Global attractor for a nonlocal p-Laplacian equation without uniqueness of solution

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Abstract

In this paper, the existence of solution for a p-Laplacian parabolic equation with nonlocal diffusion is established. To do this, we make use of a change of variable which transforms the original problem into a nonlocal one but with local diffusion. Since the uniqueness of solution is unknown, the asymptotic behaviour of the solutions is analysed in a multi-valued framework. Namely, the existence of the compact global attractor in $L^2(\Omega)$ is ensured.

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1 Introduction and setting of the problem

Nonlocal problems have been analysed in the last few decades by a large number of authors in many scientific branches, for instance in Physics and Biology. Starting with [20], Furter and Grindfeld analyse some models of populations with nonlocal effects; in an ecological context, there does not exist a reason why interactions in single-species population dynamics must be local and they provide some examples to strengthen their arguments. In the same direction, Chipot and Rodrigues [10] study the behaviour of a population of bacterias within a container. This is modeled by the nonlocal elliptic problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega u\right)\Delta u + \lambda u = f \quad \text{in } \Omega, \\ \partial_n u + \gamma\left(\displaystyle\int_{\Omega'} u\right) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\Omega' \subset \Omega$, $\lambda > 0$, function $a \in C(\mathbb{R};(0,\infty))$, $\gamma \in C(\mathbb{R};\mathbb{R}_+)$, $f \in L^2(\Omega)$ and $\partial_n u$ is the normal derivative of u. Since then, many authors have been interested in analysing variations of this problem. Much attention has been paid to the nonlocal parabolic equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f, \tag{1}$$

where the function a is continuous and there exist positive constants m, $M > 0$ such that

$$0 < m \leq a(s) \leq M \quad \forall s \in \mathbb{R}. \tag{2}$$

In particular this non-degeneracy of a avoids the extinction and only existence of the solutions in finite time intervals (for more details see [25]). For instance, in [8, 9, 15], equation (1) fulfilled with homogeneous Dirichlet boundary conditions has been analysed. In [8, 9], Chipot, Lovat and Molinet study the asymptotic behaviour of weak solutions using a suitable order between stationary solutions and dynamical systems. In [15], Chipot and Zheng analyse the convergence to one of the equilibria without assuming uniqueness of stationary solution. Similarly, Chipot and Siegwart [13] study the asymptotic behaviour of weak solutions using mixed boundary conditions.

Up to date, in the cited papers, the function f does not depend on time or on the unknown, but there is a wide range of papers which deal with these variations. In [29], assuming that f is globally Lipchitz and does depend on the unknown, dealing also with an additional non-autonomous term, Menezes studies the existence and uniqueness of weak and radial solutions. Later, in [3, 4, 5], considering a more general function f, we study the existence of minimal pullback attractors in $L^2(\Omega)$ and when the uniqueness of solution is guaranteed, the regularity issue in $H_0^1(\Omega)$ is addressed. In addition, in [4] the upper-semicontinuity of attractors w.r.t. a parameter is also analysed.

Other different nonlocal terms, not only $a(l(u))$, have also been considered. For instance, Hilhorst and Rodrigues [22] analyse the parabolic equation

$$\frac{\partial u}{\partial t} = a\left(\frac{1}{meas(\Omega)}\int_\Omega u(x)dx\right)\Delta u + f\left(u, \frac{1}{meas(\Omega)}\int_\Omega u(x)dx\right).$$

Later, Corrêa [17] considers $a\left(\int_\Omega |u(x)|^q dx\right)$ and proves the existence of positive solution in the elliptic framework. An analogous result is also proved by Corrêa et al. [18] when the nonlocal term is $a\left(\int_\Omega u(x)dx\right)$. Furthermore, the nonlocal operator could be a functional acting on $\Omega \times L^p(\Omega)$ as it is analysed by Chipot, Corrêa and Roy in [7, 11]. Besides, Andami Ovono and Rougirel [1, 2] study the existence of radial solutions, global attractor, bifurcation, branches of solutions and their stability making use of a local nonlocal operator, i.e. the operator is not defined in the whole domain but in a ball centered in each position point. In addition, in [14], Chipot, Valente and Vergara Caffarelli consider $a(|\nabla u|^2)$ instead of $a(l(u))$. The main advantage of considering this new variation is that it allows to study the long-time behaviour of weak solutions making use of global minimizers.

Recently, Chipot and Savistka [12] consider a different nonlocal operator with a more general diffusion term involving the p-Laplacian, $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$. This operator appears in wide range of scientific fields, for instance, in Fluid Dynamics (e.g., flow through porous media), Nonlinear Elasticity, Glaciology, Image Restoration (e.g., cf. [31, 6, 32]), and so on. The nonlocal problem treated in [12] is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_\tau(x) \quad \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $1 < p < \infty$, the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils

$$0 < m \le a(s) \quad \forall s \in \mathbb{R}, \tag{3}$$

and $f \in W^{-1,q}(\Omega)$, where q is the conjugate exponent of p. The existence and uniqueness of weak solution is proved making use of a change of variable (see (10)), Galerkin approximations and compactness arguments. Although this change of variable has already been used by Chipot et al. [14] in order to prove the uniqueness of solution, as far as we know, [12] is the first time that this is used to prove its existence. The main reason is that in the previous papers (cf. [10, 8, 9, 13, 18, 15, 29, 3, 4, 5]), the diffusion term contained the Laplacian, which is linear. Then, although the nonlocal term generated a nonlocal diffusion, making use of [24, Lemme 1.3, p. 12], it is not difficult to ensure the existence of solution. However, for the p-Laplacian, it does not seem possible to argue in the same way, nor even using monotonicity arguments.

In this paper, we consider the nonlocal problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta_p u = f \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $p \geq 2$, $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (3), $l \in (L^2(\Omega))'$, $u_0 \in L^2(\Omega)$ and $f \in W^{-1,q}(\Omega)$.

