Some aspects concerning the dynamics of stochastic chemostats

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Abstract In this paper we study a simple chemostat model influenced by white noise which makes this kind of models more realistic. We use the theory of random attractors and, to that end, we first perform a change of variable using the Ornstein-Uhlenbeck process, transforming our stochastic model into a system of differential equations with random coefficients. After proving that this random system possesses a unique solution for any initial value, we analyze the existence of random attractors. Finally we illustrate our results with some numerical simulations.

1 Introduction

Modeling chemostats is a really interesting and important problem with special interest in mathematical biology, since they can be used to study recombinant problems in genetically altered microorganisms [13, 14], waste water treatment [10, 18] and play an important role in theoretical ecology [2, 9, 12, 17, 22, 23, 24, 26]. Derivation and analysis of chemostat models are well documented in [19, 20, 25] and references therein.

Two standard assumptions for simple chemostat models are as follows: (1) the availability of the nutrient and its supply rate are fixed and (2) the tendency of the

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microorganisms to adhere to surfaces is not taken into account. However, these are very strong restrictions as the real world is non-autonomous and stochastic, and this justifies the analysis of stochastic chemostat models.

Let us first consider one of the simplest chemostat models,

$$\frac{dS}{dt} = (S^0 - S)D - \frac{mSx}{a+S}, \tag{1}$$

$$\frac{dx}{dt} = x\left(\frac{mS}{a+S} - D\right), \tag{2}$$

where $S(t)$ and $x(t)$ denote concentrations of the nutrient and the microbial biomass, respectively; S^0 denotes the volumetric dilution rate, a is the half-saturation constant, D is the dilution rate and m is the maximal consumption rate of the nutrient and also the maximal specific growth rate of microorganisms. We notice that all parameters are positive and we use a function Holling type-II as functional response of the microorganism describing how the nutrient is consumed by the species (see [21] for more details and biological explanations about this model).

However, we can consider a more realistic model by introducing a white noise in one of the parameters, therefore we replace the dilution rate D by $D + \alpha\dot{W}(t)$, where $W(t)$ is a white noise, i.e., is a Brownian motion, and $\alpha \geq 0$ represents the intensity of noise. Then, system (1)-(2) is replaced by the following system of stochastic differential equations

$$dS = \left[(S^0 - S)D - \frac{mSx}{a+S}\right]dt + \alpha(S^0 - S)dW(t), \tag{3}$$

$$dx = x\left(\frac{mS}{a+S} - D\right)dt - \alpha x dW(t). \tag{4}$$

System (3)-(4) has been analyzed in [27] by using the classic techniques from stochastic analysis and some stability results are provided there. However, as in our opinion there are some unclear points in the analysis carried out in [27], our aim in this paper is to use an alternative approach to this problem, specifically the theory of random dynamical systems, which will allow us to partially improve the results in [27]. In addition, we will provide some results which hold with probability one while those from [27] are said to hold in probability.

System (3)-(4) is understood in the Itô sense. Then we first consider its equivalent Stratonovich formulation which is given by

$$dS = \left[(S^0 - S)\left(D + \frac{\alpha^2}{2}\right) - \frac{mSx}{a+S}\right]dt + \alpha(S^0 - S)\circ dW(t), \tag{5}$$

$$dx = x\left(\frac{mS}{a+S} - D + \frac{\alpha^2}{2}\right)dt - \alpha x \circ dW(t). \tag{6}$$

In Section 2 we recall some basic results on random dynamical systems. In Section 3 we start with the study of equilibria and we prove a result related to the existence and uniqueness of global solution of (5)-(6), by using the so-called Ornstein-Uhlenbeck process. Then, we define a random dynamical system and prove the existence of a random attractor for system (5)-(6) giving an explicit expression for it. Finally, in Section 3.5 we show some numerical simulations with different values of α and we can see what happens when α increases.

2 Random dynamical systems

In this section we present some basic results related to random dynamical systems (RDSs) and random attractors which will be necessary for our analysis. For more detailed information about RDSs and their importance, see [1].

