_2 = 0$ or $\Delta\tau_1 = 0, |\Delta\tau_2| = 1$. Since our Hamiltonian does not have the structure of the algebra $O^{BF}(6)$ [because of the appearance of the symmetry-breaking Casimir of $U_{BF}(5)$], the $(\sigma_1, \sigma_2, \sigma_3)$ quantum numbers are only approximately valid, and interband transitions are allowed between bands with different asymptotic values, as, for example, between states of bands with asymptotic numbers (8,0,0) and (6,0,0). No transitions, finally, are allowed between states belonging to

TABLE V. $B(E2)$ and reduced matrix elements of the quadrupole operator for the indicated transitions. All $B(E2)$ values are normalized to 100 for the first 5/2-1/2 transition. States are labeled by the "asymptotic" quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$, as well as the good quantum numbers $[N_1, N_2], (\tau_1, \tau_2), L_{BF}, J_{BF}$. For the system at the critical point, the contributions to the reduced matrix element from the boson and fermion components in the quadrupole operator are separately shown. The second column gives the values obtained for the analogous transitions in the $O^{BF}(6)$ limit. The third and sixth columns give the values obtained in the E(5/12) model (normalized to the first transition in the IBFM case). In the E(5/12) case, only the collective part has been used for the transition operator. Note also that in the E(5/12) model the states are not characterized by the quantum numbers given in the first column, but they can be clearly identified by comparing Figs. 3 and 16.

Transition	$B(E2)$			Reduced matrix element		
$[N_1, N_2], (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), L_{BF}, J_{BF} \rightarrow$ $[N_1', N_2'], (\sigma_1', \sigma_2', \sigma_3'), (\tau_1', \tau_2'), L_{BF}', J_{BF}'$	Critical point IBFM	$O^{BF}(6)$ limit IBFM	E(5/12)	Critical point (Boson)	Critical point (Fermion)	E(5/12) (Coll.)
[8, 0], (8, 0, 0), (1, 0), 2, 5/2 → [8, 0], (8, 0, 0), (0, 0), 0, 1/2	100	100	100	8.3	1.2	8.3
[8, 0], (8, 0, 0), (2, 0), 4, 9/2 → [8, 0], (8, 0, 0), (1, 0), 2, 5/2	144	136	167	12.9	1.8	13.9
[8, 0], (6, 0, 0), (0, 0), 0, 1/2 → [8, 0], (8, 0, 0), (1, 0), 2, 5/2	23	0	52	2.3	0.3	3.5
[7, 1], (7, 1, 0), (2, 0), 4, 9/2 → [7, 1], (7, 1, 0), (1, 0), 2, 5/2	66	67	118	10.8	−0.8	11.6
[7, 1], (7, 1, 0), (1, 1), 3, 7/2 → [7, 1], (7, 1, 0), (1, 0), 2, 5/2	86	94	111	−9.3	−0.9	10.1
[7, 1], (6, 0, 0), (0, 0), 0, 1/2 → [7, 1], (7, 1, 0), (1, 0), 2, 5/2	17	0	74	−3.2	0.9	4.1

$[N_1 = N+1, 0]$ and $[N_1 = N, N_2 = 1]$ representations, since the quadrupole operator is a generator of $U^{BF}(6)$.

Selected transitions are reported in Table V. All $B(E2)$ values are normalized to 100 for the first 5/2-1/2 transition. For a better illustration we separately quote the contributions at the critical point to the reduced matrix element coming from the boson and fermion parts of the quadrupole operator. For comparison, we also report in the second column the values for the corresponding $B(E2)$ in the $O^{BF}(6)$ limit. The order of magnitude of the transitions at the critical point are the same as at the $O^{BF}(6)$ case. Transitions that are forbidden in that limit remain small at the critical point. Furthermore, in all cases the boson quadrupole matrix element by far exceeds the fermion contribution.