The aim of the paper is twofold. On the one hand, due to the assumptions on the viscosity term a, we prove existence (but not uniqueness) of solutions to (4) combining the change of variable cited above and monotonicity techniques. On the other hand, for a suitable defined dynamical system associated to this problem, the existence of attractor is ensured in this multi-valued framework.

The content of the paper is as follows. Section 2 is devoted to study the existence of solutions. In Section 3 we briefly recall some abstract results of dynamical systems for multi-valued semiflows. Then, this is applied in Section 4 where the existence of the compact global attractor in $L^2(\Omega)$ is established.

Before to start, let us introduce some notation that will be used all through the paper, as well as the notion of a solution to (4). As usual, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $|\cdot|$ its associated norm; since no confusion arises, these symbols also denote the action amongst $L^p(\Omega)$ and $L^q(\Omega)$ elements and the Lebesgue measure of a subset of $\mathbb{R}^N$ respectively. Thanks to the Poincaré inequality, we will use as norm in $W_0^{1,p}(\Omega)$, which will be denoted by $\|\cdot\|_p$, the $L^p(\Omega)$ norm of the gradient of an element.

By $\langle\cdot,\cdot\rangle$ we denote the duality product between $W_0^{1,p}(\Omega)$ and $W^{-1,q}(\Omega)$ and by $\|\cdot\|_*$, the norm in $W^{-1,q}(\Omega)$. In particular, we recall that the p-Laplacian operator is a one-to-one mapping from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$, given by

$$\langle -\Delta_p u, v\rangle = (|\nabla u|^{p-2}\nabla u, \nabla v) \quad \forall u, v \in W_0^{1,p}(\Omega),$$

where for short we are denoting $(|\nabla u|^{p-2}\nabla u, \nabla v) = \sum_{i=1}^N (|\partial_i u|^{p-2}\partial_i u, \partial_i v)$. Identifying $L^2(\Omega)$ with its dual, we have the usual chain of dense and compact embeddings $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,q}(\Omega)$. Observe that, by the Riesz theorem, we can obtain $\tilde{l} \in L^2(\Omega)$ with $\langle l, u\rangle_{(L^2(\Omega))',L^2(\Omega)} = (\tilde{l}, u)$; here on, thanks to the identification $(L^2(\Omega))' \equiv L^2(\Omega)$, we make the abuse of notation of using l instead of $\tilde{l}$, but at the same time we keep the usual notation in the existing previous literature $l(u)$ instead of (l,u) for the operator l acting on u.

Definition 1. *A (weak) solution to (4) is a function u that belongs to $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ for all $T > 0$, with $u(0) = u_0$, such that*

$$\frac{d}{dt}(u(t),v) + a(l(u(t)))(|\nabla u(t)|^{p-2}\nabla u(t), \nabla v) = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega), \tag{5}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(0,\infty)$.

Remark 2. *If u is a solution to (4), then, bearing in mind assumptions (3) and (5), it fulfils that $u' \in L^q(0,T;W^{-1,q}(\Omega))$ for any $T > 0$. Therefore, $u \in C([0,\infty);L^2(\Omega))$ and the initial datum $u_0 \in L^2(\Omega)$ in (4) makes complete sense. In addition, it satisfies the energy equality*

$$|u(t)|^2 + 2\int_s^t a(l(u(r)))\|u(r)\|_p^p dr = |u(s)|^2 + 2\int_s^t \langle f, u(r)\rangle dr \tag{6}$$

for all $0 \leq s \leq t$ (cf. [19, Théorème 2, p. 575], [33, Lemma 3.2, p. 71] for more details).

2 Existence of solution

In this section, we will prove the existence of solutions to (4). To that end, we will combine the Galerkin approximations, a change of variable (see (10) below) which has been already used by Chipot and his collaborators (cf. [14, 12]) and compactness arguments.

Theorem 3. *Assume that function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (3), $f \in W^{-1,q}(\Omega)$ and $l \in L^2(\Omega)$. Then, for each $u_0 \in L^2(\Omega)$, there exists at least a solution to (4).*

Proof. We will prove the existence of solution to (4) in an interval $[0,\widetilde{T}]$ (to be specified later). An inductive concatenation procedure will provide the desired global-in-time solution. We split the proof into several steps.

Step 1: Galerkin approximations, a priori estimates and compactness arguments.

Consider a special basis of $L^2(\Omega)$ composed by elements $\{v_j\} \subset H_0^s(\Omega)$ with $s \geq (2p + N(p-2))/(2p)$ in the sense of [24, p. 161]. Then, $H_0^s(\Omega) \subset W_0^{1,p}(\Omega)$. In what follows, we denote by $V_n = span[v_1,\ldots,v_n]$. Observe that in this way $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $W_0^{1,p}(\Omega)$.

Fix an arbitrary positive value $T > 0$. For each $n \in \mathbb{N}$, consider $u_n(t;u_0) = \sum_{j=1}^n \varphi_{nj}(t)v_j$ (for short denoted $u_n(t)$), local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t),v_j) + a(l(u_n(t)))(|\nabla u_n(t)|^{p-2}\nabla u_n(t),\nabla v_j) = \langle f, v_j\rangle & \text{a.e } t\in(0,T), \\ (u_n(0),v_j) = (u_0,v_j), & j = 1,\ldots,n. \end{cases} \tag{7}$$

Existence (but not necessarily uniqueness) of local solution is guaranteed by the Caratheodory theorem [16, Theorem 1.1, p. 43] in some interval $[0,t_n)$.

Now, multiplying in (7) by $\varphi_{nj}(t)$ and summing from $j = 1$ to n, we have

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + m\|u_n(t)\|_p^p \leq \langle f, u_n(t)\rangle \quad \text{a.e. } t\in(0,t_n). \tag{8}$$

From the Young inequality, we deduce

$$\langle f, u_n(t)\rangle \leq \|f\|_*\|u_n(t)\|_p \leq \frac{1}{q}\left(\frac{2}{mp}\right)^{q/p}\|f\|_*^q + \frac{m}{2}\|u_n(t)\|_p^p.$$

Plugging this into (8) we obtain

$$\frac{d}{dt}|u_n(t)|^2 + m\|u_n(t)\|_p^p \leq \frac{2}{q}\left(\frac{2}{mp}\right)^{q/p}\|f\|_*^q \quad \text{a.e. } t\in(0,t_n).$$

This provides a priori estimates that prevent the blow-up and, using standard arguments of continuation of solutions, we deduce the existence of solutions to (7) in the interval $[0,T]$. Moreover, the sequence $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Whence the sequence $\{-\Delta_p u_n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$.