Let $(X, \|\cdot\|_X)$ be a separable Banach space and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space where $\mathscr{F}$ is the $\sigma-$algebra of measurable subsets of Ω (called "events") and $\mathbb{P}$ is the probability measure. To connect the state ω in the probability space Ω at time 0 with its state after a time of t elapses, we define a flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on Ω with each θ_t being a mapping $\theta_t : \Omega \to \Omega$ that satisfies

(1) $\theta_0 = \mathrm{Id}_\Omega$,
(2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s,t \in \mathbb{R}$,
(3) the mapping $(t,\omega) \mapsto \theta_t\omega$ is measurable,
(4) the probability measure $\mathbb{P}$ is preserved by θ_t, i.e., $\theta_t\mathbb{P} = \mathbb{P}$.

This set-up establishes a time-dependent family θ that tracks the noise, and $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ is called a *metric dynamical system* [1].

Definition 1. A stochastic process $\{\varphi(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ is said to be a continuous RDS over $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with state space X if $\varphi : [0,+\infty) \times \Omega \times X \to X$ is $(\mathscr{B}[0,+\infty) \times \mathscr{F} \times \mathscr{B}(X), \mathscr{B}(X))$- measurable, and for each $\omega \in \Omega$,

(i) the mapping $\varphi(t,\omega) : X \to X$, $x \mapsto \varphi(t,\omega)x$ is continuous for every $t \geq 0$,
(ii) $\varphi(0,\omega)$ is the identity operator on X,
(iii) (cocycle property) $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\varphi(s,\omega)$ for all $s,t \geq 0$.

Definition 2. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A random set K is a measurable subset of $X \times \Omega$ with respect to the product $\sigma-$algebra $\mathscr{B}(X) \times \mathscr{F}$.

The $\omega-$section of a random set K is defined by

$$K(\omega) = \{x : (x,\omega) \in K\}, \quad \omega \in \Omega.$$

In the case that a set $K \subset X \times \Omega$ has closed or compact $\omega-$sections it is a random set as soon as the mapping $\omega \mapsto d(x, K(\omega))$ is measurable (from Ω to $[0,\infty)$) for every $x \in X$, see [8]. Then K will be said to be a closed or a compact, respectively, random set. It will be assumed that closed random sets satisfy $K(\omega) \neq \emptyset$ for all or at least for $\mathbb{P}-$almost all $\omega \in \Omega$.

Remark 1. It should be noted that in the literature very often random sets are defined provided that $\omega \mapsto d(x, K(\omega))$ is measurable for every $x \in X$. Obviously this is satisfied, for instance, when $K(\omega) = N$ for all ω, where N is some non-measurable subset of X, and also when $K = (U \times F) \cup (\overline{U} \times F^c)$ for some open set $U \subset X$ and $F \notin \mathscr{F}$. In both cases $\omega \mapsto d(x, K(\omega))$ is constant, hence measurable, for every $x \in X$. However, both cases give $K \subset X \times \Omega$ which is not an element of the product $\sigma-$algebra $\mathscr{B}(X) \times \mathscr{F}$.

Definition 3. A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{x\in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{t\in\mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

In what follows we use $\mathscr{D}(X)$ to denote the set of all tempered random sets of X.

Definition 4. A random set $B(\omega) \subset X$ is called a random absorbing set in $\mathscr{D}(X)$ if for any $D \in \mathscr{D}(X)$ and a.e. $\omega \in \Omega$, there exists $T_D(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) \subset B(\omega), \quad \forall t \geq T_D(\omega).$$

Definition 5. Let $\{\varphi(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ be an RDS over $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with state space X and let $A(\omega) (\subset X)$ be a random set. Then $\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega}$ is called a global random $\mathscr{D}-$attractor (or pullback $\mathscr{D}-$attractor) for $\{\varphi(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ if

(i) (compactness) $A(\omega)$ is a compact set of X for any $\omega \in \Omega$;
(ii) (invariance) for any $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t,\omega) A(\omega) = A(\theta_t \omega);$$

(iii) (attracting property) for any $D \in \mathscr{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \operatorname{dist}_X(\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega), A(\omega)) = 0,$$

where

$$\operatorname{dist}_X(G,H) = \sup_{g\in G} \inf_{h\in H} \|g-h\|_X$$

is the Hausdorff semi-metric for $G, H \subseteq X$.