V. SUMMARY AND CONCLUSIONS

In this paper, we have considered within the Bohr Hamiltonian the case of the coupling of an odd particle moving in the $j = 1/2$, 3/2, and 5/2 orbitals to a boson core at the critical point from spherical to γ-unstable shapes. This critical point is modeled as an infinite square well in the deformation parameter β. We have deduced the energy spectrum as well as electromagnetic transitions and moments ($E2$ and $M1$). We have designed an IBFM Hamiltonian to mimic the situation studied in the E(5/12) model. A consistent picture between IBFM numerical calculations and E(5/12) analytical results is obtained for spectrum and electromagnetic transitions and moments. In addition, we have considered within the IBFM the case of the coupling of an odd particle moving in the $j = 1/2, 3/2$, and 5/2 orbitals to a boson core undergoing a transition from U(5) to O(6) situations via a more realistic quadrupole-quadrupole boson-fermion interaction. We have followed along the transition the evolution of the spectrum and studied in detail energies and transitions in correspondence to the critical point. Results of these additional IBFM calculations with the E(5/12) model show also a remarkable agreement, giving robust indication of criticality in odd-even nuclei. What have been taken as benchmarks for criticality in the even-even case are energy and $B(E2)$ ratios between a few low-lying energy levels. These can be used too for odd-even systems and can be extracted for the E(5/12) model from Tables IV and V and Figs. 5, 7, 9, and 10. Thus, the results presented here can be used as benchmarks for identifying odd-even nuclei close to the critical point in the transition from spherical to γ-unstable shapes. Good candidates for looking for critical nuclei close to E(5/12) are the odd-neutron Pt isotopes in which the relevant active orbitals around the Fermi surface are $3p_{1/2}$, $3p_{3/2}$, and $2f_{5/2}$; odd-proton Ir isotopes with active orbits $2d_{5/2}$, $2d_{3/2}$, and $3s_{1/2}$; and isotopes in the region around $A = 130$ (e.g., odd-neutron Ba isotopes with relevant orbits $2d_{5/2}$, $2d_{3/2}$, and $3s_{1/2}$).

ACKNOWLEDGMENTS

Andrea Vitturi acknowledges financial support from the Spanish Ministerio de Educación y Ciencia for a sabbatical year at the University of Seville. This work has been partially supported by the Spanish Ministerio de Educación y Ciencia, the European regional development fund (FEDER) under Project No. FIS2005-01105, and the INFN.

[1] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, Reading, MA, 1975), vol. 2.
[2] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).
[3] D. R. Bès, Nucl. Phys. **10**, 373 (1959).
[4] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
[5] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).
[6] F. Iachello, Phys. Rev. Lett. **91**, 132502 (2003).
[7] M. A. Caprio and F. Iachello, Nucl. Phys. **A781**, 26 (2007).
[8] J. M. Arias, Phys. Rev. C **63**, 034308 (2001).
[9] M. A. Caprio, Phys. Rev. C **65**, 031304(R) (2002).
[10] M. A. Caprio, Phys. Rev. C **69**, 044307 (2004).
[11] L. Fortunato and A. Vitturi, J. Phys. G: Nucl. Part. Phys. **30**, 627 (2004).
[12] L. Fortunato, Phys. Rev. C **70**, 011302(R) (2004).
[13] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85**, 3584 (2000).
[14] N. V. Zamfir *et al.*, Phys. Rev. C **65**, 044325 (2002).
[15] R. M. Clark *et al.*, Phys. Rev. C **69**, 064322 (2004).
[16] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87**, 052503 (2001).
[17] R. Krücken *et al.*, Phys. Rev. Lett. **88**, 232501 (2002).
[18] R. M. Clark *et al.*, Phys. Rev. C **68**, 037301 (2003).
[19] D. Tonev *et al.*, Phys. Rev. C **69**, 034334 (2004).
[20] E. A. McCutchan *et al.*, Phys. Rev. C **69**, 024308 (2004).
[21] E. A. McCutchan *et al.*, Phys. Rev. C **71**, 024309 (2005).
[22] A. Dewald *et al.*, J. Phys. G: Nucl. Part. Phys. **31**, S1427 (2005).
[23] N. Pietralla and O. M. Gorbachenko, Phys. Rev. C **70**, 011304(R) (2004).
[24] D. Bonatsos, D. Lenis, D. Petrellis, P. A. Terziev, and I. Yigitoglu, Phys. Lett. **B632**, 238 (2006).
[25] D. Bonatsos, D. Lenis, D. Petrellis, P. A. Terziev, and I. Yigitoglu, Phys. Lett. **B621**, 102 (2005).
[26] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, and P. A. Terziev, Phys. Rev. C **69**, 014302 (2004).
[27] G. Levai and J. M. Arias, Phys. Rev. C **69**, 014304 (2004).
[28] F. Iachello, Phys. Rev. Lett. **95**, 052503 (2005).
[29] M. S. Fetea *et al.*, Phys. Rev. C **73**, 051301(R) (2006).
[30] C. E. Alonso, J. M. Arias, and A. Vitturi, Phys. Rev. Lett. **98**, 052501 (2007).
[31] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University, Cambridge, England, 1987).
[32] J. N. Ginocchio and M. W. Kirson, Nucl. Phys. **A350**, 31 (1980).
[33] A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980).
[34] A. Bohr and B. Mottelson, Phys. Scr. **22**, 468 (1980).
[35] R. F. Casten, *Nuclear Structure from a Simple Perspective* (Oxford University, New York, 2000).
[36] D. H. Feng, R. Gilmore, and S. R. Deans, Phys. Rev. C **23**, 1254 (1981).
[37] A. Frank, Phys. Rev. C **39**, 652 (1989).
[38] E. López-Moreno and O. Castaños, Phys. Rev. C **54**, 2374 (1996).