Now, defining $P_n : H^{-s}(\Omega) \ni f \mapsto P_n f := \sum_{j=1}^n \langle f, v_j\rangle v_j \in V_n$, which is the continuous extension of the projector P_n defined as $P_n : L^2(\Omega) \ni f \mapsto P_n f := \sum_{j=1}^n (f,v_j)v_j \in V_n$, we have

$$\frac{du_n}{dt} = a(l(u_n))\Delta_p u_n + P_n f \quad \text{in } \mathcal{D}'(0,T;H^{-s}(\Omega)).$$

On the other hand, making use of the fact that $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (3), we deduce that the sequence $\{f/a(l(u_n))\}$ is bounded in $L^\infty(0,T;W^{-1,q}(\Omega))$.

Therefore, from compactness arguments, the Aubin-Lions lemma and the Dominated Convergence theorem, there exist a subsequence of $\{u_n\}$ (relabeled the same), $\xi \in L^q(0,T;W^{-1,q}(\Omega))$ and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ with $u' \in L^q(0,T;H^{-s}(\Omega))$, such that

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ u_n \to u \quad \text{strongly in } L^p(0,T;L^p(\Omega)), \\ a(l(u_n)) \overset{*}{\rightharpoonup} a(l(u)) \quad \text{weakly-star in } L^\infty(0,T), \\ -\Delta_p u_n \rightharpoonup \xi \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \\ u_n' \rightharpoonup u' \quad \text{weakly in } L^q(0,T;H^{-s}(\Omega)), \\ \dfrac{f}{a(l(u_n))} \to \dfrac{f}{a(l(u))} \quad \text{strongly in } L^s(0,T;W^{-1,q}(\Omega)) \quad \forall s \in [1,\infty). \end{cases} \tag{9}$$

The difficulty in order to apply these convergences and to pass to the limit is the presence of the nonlocal term in front of the p-Laplacian, which makes $-a(l(\cdot))\Delta_p(\cdot)$ not behave as a monotone operator. More exactly, it is not difficult to deduce that $-a(l(u_n))\Delta_p u_n$ converge to $a(l(u))\xi$ weakly in $L^q(0,T;W^{-1,q}(\Omega))$. However, we cannot identify this as $-a(l(u))\Delta_p u$. We will remove the nonlocal term in front of the p-Laplacian, and then to apply monotonicity arguments (cf. [24]).

Step 2: Local diffusion problems through a change of variable.

Following [14, 12], we can obtain formally a local diffusion problem by rescaling the time. Namely, we put

$$\alpha(t) = \int_0^t a(l(u(s)))ds, \tag{10}$$

where u is (formally) a solution to (4). Then, the change of variable $u(x,t) = w(x,\alpha(t))$ leads to the problem

$$\begin{cases} w_s(\alpha(t)) - \Delta_p w(\alpha(t)) = \dfrac{f}{a(l(w(\alpha(t))))} \quad \text{in } \Omega \times (0,T), \\ w = 0 \quad \text{on } \partial\Omega \times (0,T), \\ w(x,\alpha(0)) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

Using the rescaled time, the previous problem can be rewritten as

$$\begin{cases} w_t - \Delta_p w = \dfrac{f}{a(l(w))} \quad \text{in } \Omega \times (0,\alpha(T)), \\ w = 0 \quad \text{on } \partial\Omega \times (0,\alpha(T)), \\ w(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{11}$$

To arrive at this problem not only formally but rigorously, we consider a sequence of Galerkin approximation problems associated to (7) and the corresponding rescaled times

$$\alpha_n(t) := \int_0^t a(l(u_n(s)))ds.$$

The new unknown $w_n(t) = \sum_{j=1}^n \widetilde{\varphi}_{nj}(t)v_j$ is set such that $w_n(x,\alpha_n(t)) := u_n(x,t)$ (therefore $\widetilde{\varphi}_{nj}(\alpha_n(t)) = \varphi_{nj}(t)$ for $t \in [0,T]$). Once that the time is rescaled, w_n solves

$$\begin{cases} \dfrac{d}{dt}(w_n(t),v_j) + (|\nabla w_n(t)|^{p-2}\nabla w_n(t), \nabla v_j) = \dfrac{\langle f, v_j\rangle}{a(l(w_n(t)))} \quad \text{a.e. } t \in (0,\alpha_n(T)), \\ (w_n(0),v_j) = (u_0,v_j), \qquad j = 1,\dots,n. \end{cases} \tag{12}$$

It must be pointed out that thanks to (3) all the above problems are posed at least in the common time-interval $(0, mT)$. There, we will make the most of these local diffusion problems where the monotonicity arguments can be successfully applied.

Observe that if $\varphi \in \mathcal{D}(0, mT)$, then $\varphi \in \mathcal{D}(0, \alpha_n(T))$ and $\varphi(\alpha_n(\cdot)) \in W_0^{1,p}(0,T)$ for all $n \in \mathbb{N}$.

Then, from (7), we deduce

$$-\int_0^T (u_n(t), v)\, \varphi'(\alpha_n(t)) a(l(u_n(t)))dt + \int_0^T \left(|\nabla u_n(t)|^{p-2}\nabla u_n(t), \nabla v\right) a(l(u_n(t)))\varphi(\alpha_n(t))dt$$
$$= \int_0^T \langle f, v\rangle \varphi(\alpha_n(t))dt \tag{13}$$

for all $v \in V_n$.