Proposition 1. *[6, 11] Let $B \in \mathscr{D}(X)$ be a closed absorbing set for the continuous random dynamical system $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ that satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega)B(\theta_{-t_n}\omega)$ has a convergent subsequence in X when $t_n \to \infty$. Then φ has a unique global random attractor $\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega}$ with component subsets*

$$A(\omega) = \bigcap_{\tau\geq T_B(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)}.$$

If the pullback absorbing set is positively invariant, i.e., $\varphi(t,\omega)B(\omega) \subset B(\theta_t\omega)$ for all $t \geq 0$, then

$$A(\omega) = \bigcap_{t\geq 0} \overline{\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)}.$$

Remark 2. When the state space $X = \mathbb{R}^d$ as in this paper, the asymptotic compactness follows trivially. Note that the random attractor is path-wise attracting in the pullback sense, but does not need to be path-wise attracting in the forward sense, although it is forward attracting in probability, due to some possible large deviations, see e.g. [1].

The next result ensures when two random dynamical systems are conjugated (see also [3, 4]).

Lemma 1. *Let φ_u be a random dynamical system on X. Suppose that the mapping $T : \Omega \times X \to X$ possesses the following properties: for fixed $\omega \in \Omega$, $T(\omega,\cdot)$ is a homeomorphism on X, and for $x \in X$, the mappings $T(\cdot,x)$, $T^{-1}(\cdot,x)$ are measurable. Then the mapping*

$$(t,\omega,x) \to \varphi_v(t,\omega)x := T^{-1}(\theta_t\omega, \varphi_u(t,\omega)T(\omega,x))$$

is a (conjugated) random dynamical system.

3 Random chemostat

In this section we will investigate the stochastic system (5)-(6). To this end, we first transform it into differential equations with random coefficients and without white noise.

Let W be a two sided Wiener process. Kolmogorov's theorem ensures that W has a continuous version, that we will denote by ω, whose canonical interpretation is as follows: let Ω be defined by

$$\Omega = \{\omega \in \mathscr{C}(\mathbb{R},\mathbb{R}) : \omega(0) = 0\} = \mathscr{C}_0(\mathbb{R},\mathbb{R}),$$

$\mathscr{F}$ be the Borel $\sigma-$algebra on Ω generated by the compact open topology (see [1] for details) and $\mathbb{P}$ the corresponding Wiener measure on $\mathscr{F}$. We consider the Wiener shift flow given by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. Now let us introduce the following Ornstein-Uhlenbeck process on $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$

$$z^*(\theta_t\omega) = -\int_{-\infty}^{0} e^s \theta_t\omega(s) ds, \quad t \in \mathbb{R}, \quad \omega \in \Omega,$$

which solves the following Langevin equation [1, 5]

$$dz + zdt = d\omega(t), \quad t \in \mathbb{R}.$$

Proposition 2. *([1, 5]) There exists a θ_t-invariant set $\widetilde{\Omega} \in \mathscr{F}$ of Ω of full $\mathbb{P}$ measure such that for $\omega \in \widetilde{\Omega}$, we have*

(i) the random variable $|z^*(\omega)|$ is tempered.
(ii) the mapping

$$(t, \omega) \to z^*(\theta_t\omega) = -\int_{-\infty}^{0} e^s \omega(t+s) \mathrm{d}s + \omega(t)$$

is a stationary solution of (7) *with continuous trajectories;*

(iii) in addition, for any $\omega \in \tilde{\Omega}$:

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{t} = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*(\theta_s\omega) ds = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z^*(\theta_s\omega)| ds = \mathbb{E}[z^*] < \infty.$$