[39] J. Jolie, P. Cejnar, R. F. Casten, S. Heinze, A. Linnemann, and V. Werner, Phys. Rev. Lett. **89**, 182502 (2002).
[40] J. Jolie, S. Heinze, P. Van Isacker, and R. F. Casten, Phys. Rev. C **70**, 011305(R) (2004).
[41] J. M. Arias, C. E. Alonso, A. Vitturi, J. E. García-Ramos, J. Dukelsky, and A. Frank, Phys. Rev. C **68**, 041302(R) (2003).
[42] J. E. García-Ramos, J. Dukelsky, and J. M. Arias, Phys. Rev. C **72**, 037301 (2005).
[43] P. Cejnar and J. Jolie, Phys. Rev. E **61**, 6237 (2000).
[44] J. Jolie, P. Cejnar, R. F. Casten, S. Heinze, A. Linnemann, and V. Werner, Phys. Rev. Lett. **89**, 182502 (2002).
[45] P. Cejnar, S. Heinze, and J. Jolie, Phys. Rev. C **68**, 034326 (2003).
[46] J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. Lett. **91**, 162502 (2003).
[47] P. Cejnar, S. Heinze, and J. Dobes, Phys. Rev. C **71**, 011304(R) (2005).
[48] S. Heinze, P. Cejnar, J. Jolie, and M. Macek, Phys. Rev. C **73**, 014306 (2006).
[49] A. Frank, C. E. Alonso, and J. M. Arias, Phys. Rev. C **65**, 014301 (2001).
[50] J. E. García-Ramos, J. M. Arias, J. Barea, and A. Frank, Phys. Rev. C **68**, 024307 (2003).
[51] S. Dusuel, J. Vidal, J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. C **72**, 011301(R) (2005).
[52] S. Dusuel, J. Vidal, J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. C **72**, 064332 (2005).
[53] J. Vidal, J. M. Arias, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. C **73**, 054305 (2006).
[54] J. M. Arias, J. Dukelsky, J. E. García-Ramos, and J. Vidal, Phys. Rev. C **75**, 014301 (2007).
[55] A. Leviatan, Phys. Rev. C **74**, 051301(R) (2006).
[56] A. Leviatan, Phys. Rev. C **72**, 031305(R) (2005).
[57] A. Leviatan and J. N. Ginocchio, Phys. Rev. Lett. **90**, 212501 (2003).
[58] D. J. Rowe, Nucl. Phys. **A745**, 47 (2004).
[59] P. S. Turner and D. J. Rowe, Nucl. Phys. **A756**, 333 (2005).
[60] G. Rosensteel and D. J. Rowe, Nucl. Phys. **A759**, 92 (2005).
[61] D. J. Rowe and G. Thiamova, Nucl. Phys. **A760**, 59 (2005).
[62] R. F. Casten, Nature Phys. **2**, 811 (2006).
[63] F. Iachello and P. Van Isacker, *The Interacting Boson-Fermion Model* (Cambridge University, Cambridge, England, 1991).
[64] C. E. Alonso, J. M. Arias, L. Fortunato, and A. Vitturi, Phys. Rev. C **72**, 061302(R) (2005).
[65] C. E. Alonso, J. M. Arias, and A. Vitturi, Phys. Rev. C **74**, 027301 (2006).
[66] P. Van Isacker, A. Frank, and H-Z. Sun, Ann. Phys. (NY) **157**, 183 (1984).
[67] E. Chacón, M. Moshinsky, and R. T. Sharp, J. Math. Phys. **17**, 668 (1976).