Since the sequence $\{u_n\}$ is bounded in $L^\infty(0,T; L^2(\Omega))$ and each $u_n \in C([0,T]; L^2(\Omega))$, there exists a positive constant $C_\infty > 0$ such that

$$|u_n(t)| \le C_\infty \quad \forall t \in [0,T].$$

From this, bearing in mind that $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (3) and $l \in L^2(\Omega)$, there exists a positive constant $M(C_\infty) > 0$ such that

$$0 < m \le a(l(u_n(t))) \le M(C_\infty) \quad \forall t \in [0,T], \forall n \ge 1.$$

Now, replacing $u_n(x,t)$ by $w_n(x, \alpha_n(t))$ in (13) and using [14, Lemma 2.2], it yields

$$-\int_0^{\alpha_n(T)} (w_n(t), v)\, \varphi'(t)dt + \int_0^{\alpha_n(T)} \left(|\nabla w_n(t)|^{p-2}\nabla w_n(t), \nabla v\right) \varphi(t)dt = \int_0^{\alpha_n(T)} \frac{\langle f, v\rangle}{a(l(w_n(t)))}\varphi(t)dt$$

for all $v \in V_n$.

Since $\mathrm{supp}(\varphi) \subset (0, mT)$ and $0 < mT \le \alpha_n(T)$ for all $n \ge 1$, all integrals above can be considered in $(0, mT)$. Then, taking limit when $n \to \infty$, from (9) (and consequently, the analogous set of convergences of $\{w_n\}$ towards w), we deduce that

$$-\int_0^{mT} (w(t), v)\, \varphi'(t)dt + \int_0^{mT} \langle \widehat{\xi}(t), v\rangle \varphi(t)dt = \int_0^{mT} \frac{\langle f, v\rangle}{a(l(w(t)))}\varphi(t)dt,$$

where

$$\widehat{\xi}(x, \alpha(t)) = \xi(x,t) \quad \text{a.e. } t \in (0, \alpha^{-1}(mT)).$$

This implies that

$$w'(t) + \widehat{\xi}(t) = \frac{f}{a(l(w(t)))} \quad \text{in } W^{-1,q}(\Omega)), \text{ a.e. } t \in (0, mT). \tag{14}$$

At this point we are almost done. It remains to check that $\widehat{\xi}$ coincides with $-\Delta_p w$, to obtain that w solves (11) in a certain time-interval, whose proof combines monotonicity and compactness arguments.

Step 3: Monotonicity and compactness arguments.

From (14) it yields the energy equality

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \langle \widehat{\xi}(t), w(t)\rangle = \frac{\langle f, w(t)\rangle}{a(l(w(t)))} \quad \text{a.e. } t \in (0, mT).$$

Therefore, integrating in $(0, mT)$, we have

$$\int_0^{mT} \langle \widehat{\xi}(t), w(t)\rangle dt = \int_0^{mT} \frac{\langle f, w(t)\rangle}{a(l(w(t)))}dt + \frac{|w(0)|^2}{2} - \frac{|w(mT)|^2}{2}. \tag{15}$$

Claim 3.1: It holds that $w(0) = u_0$.

Indeed, consider $\varphi \in W^{1,p}(0, mT)$ with $\varphi(0) \neq 0$ and $\varphi(mT) = 0$, and $v \in V_n$. Taking into account (14), we deduce

$$-(w(0), v)\varphi(0) - \int_0^{mT} (w(t), v)\varphi'(t)dt + \int_0^{mT} \langle \widehat{\xi}(t), v\rangle \varphi(t)dt = \int_0^{mT} \frac{\langle f, v\rangle}{a(l(w(t)))}\varphi(t)dt.$$

Again, from (12), multiplying by φ and integrating in $(0, mT)$, we deduce

$$-(u_0, v)\varphi(0) - \int_0^{mT} (w_n(t), v)\varphi'(t)dt + \int_0^{mT} (|\nabla w_n(t)|^{p-2}\nabla w_n(t), \nabla v)\varphi(t)dt = \int_0^{mT} \frac{\langle f, v\rangle}{a(l(w_n(t)))}\varphi(t)dt$$

for all $v \in V_n$. Taking limit when $n \to \infty$ and making use of (9), we deduce from the above expressions that $w(0) = u_0$.

Claim 3.2: The following estimate holds

$$\liminf_{n\to\infty} |w_n(mT)| \geq |w(mT)|. \tag{16}$$

Actually, we prove that $w_n(mT)$ converge weakly to $w(mT)$ in $L^2(\Omega)$. Indeed, from (12), integrating in $(0, mT)$, we have

$$(w_n(mT), v) = (u_0, v) + \int_0^{mT} \left[(|\nabla w_n(t)|^{p-2}\nabla w_n(t), \nabla v) + \langle \frac{f}{a(l(w_n(t)))}, v\rangle \right] dt,$$

for all $v \in V_n$.

Now, taking limit when $n \to \infty$, making use of (9) and integrating (14) in $(0, mT)$, we obtain the announced weak convergence. Therefore, (16) holds.

Claim 3.3: Identification of $\widehat{\xi}$ as $-\Delta_p w$.

Multiplying (12) by $\widetilde{\varphi}_{nj}(t)$, summing from $j = 1$ until n, and taking limit when $n \to \infty$, bearing in mind (9) and (16), we deduce

$$\limsup_{n\to\infty} \int_0^{mT} \|w_n(t)\|_p^p dt \leq \int_0^{mT} \frac{\langle f, w(t)\rangle}{a(l(w(t)))}dt + \frac{|u_0|^2}{2} - \frac{|w(mT)|^2}{2}. \tag{17}$$

Now, consider $v \in L^p(0, mT; W_0^{1,p}(\Omega))$. Then, from the well-known inequality

$$\int_0^{mT} (|\nabla w_n(t)|^{p-2}\nabla w_n(t) - |\nabla v(t)|^{p-2}\nabla v(t), \nabla(w_n(t) - v(t)))dt \geq 0,$$

combined with (9) and (17), we have

$$\frac{|u_0|^2}{2} - \frac{|w(mT)|^2}{2} + \int_0^{mT} \left[\frac{\langle f, w(t)\rangle}{a(l(w(t)))} - \langle \widehat{\xi}(t), v(t)\rangle - (|\nabla v(t)|^{p-2}\nabla v(t), \nabla(w(t) - v(t))) \right] dt \geq 0.$$

Now, plugging (15) into the above inequality, we obtain

$$\int_0^{mT} \left[\langle \widehat{\xi}(t), w(t) - v(t)\rangle + (|\nabla v(t)|^{p-2}\nabla v(t), \nabla(w(t) - v(t))) \right] dt \geq 0$$

for all $v \in L^p(0, mT; W_0^{1,p}(\Omega))$.