In what follows we will consider the restriction of the Wiener shift θ to the set $\tilde{\Omega}$, and we restrict accordingly the metric dynamical system to this set, that is also a metric dynamical system, see [4]. For simplicity, we will still denote the restricted metric dynamical system by the old symbols $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

3.1 Stochastic chemostat becomes a random chemostat

In what follows we use the Ornstein-Uhlenbeck process to transform (5)-(6) into a random system. Let us note that analyzing the equilibria we obtain that the only one is the axial equilibrium $(S^0, 0)$ and then we define two new variables σ and κ by

$$\sigma(t) = (S(t) - S^0)e^{\alpha z^*(\theta_t \omega)}, \tag{7}$$

$$\kappa(t) = x(t)e^{\alpha z^*(\theta_t \omega)}. \tag{8}$$

For the sake of simplicity we will write z^* instead of $z^*(\theta_t \omega)$, and σ and κ instead of $\sigma(t)$ and $\kappa(t)$.

On the one hand, by differentiation, we have

$$\begin{aligned}
d\sigma &= e^{\alpha z^*} dS + (S - S^0)e^{\alpha z^*} \alpha dz^* \\
&= \left\{ \left[(S^0 - S)\left(D + \frac{\alpha^2}{2} \right) - \frac{mSx}{a+S} \right] dt + \alpha(S^0 - S) \circ dW(t) \right\} e^{\alpha z^*} \\
&\quad + (S - S^0)e^{\alpha z^*} \alpha \{-z^* dt + dW(t)\} \\
&= (S^0 - S)\left(D + \frac{\alpha^2}{2} \right) e^{\alpha z^*} dt - \frac{mSx}{a+S} e^{\alpha z^*} dt + \alpha(S^0 - S)e^{\alpha z^*} \circ dW(t) \\
&\quad - (S - S^0)\alpha e^{\alpha z^*} z^* dt + (S - S^0)e^{\alpha z^*} \alpha \circ dW(t) \\
&= \left[-\left(D + \frac{\alpha^2}{2} \right) \sigma - \frac{mS\kappa}{a+S} - \alpha \sigma z^* \right] dt \\
&= \left[-\left(D + \frac{\alpha^2}{2} \right) \sigma - \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa - \alpha \sigma z^* \right] dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d\kappa &= e^{\alpha z^*} dx + x e^{\alpha z^*} \alpha dz^* \\
&= \left[x\left(\frac{mS}{a+S} - D + \frac{\alpha^2}{2} \right) dt - \alpha x \circ dW(t) \right] e^{\alpha z^*} + \alpha x e^{\alpha z^*} \left[-z^* dt + dW(t) \right] \\
&= \frac{xmS}{a+S} e^{\alpha z^*} dt + x\left(-D + \frac{\alpha^2}{2} \right) e^{\alpha z^*} dt - \alpha x e^{\alpha z^*} \circ dW(t) \\
&\quad - \alpha x z^* e^{\alpha z^*} dt + \alpha x e^{\alpha z^*} \circ dW(t)
\end{aligned}$$

$$= \left[\frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa - \left(D - \frac{\alpha^2}{2} \right) \kappa - \alpha z^* \kappa \right] dt.$$

Thus, we have obtained the following random system

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa, \tag{9}$$

$$\frac{d\kappa}{dt} = -(\widetilde{D} + \alpha z^*)\kappa + \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa, \tag{10}$$

where $\bar{D} := D + \frac{\alpha^2}{2}$ and $\widetilde{D} := D - \frac{\alpha^2}{2}$.

3.2 Random chemostat generates an RDS

Next we prove that the random chemostat system (9)-(10) generates an RDS. From now on, we denote $\mathscr{X} := \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \geq 0\}$, the upper half-plane.

Lemma 2. *Assume that*

$$D \geq \frac{\alpha^2}{2}, \qquad \tilde{\lambda} := \frac{\tilde{D}a}{m - \tilde{D}} \geq S^0. \tag{11}$$

Then for any $\omega \in \Omega$ and any initial value $u_0 := (\sigma_0, \kappa_0) \in \mathscr{X}$, where $\sigma_0 := \sigma(0)$ and $\kappa_0 := \kappa(0)$, system (9)-(10) *possesses a unique global solution $u(\cdot;\omega,u_0) := (\sigma(\cdot;\omega,u_0), \kappa(\cdot;\omega,u_0)) \in \mathscr{C}^1([0,+\infty), \mathscr{X})$ with $u(0;\omega,u_0) = u_0$. Moreover the solution mapping generates a random dynamical system $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathscr{X} \to \mathscr{X}$ defined as*

$$\varphi_u(t,\omega)u_0 = u(t;\omega,u_0), \quad \forall t \in \mathbb{R}^+, u_0 \in \mathscr{X}, \omega \in \Omega.$$

Proof. Observe that we can rewrite one of the terms in the previous equations as

$$\frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa = \frac{m(S^0 + \sigma e^{-\alpha z^*} + a - a)}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa = m\kappa - \frac{ma\kappa}{a + S^0 + \sigma e^{-\alpha z^*}}$$

and therefore system (9)-(10) turns into

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - m\kappa + \frac{ma}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa, \tag{12}$$

$$\frac{d\kappa}{dt} = -(\widetilde{D} + \alpha z^*)\kappa + m\kappa - \frac{ma}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa. \tag{13}$$

Denoting $u(\cdot;\omega,u_0) := (\sigma(\cdot;\omega,u_0),\kappa(\cdot;\omega,u_0))$, system (12)-(13) can be rewritten as

$$\frac{du}{dt} = L(\theta_t\omega)\cdot u + F(u,\theta_t\omega),$$

where

$$L(\theta_t\omega) = \begin{pmatrix} -(\bar{D}+\alpha z^*) & -m \\ 0 & -(\widetilde{D}+\alpha z^*)+m \end{pmatrix}$$

and $F : \mathscr{X}\times[0,+\infty) \longrightarrow \mathbb{R}^2$ is given by

$$F(\xi,\theta_t\omega) = \begin{pmatrix} \dfrac{ma}{a+S^0+\xi_1 e^{-\alpha z^*}}\xi_2 \\ \dfrac{-ma}{a+S^0+\xi_1 e^{-\alpha z^*}}\xi_2 \end{pmatrix},$$

where $\xi = (\xi_1,\xi_2)\in\mathscr{X}$.

Since $z^*(\theta_t\omega)$ is continuous, L generates an evolution system on $\mathbb{R}^2$. Moreover, we notice that

$$\frac{\partial}{\partial\xi_2}\left[\pm\frac{am}{a+S^0+\xi_1 e^{-\alpha z^*}}\xi_2\right] = \pm\frac{am}{a+S^0+\xi_1 e^{-\alpha z^*}}$$

and

$$\frac{\partial}{\partial\xi_1}\left[\pm\frac{am}{a+S^0+\xi_1 e^{-\alpha z^*}}\xi_2\right] = \mp\frac{ame^{-\alpha z^*}}{(a+S^0+\xi_1 e^{-\alpha z^*})^2}\xi_2$$

so $F(\cdot,\theta_t\omega)\in\mathscr{C}(\mathscr{X}\times[0,+\infty);\mathbb{R}^2)$ and is continuously differentiable with respect to the variables (ξ_1,ξ_2), which implies that it is locally Lipschitz with respect to $(\xi_1,\xi_2)\in\mathscr{X}$.

Therefore, thanks to classical results from the theory of ordinary differential equations, system (12)-(13) possesses a unique local solution. Let us check now that in fact this solution is a global one. In order to do that, we split our analysis into two different cases: first, we assume $\sigma(t)\geq 0$ for all $t\geq 0$. Thus, from (9)-(10)

$$\frac{d}{dt}(\sigma+\kappa) = -\bar{D}\sigma - \alpha z^*\sigma - \widetilde{D}\kappa - \alpha z^*\kappa$$

$$\leq -\widetilde{D}\sigma - \alpha z^*\sigma - \widetilde{D}\kappa - \alpha z^*\kappa$$
$$= -(\widetilde{D} + \alpha z^*)(\sigma + \kappa).$$

Hence

$$\sigma(t) + \kappa(t) \leq (\sigma(0) + \kappa(0))e^{-\widetilde{D}t - \alpha\int_0^t z^*(\theta_s\omega)ds},$$

so $\sigma + \kappa$ tends to zero when t goes to infinity since $D \geq \frac{\alpha^2}{2}$, i.e., $\widetilde{D} \geq 0$.