Then, taking $v = w - \delta z$ with $\delta > 0$ and $z \in L^p(0, mT; W_0^{1,p}(\Omega))$, we conclude

$$\int_0^{mT} \left[\langle \widehat{\xi}(t), z(t) \rangle - (|\nabla(w(t) - \delta z(t))|^{p-2} \nabla(w(t) - \delta z(t)), \nabla z(t)) \right] dt \geq 0.$$

Since $\delta > 0$ is arbitrary, we deduce that $\widehat{\xi}(x,t) = -\Delta_p w(x,t)$ a.e. $t \in (0, mT)$ (in particular $\xi(x,t) = -\Delta_p u(x,t)$ a.e. $t \in (0, \alpha^{-1}(mT))$). Thus, w solves (11) in $(0, mT)$ and $u(x,t) = w(x, \alpha(t))$ is a solution to (4) in $[0, \widetilde{T}]$ with $\widetilde{T} = \alpha^{-1}(mT)$. Applying the same arguments to intervals of the form $[k\widetilde{T}, (k+1)\widetilde{T}]$ with $k \in \mathbb{N}$ and concatenation, we obtain a global-in-time solution. □

Remark 4. *If $f \in L^2(\Omega)$, any solution to (4) is slightly more regular. Namely, for any solution to (4) it holds that $u \in L^\infty(\varepsilon, T; W_0^{1,p}(\Omega))$ for any $0 < \varepsilon < T$ with $u' \in L^\infty(\varepsilon, T; W^{-1,q}(\Omega))$, and therefore $u \in C_w([\varepsilon, T]; W_0^{1,p}(\Omega))$. Actually, if $u_0 \in W_0^{1,p}(\Omega)$, the above regularity holds for $\varepsilon = 0$. See Proposition 16 below for more details.*

The following result is a generalization of Theorem 3, where the operator l is allowed to belong to a bigger space, namely $L^q(\Omega)$, provided that a natural restriction on a is imposed. The proof is analogous to the previous one with minor changes, so it is omitted.

Corollary 5. *Assume that function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (2), $f \in W^{-1,q}(\Omega)$ and $l \in L^q(\Omega)$. Then, for each $u_0 \in L^2(\Omega)$, there exists at least a solution to (4).*

3 Set-valued dynamical systems and global attractors

In this section, we recall some abstract results on multi-valued autonomous dynamical systems (cf. [28] and the references therein) which allow to prove the main result of this paper, that is, the existence of the global attractor in $L^2(\Omega)$ for a suitable dynamical system associated to problem (4).

To set our abstract framework, consider a metric space (X, d_X) and denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X.

Definition 6. *A multi-valued map $S : \mathbb{R}_+ \times X \mapsto \mathcal{P}(X)$ is a multi-valued semiflow on X, and is denoted by $(X, \{S(t)\}_{t \geq 0})$, if*

(i) $S(0) = I_X$, the identical map on X;

(ii) $S(t+s)x \subset S(t)(S(s)x)$ for all $0 \leq s \leq t$ and any $x \in X$, where

$$S(t)W := \bigcup_{y \in W} S(t)y \quad \forall W \subset X.$$

When the relationship established in (ii) is an equality instead of an inclusion, the multi-valued semiflow S is called strict.

Definition 7. *A multi-valued semiflow $(X, \{S(t)\}_{t \geq 0})$ is upper-semicontinuous if for all $t \in \mathbb{R}_+$ the mapping $S(t)$ is upper-semicontinuous from X into $\mathcal{P}(X)$, that is, for each $x \in X$ and any neighbourhood $\mathcal{N}(S(t)x)$ of $S(t)x$, there exists a neighbourhood $\mathcal{M}$ of x such that $S(t)y \subset \mathcal{N}(S(t)x)$ for any $y \in \mathcal{M}$.*

A multi-valued semiflow $(X, \{S(t)\}_{t \geq 0})$ is asymptotically compact if for any bounded subset B of X and any sequence $\{t_n\} \subset \mathbb{R}_+$ with $t_n \to \infty$, it fulfils that any sequence $\{y_n\}$, with $y_n \in S(t_n)B$, is relatively compact in X.

In what follows, we consider the Hausdorff semi-distance in X between two subsets $\mathcal{O}_1$ and $\mathcal{O}_2$, which is denoted by $dist_X(\mathcal{O}_1, \mathcal{O}_2)$ and defined as

$$dist_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x, y) \quad \text{for } \mathcal{O}_1, \, \mathcal{O}_2 \subset X.$$

Definition 8. *A subset $B_0 \subset X$ is absorbing for a multi-valued semiflow $(X, \{S(t)\}_{t\geq 0})$ if given any bounded subset B of X, there exists $t(B) > 0$ such that*

$$S(t)B \subset B_0 \quad \forall t \geq t(B).$$

A subset $B_0 \subset X$ is attracting for a multi-valued semiflow $(X, \{S(t)\}_{t\geq 0})$ if for any bounded subset B of X, it fulfils

$$\lim_{t\to\infty} dist(S(t)B, B_0) = 0.$$

Definition 9. *A subset $\mathcal{A} \subset X$ is called a compact global attractor of a multi-valued semiflow $(X, \{S(t)\}_{t\geq 0})$ if it is nonempty, compact, attracting for S and negatively invariant for S, i.e., $\mathcal{A} \subset S(t)\mathcal{A}$ for all $t \geq 0$.*

Remark 10. *It is not difficult to check that a global attractor $\mathcal{A}$ for S is minimal in the sense that if B_0 is also attracting for S, then $\mathcal{A} \subset \overline{B_0}$. In particular, even just being bounded and closed, the global attractor for a multi-valued semiflow is unique. Other definitions and properties of an attractor for a multi-valued semiflow are possible for more general cases (e.g., cf. [28] and the references therein). However, we reduce to this setting since these properties are obtained in our study.*

The existence of the compact global attractor for a multi-valued semiflow $(X, \{S(t)\}_{t\geq 0})$ is ensured by the following result (cf. [28]).