Moreover, since $S^0 + \sigma e^{-\alpha z^*} = S \geq 0$, we have

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}}\kappa$$
$$\leq -(\bar{D} + \alpha z^*)\sigma$$

and solving this differential equation we obtain

$$\sigma(t) \leq \sigma(0)e^{-\bar{D}t + \alpha\int_0^t z^*(\theta_s\omega)ds},$$

which implies that σ always tends to zero when t goes to infinity, because $\bar{D} \geq 0$.

Summing up, we have

$$0 \leq \sigma(t) \longrightarrow 0, \quad \text{when} \quad t \uparrow +\infty,$$
$$0 \leq \sigma(t) + \kappa(t) \longrightarrow 0, \quad \text{when} \quad t \uparrow +\infty,$$

since $D \geq \frac{\alpha^2}{2}$, so we have

$$0 \leq \kappa(t) = (\sigma(t) + \kappa(t)) - \sigma(t) \longrightarrow 0, \quad \text{when} \quad t \uparrow +\infty.$$

In particular, σ and κ are bounded.

Now, we assume there exists some $\widetilde{t} \geq 0$ such that $\sigma(\widetilde{t}) < 0$. In this case, there exists t^* such that $\sigma(t^*) = 0$ and then

$$\frac{d\sigma}{dt}(t^*) = \left[-(\bar{D} + \alpha z^*)\sigma - \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}}\kappa\right](t^*)$$
$$= -\frac{mS^0}{a + S^0}\kappa(t^*) < 0.$$

Therefore, we have $\sigma(t) < 0$ for all $t > t^*$ and from (7) we get that $S(t) < S^0$, for all $t > t^*$.

Now, since the mapping $f(S) := \frac{mS}{a+S}$ is an increasing function, then $f(S(t)) < f(S^0)$, for all $t > t^*$, i.e., we have

$$\frac{mS}{a+S} \leq \frac{mS^0}{a+S^0}.$$

Hence, from (9)-(10)

$$\begin{aligned}\frac{d\sigma}{dt} &= -(\bar{D}+\alpha z^*)\sigma - \frac{m(S^0+\sigma e^{-\alpha z^*})}{a+S^0+\sigma e^{-\alpha z^*}}\kappa \\ &\leq -(\bar{D}+\alpha z^*)\sigma\end{aligned}$$

and for $t > t^*$

$$\begin{aligned}\frac{d\kappa}{dt} &= -(\widetilde{D}+\alpha z^*)\kappa + \frac{m(S^0+\sigma e^{-\alpha z^*})}{a+S^0+\sigma e^{-\alpha z^*}}\kappa \\ &= -(\widetilde{D}+\alpha z^*)\kappa + \frac{mS}{a+S}\kappa \\ &\leq -(\widetilde{D}+\alpha z^*)\kappa + \frac{mS^0}{a+S^0}\kappa \\ &= -\left(\widetilde{D} - \frac{mS^0}{a+S^0} + \alpha z^*\right)\kappa,\end{aligned}$$

thus

$$\sigma(t) \leq \sigma(0)e^{-\bar{D}t-\alpha\int_0^t z^*(\theta_s\omega)ds}$$

and for $t > t^*$

$$\kappa(t) \leq \kappa(0)e^{-\left(\widetilde{D}-\frac{mS^0}{a+S^0}\right)t-\alpha\int_0^t z^*(\theta_s\omega)ds}. \tag{14}$$

Summing up, in this second case σ and κ also keep bounded because of the assumption $\tilde{\lambda} \geq S^0$.

Therefore, the unique local solution to system (12)-(13) can be extended to a unique global solution.

Notice that, although σ remains negative, it will never make vanish the denominator $a+S^0+\sigma e^{-\alpha z^*}$. Indeed, if we suppose that there exists $\bar{t} > t^* > 0$ such that

$$a+S^0+\sigma(\bar{t})e^{-\alpha z^*(\theta_{\bar{t}}\omega)} = 0,$$

then for every $M > 0$ given, there exists $t_M \in (t^*, \bar{t})$ such that

$$\frac{m(S^0 + \sigma(t)e^{\alpha z^*(\theta_t \omega)}}{a + S^0 + \sigma(t)e^{-\alpha z^*(\theta_t \omega)}} \geq M$$

for all $t \in (t_M, \bar{t}]$.

Hence, κ satisfies the following differential equation

$$\begin{aligned} \frac{d\kappa}{dt} &= -(\widetilde{D} + \alpha z^*) + \frac{m(S^0 + \sigma e^{-\alpha z^*})}{a + S^0 + \sigma e^{-\alpha z^*}} \kappa \\ &\geq -(\widetilde{D} + \alpha z^*)\kappa + M\kappa \\ &= -(\widetilde{D} - M + \alpha z^*)\kappa, \end{aligned} \tag{15}$$

thus, if we evaluate the solution of (15) in $t = \bar{t}$, we obtain

$$\kappa(\bar{t}) \geq \kappa(t_M) e^{-(\widetilde{D}-M)(\bar{t}-t_M) - \alpha \int_{t_M}^{\bar{t}} z^*(\theta_s \omega) ds}.$$

Now, by taking $M > \frac{mS^0}{a+S^0}$, it is straightforward to check that

$$\begin{aligned} \kappa(\bar{t}) &\geq \kappa(t_M) e^{-(\widetilde{D}-M)(\bar{t}-t_M) - \alpha \int_{t_M}^{\bar{t}} z^*(\theta_s \omega) ds} \\ &> \kappa(t_M) e^{-\left(\widetilde{D} - \frac{mS^0}{a+S^0}\right)(\bar{t}-t_M) - \alpha \int_{t_M}^{\bar{t}} z^*(\theta_s \omega) ds}. \end{aligned} \tag{16}$$

On the other hand, by solving (14) and evaluating in $t = \bar{t}$, we have

$$\kappa(\bar{t}) \leq \kappa(t_M) e^{-\left(\widetilde{D} - \frac{mS^0}{a+S^0}\right)(\bar{t}-t_M) - \alpha \int_{t_M}^{\bar{t}} z^*(\theta_s \omega) ds}, \tag{17}$$

therefore, thanks to (16) and (17) we conclude that

$$\begin{aligned} \kappa(\bar{t}) &> \kappa(t_M) e^{-\left(\widetilde{D} - \frac{mS^0}{a+S^0}\right)(\bar{t}-t_M) - \alpha \int_{t_M}^{\bar{t}} z^*(\theta_s \omega) ds} \\ &\geq k(\bar{t}), \end{aligned}$$

in other words, we obtain

$$\kappa(\bar{t}) > \kappa(\bar{t}),$$

which is clearly a contradiction.

As a consequence of this we deduce that for all values of t

$$\sigma > -(a + S^0)e^{\alpha z^*}.$$

Now we would like to check that this global solution belongs to the set $\mathscr{X}$ for any $t \in \mathbb{R}^+$. If there exists $t \in \mathbb{R}^+$ such that $\kappa(t) = 0$, assuming $\sigma(0) > 0$, we have

$$\begin{aligned} \frac{d\sigma}{dt}(t) &= \left[-(\bar{D}+\alpha z^*)\sigma - \frac{m(S^0+\sigma e^{-\alpha z^*})}{a+S^0+\sigma e^{-\alpha z^*}}\kappa \right](t) \\ &= -(\bar{D}+\alpha z^*)\sigma(t), \end{aligned}$$

and therefore

$$\sigma(t) = \sigma(0)e^{-\bar{D}t-\alpha\int_0^t z^*(\theta_s\omega)ds},$$

which, since $\bar{D} \geq 0$, implies that

$$\lim_{t\uparrow+\infty} \sigma(t) = 0 \quad \text{and} \quad \lim_{t\downarrow-\infty} \sigma(t) = +\infty,$$