Theorem 11. *Consider a multi-valued semiflow $(X, \{S(t)\}_{t\geq 0})$ which is asymptotically compact, upper-semicontinuous with closed values and possesses a bounded absorbing set B_0. Then, there exists the compact global attractor $\mathcal{A}$ and it is given by*

$$\mathcal{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B_0}^X.$$

In addition, if S is strict, then, the global attractor $\mathcal{A}$ is invariant, i.e., $\mathcal{A} = S(t)\mathcal{A}$ for all $t \geq 0$.

4 Existence of the global attractor

The main goal of this section is to ensure the existence of the compact global attractor in $L^2(\Omega)$ for a suitable dynamical system associated to problem (4) using Theorem 11.

In what follows, we denote by $\Phi(u_0)$ the set of solutions to (4) in $[0, \infty)$ with initial datum $u_0 \in L^2(\Omega)$. This is a nonempty and well-defined set, thanks to Theorem 3.

Then, we can define a multi-valued map $S : \mathbb{R}_+ \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ as

$$S(t)u_0 = \{u(t) : u \in \Phi(u_0)\}, \quad u_0 \in L^2(\Omega). \tag{18}$$

Lemma 12. *Assume that function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (3), $f \in W^{-1,q}(\Omega)$ and $l \in L^2(\Omega)$. Then, the multi-valued map S defined in (18) is a strict multi-valued semiflow in $L^2(\Omega)$.*

Now, to study more properties of the multi-valued semiflow S, we need the following result. To prove it, we use an energy method which relies on the continuity of the solutions (cf. [23, 26, 27, 21]).

Lemma 13. *Under the assumptions of Lemma 12, given u_0 and a sequence of initial data $\{u_0^n\} \subset L^2(\Omega)$ with u_0^n converging to u_0 in $L^2(\Omega)$, it holds that for any sequence $\{u^n\}$ where $u^n \in \Phi(u_0^n)$, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi(u_0)$, such that*

$$u^n(t) \to u(t) \quad \textit{strongly in } L^2(\Omega) \quad \forall t \geq 0.$$

Proof. Consider $T > 0$ fixed. From (6) and making use of (3), we obtain

$$\frac{1}{2}\frac{d}{dt}|u^n(t)|^2 + m\|u^n(t)\|_p^p \leq \langle f, u^n(t)\rangle \quad \text{a.e. } t \in (0,T).$$

Since

$$\langle f, u^n(t)\rangle \leq \left(\frac{2}{mp}\right)^{q/p} \frac{\|f\|_*^q}{q} + \frac{m}{2}\|u^n(t)\|_p^p,$$

we have

$$\frac{d}{dt}|u^n(t)|^2 + m\|u^n(t)\|_p^p \leq \frac{2}{q}\left(\frac{2}{mp}\right)^{q/p} \|f\|_*^q \quad \text{a.e. } t \in (0,T).$$

Therefore, the sequence $\{u^n\}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Since each $u^n \in C([0,T];L^2(\Omega))$, there exists a positive constant $C_\infty > 0$ such that

$$|u^n(t)| \leq C_\infty \quad \forall t \in [0,T].$$

From this, taking into account that $a \in C(\mathbb{R};\mathbb{R}_+)$ and $l \in L^2(\Omega)$, there exists a positive constant $M(C_\infty) > 0$ such that

$$a(l(u^n(t))) \leq M(C_\infty) \quad \forall t \in [0,T].$$

Then, bearing in mind this together with the boundedness of $\{u^n\}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce that the sequences $\{-a(l(u^n))\Delta_p u^n\}$ and $\{(u^n)'\}$ are bounded in $L^q(0,T;W^{-1,q}(\Omega))$. Now, applying the Aubin Lions lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ with $u' \in L^q(0,T;W^{-1,q}(\Omega))$, such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ u^n \to u \quad \text{strongly in } L^p(0,T;L^p(\Omega)), \\ u^n(s) \to u(s) \quad \text{strongly in } L^2(\Omega) \text{ a.e. } t \in (0,T), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \\ -a(l(u^n))\Delta_p u^n \rightharpoonup -a(l(u))\Delta_p u \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \end{cases}$$

where the last convergence has been obtained arguing as in the proof of the existence of solution (cf. Theorem 3). Indeed, in that way we deduce that u solves (4) with $u(0) = u_0$.

Now we can prove the convergence given in the statement. We split the proof into two parts.

Step 1. There exists a subsequence (relabeled the same) $\{u^n\}$ such that

$$u^n(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega) \quad \forall t \in [0,T].$$

To do this, we apply the Ascoli-Arzelà theorem to the sequence $\{u^n\}$. Observe that the sequence $\{u^n\}$ is equicontinuous in $W^{-1,q}(\Omega)$ on $[0,T]$ and bounded in $C([0,T];L^2(\Omega))$. In addition, since the embedding $L^2(\Omega) \hookrightarrow W^{-1,q}(\Omega)$ is compact, by the Ascoli-Arzelà theorem, a subsequence fulfils

$$u^n \to u \quad \text{strongly in } C([0,T];W^{-1,q}(\Omega)).$$

From this, taking into account the boundedness of $\{u^n\}$ in $C([0,T];L^2(\Omega))$, the claim is proved.

Step 2. The sequence $\{u^n\}$ satisfies

$$\limsup_{n\to\infty} |u^n(t)| \le |u(t)| \quad \forall t \in [0,T].$$

Observe that from the energy equality (6), we deduce

$$|z(t)|^2 \le |z(s)|^2 + (t-s)\frac{2}{q}\left(\frac{1}{mp}\right)^{q/p} \|f\|_*^q \quad \forall 0 \le s \le t,$$

where z is replaced by u or any u^n.

Now, we define the continuous and non-increasing functions on $[0,T]$

$$J_n(t) = |u^n(t)|^2 - (t-s)\frac{2}{q}\left(\frac{1}{mp}\right)^{q/p} \|f\|_*^q,$$

$$J(t) = |u(t)|^2 - (t-s)\frac{2}{q}\left(\frac{1}{mp}\right)^{q/p} \|f\|_*^q.$$

Observe that since

$$u^n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \text{ a.e. } t \in (0,T),$$

we have

$$J_n(t) \to J(t) \quad \text{a.e. } t \in (0,T). \tag{19}$$

In fact, making use of the continuity of the functional J on $[0,T]$, the non-increasing character of the function J_n on $[0,T]$, together with (19), we obtain

$$J_n(t) \to J(t) \quad \forall t \in (0,T).$$

From this, taking into account the expressions of J and J_n, the claim is proved.

From Steps 1 and 2 we deduce that $u^n(t)$ converge to $u(t)$ strongly in $L^2(\Omega)$ for any $t \in [0,T]$. A diagonal procedure allows now to conclude the desired convergence for all times. □

Proposition 14. *Under the assumptions of Lemma 12, the multi-valued semiflow S is upper-semi-continuous with closed values.*

Proof. First, we will show that the multi-valued semiflow S is upper-semicontinuous. We argue by contradiction. Assume that there exist $t \in \mathbb{R}_+$, $u_0 \in L^2(\Omega)$, a neighbourhood $\mathcal{N}(S(t)u_0)$ and a sequence $\{y_n\}$ which fulfils that each $y_n \in S(t)u_0^n$, where u_0^n converge strongly to u_0 in $L^2(\Omega)$ and $y_n \notin \mathcal{N}(S(t)u_0)$ for all $n \in \mathbb{N}$.

Observe that, since $y_n \in S(t)u_0^n$ for all n, there exists $u^n \in \Phi(u_0^n)$ such that $y_n = u^n(t)$. Now, since $\{u_0^n\}$ is a convergent sequence of initial data, making use of Lemma 13, there exists a subsequence of $\{u^n(t)\}$ (relabeled the same) which converges to a function $u(t) \in S(t)u_0$. This is a contradiction because $y_n \notin \mathcal{N}(S(t)u_0)$ for any $n \in \mathbb{N}$.

Finally, the multi-valued semiflow S has closed values thanks to Lemma 13. □

Now we establish the existence of an absorbing set for $(L^2(\Omega), \{S(t)\}_{t\ge 0})$.

Proposition 15. *Under the assumptions of Lemma 12, there exists $R_1 > 0$ depending on f, m, Ω and p, such that the set $\overline{B}_{L^2}(0,R_1)$, which is the closed ball in $L^2(\Omega)$ of center 0 and radius R_1, is absorbing for the multi-valued semiflow $(L^2(\Omega), \{S(t)\}_{t\ge 0})$.*

Proof. Consider a nonempty bounded subset B of $L^2(\Omega)$.

It will be proved that there exists $t(B) > 0$ such that

$$|u(t)| \leq R_1 \quad \forall t \geq t(B), \quad \forall u_0 \in B$$

for any $u \in \Phi(u_0)$.

At light of (6) and (3), we have

$$\frac{d}{dt}|u(t)|^2 + 2m\|u(t)\|_p^p \leq 2\langle f, u(t)\rangle \quad \text{a.e. } t > 0.$$

Now, denoting by C_I the constant of the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, adding $\pm\mu|u(t)|^2$ in the above inequality, multiplying by $e^{\mu t}$ (with $\mu \in (0, 2m)$ to be specified later) and taking into account

$$|u(t)|^2 \leq \frac{(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{2/(p-2)} + \|u(t)\|_p^p,$$

we deduce

$$\frac{d}{dt}(e^{\mu t}|u(t)|^2) \leq C_1\mu e^{\mu t} + C_2 e^{\mu t}\|f\|_*^q \quad \text{a.e. } t > 0,$$

where for short we have denoted

$$C_1 = \frac{(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{2/(p-2)} \quad \text{and} \quad C_2 = \frac{1}{q}\left(\frac{2^p}{p(2m-\mu)}\right)^{q/p}.$$

Now, integrating in $(0, t)$, we conclude

$$|u(t)|^2 \leq |u_0|^2 e^{-\mu t} + C_1 + C_2\mu^{-1}\|f\|_*^q,$$

whence the absorbing property follows. Namely, the explicit expression of an absorbing radius is given by $R_1 = 1 + C_1 + C_2\mu_*^{-1}\|f\|_*^q$ with $\mu_* = (2^{p+1}m)/(q + 2^p)$. □

Now, imposing more regularity on f, we make the most of additional regularity of any solution to (4) (cf. Remark 4), and the existence of an absorbing set in $W_0^{1,p}(\Omega)$ for S will be established. In particular, since this set will be compact in $L^2(\Omega)$, the asymptotic compactness of $(L^2(\Omega), \{S(t)\}_{t\geq 0})$ will follow.

Proposition 16. *Under the assumptions of Lemma 12, if $f \in L^2(\Omega)$, there exists $R_2 > 0$ depending on f, m, Ω and p, such that the set $\overline{B}_{W_0^{1,p}}(0, R_2)$, which is the closed ball in $W_0^{1,p}(\Omega)$ of center 0 and radius R_2, is absorbing for the multi-valued semiflow $(L^2(\Omega), \{S(t)\}_{t\geq 0})$.*

Proof. Consider a nonempty bounded subset B of $L^2(\Omega)$. We aim to prove that there exists $t'(B) > 0$ such that

$$\|u(t)\|_p \leq R_2 \quad \forall t \geq t'(B), \quad \forall u_0 \in B$$

for any $u \in \Phi(u_0)$.

Fix one such solution to (4), $u \in \Phi(u_0)$. Observe that the problem

$$(P_u)\begin{cases} \dfrac{\partial y}{\partial t} - a(l(u))\Delta_p y = f & \text{in } \Omega \times (0, \infty), \\ y = 0 & \text{on } \partial\Omega \times (0, \infty), \\ y(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

possesses a unique solution because of the monotonicity of the p-Laplacian (cf. [24, Chapitre II]). Therefore, more regular (a posteriori) estimates as well as using the Galerkin approximations make complete sense. In addition, observe that since u is a solution to (4), by the uniqueness of solution to (P_u), it follows that $y = u$.

Then, we consider the Galerkin formulation associated to problem (P_u)

$$\begin{cases} \dfrac{d}{dt}(u_n(t), v_j) + a(l(u))(|\nabla u_n(t)|^{p-2}\nabla u_n(t), \nabla v_j) = (f, v_j) & \text{a.e. } t > 0, \\ \\ (u_n(0), v_j) = (u_0, v_j), & j = 1, \dots, n, \end{cases} \tag{20}$$

with $u_n(t; u_0) = \sum_{j=1}^{n} \varphi_{nj}(t) v_j$, which is denoted by $u_n(t)$ in what follows.

Arguing analogously as in the proof of Lemma 13 we obtain that u_n satisfies

$$\frac{d}{dt}|u_n(t)|^2 + m\|u_n(t)\|_p^p \le \frac{2}{q}\left(\frac{2}{mp}\right)^{q/p} \|f\|_*^q \quad \text{a.e. } t > 0.$$

Now, integrating in $(t-1, t)$,

$$|u_n(t)|^2 + m\int_{t-1}^{t} \|u_n(s)\|_p^p ds \le |u_n(t-1)|^2 + \frac{2}{q}\left(\frac{2}{mp}\right)^{q/p} \|f\|_*^q.$$

In particular, reasoning as in Proposition 15, we obtain

$$\int_{t-1}^{t} \|u_n(s)\|_p^p ds \le \frac{R_1^2}{m} + \frac{2}{mq}\left(\frac{2}{mp}\right)^{q/p} \|f\|_*^q \quad \forall t \ge t'(B) := t(B) + 1. \tag{21}$$

On the other hand, multiplying (20) by $\varphi'_{nj}(t)/a(l(u(t)))$ and summing from $j = 1$ until n, we have

$$\frac{|u'_n(t)|^2}{a(l(u(t)))} + \frac{1}{p}\frac{d}{dt}\|u_n(t)\|_p^p = \frac{(f, u'_n(t))}{a(l(u(t)))} \quad \text{a.e. } t > 0.$$

Then, making use of the Cauchy inequality and (3), we deduce

$$\frac{1}{p}\frac{d}{dt}\|u_n(t)\|_p^p \le \frac{|f|^2}{4m} \quad \text{a.e. } t > 0.$$

Now, integrating in (r, t), with $0 \le t - 1 \le r \le t$,

$$\|u_n(t)\|_p^p \le \|u_n(r)\|_p^p + \frac{p}{4m}|f|^2.$$

Then, integrating in $r \in (t-1, t)$, we have

$$\|u_n(t)\|_p^p \le \int_{t-1}^{t} \|u_n(r)\|_p^p dr + \frac{p}{4m}|f|^2.$$

Taking into account (21), from the previous expression we deduce

$$\|u_n(t)\|_p^p \le \frac{R_1^2}{m} + \frac{2}{mq}\left(\frac{2}{mp}\right)^{q/p} \|f\|_*^q + \frac{p}{4m}|f|^2 =: R_2^p \quad \forall t \ge t'(B).$$

Therefore, the sequence $\{u_n\}$ is bounded in $L^\infty(t'(B), \infty; W_0^{1,p}(\Omega))$. In particular, there exists a subsequence of $\{u_n\}$ which converges to u weakly in $L^p(t'(B), T; W_0^{1,p}(\Omega))$ for any $T > t'(B)$, since u is the unique solution to (P_u). As $u \in C([t'(B), \infty); L^2(\Omega))$, making use of [30, Lemma 11.2], we deduce

$$\|u(t)\|_p^p \le R_2^p \quad \forall t \ge t'(B).$$

□

To conclude, we obtain the main result of this section, the existence of the compact global attractor in $L^2(\Omega)$.

Theorem 17. *Assume that function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (3), and that both f and l belong to $L^2(\Omega)$. Then, there exists the compact global attractor $\mathcal{A}$, which is invariant and is given by*

$$\mathcal{A} := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)\overline{B}_{W_0^{1,p}}(0,R_2)}^{L^2(\Omega)}. \tag{22}$$

Proof. From Proposition 14 we deduce that the multi-valued semiflow S is upper-semicontinuous with closed values. In addition, Proposition 15 guarantees the existence of an absorbing set in $L^2(\Omega)$. Therefore, according to Theorem 11, to prove the existence of the compact global attractor, we only need to check that the multi-valued semiflow S is asymptotically compact. This is immediate thanks to Proposition 16 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, by Theorem 11, the existence of the compact global attractor $\mathcal{A}$, given by (22), holds.

In addition, since the multi-valued semiflow S is strict (cf. Lemma 12), $\mathcal{A}$ is invariant. □

As a straightforward consequence, we obtain the following generalised result ensuring the existence of attractor under a weaker assumption on l.

Corollary 18. *Assume that function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (2), $f \in L^2(\Omega)$ and $l \in L^q(\Omega)$. Then, the thesis of Theorem 17 hold.*

Remark 19. *Observe that both the existence of solutions (cf. Theorem 3) and attractor (cf. Theorem 17) have been obtained with assumption (3) instead of (2) on function a (unless generalization on l, cf. Corollaries 5 and 18). The weaker assumption (3) can also be applied for proving all the results in [4], including the robustness of the parametric attractors.*

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