Similarly, assuming $\kappa(t) = 0$ and $\sigma(0) < 0$, we obtain

$$\lim_{t\uparrow+\infty} \sigma(t) = 0 \quad \text{and} \quad \lim_{t\downarrow-\infty} \sigma(t) = -\infty.$$

By the previous analysis, we deduce that for any initial data $u_0 \in \mathscr{X}$, the solution $u(t)$ remains in $\mathscr{X}$.

Now we can define the mapping $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathscr{X} \to \mathscr{X}$ given by

$$\varphi_u(t,\omega)u_0 := u(t;\omega,u_0), \quad \forall t \geq 0,\ u_0 \in \mathscr{X},\ \omega \in \Omega.$$

Since the function F is continuous in u, t, and is measurable in ω, we obtain the $(\mathscr{B}[0,+\infty) \times \mathscr{F} \times \mathscr{B}(\mathscr{X}), \mathscr{B}(\mathscr{X}))-$measurability of the previous mapping. Items (i), (ii) and (iii) in Definition 1 follow easily by the definition of φ_u.

3.3 Existence of the random attractor

Now, we study the existence of a random attractor, describing it explicitly.

Lemma 3. *Under the assumption (11), there exists a tempered compact random absorbing set $B_\varepsilon(\omega) \in \mathscr{D}(\mathscr{X})$, for all $\varepsilon > 0$, of the random dynamical system $\{\varphi_u(t,\omega)\}_{t\geq 0,\,\omega\in\Omega}$, that is, for any $D \in \mathscr{D}(\mathscr{X})$ and each $\omega \in \Omega$, there exists $T_D(\omega) > 0$ such that*

$$\varphi_u(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B_\varepsilon(\omega) \quad \forall t \geq T_D(\omega).$$

Proof. Recall that $\varphi_u(t,\omega)u_0 = u(t;\omega,u_0)$ denotes the solution of system (12)-(13), satisfying $u(0;\omega,u_0) = u_0$, where $u_0 := u_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$.

First we assume that $\sigma(t) \geq 0$ for all $t \geq 0$ and define $\|\cdot\|_1$ as

$$\begin{aligned}\|\varphi_u(t,\theta_{-t}\omega)u_0\|_1 &= \|u(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)\|_1 \\ &= \sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) + \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)).\end{aligned}$$

Note that

$$\begin{aligned}&\sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) + \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) \\ &\quad\leq \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\sigma_0+\kappa_0\} e^{-\widetilde{D}t-\alpha\int_0^t z^*(\theta_s\theta_{-t}\omega)ds} \\ &\quad= \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\sigma_0+\kappa_0\} e^{-\widetilde{D}t-\alpha\int_{-t}^0 z^*(\theta_s\omega)ds}.\end{aligned}$$

Therefore, thanks to the temperedness of $D(\omega)$ and (11), there exists $T_D(\omega)$ such that $\|u(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)\|_1 \leq \varepsilon$, for all $\varepsilon > 0$, $u_0 \in D(\theta_{-t}\omega)$, when $t > T_D(\omega)$.

Define

$$B^1_\varepsilon(\omega) := \{(\sigma,\kappa) \in \mathscr{X} \,:\, 0 \leq \sigma+\kappa \leq \varepsilon\},$$

then $B^1_\varepsilon(\omega)$ is positively invariant according to Lemma 2 and absorbing in $\mathscr{X}$.

Now we assume that there exists some $\widetilde{t} \geq 0$ such that $\sigma(\widetilde{t}) < 0$. In this case we proved that $\sigma(t) < 0$ for all $t \geq \widetilde{t}$. We now get

$$\begin{aligned}\sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) &\leq \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\sigma_0\} e^{-\bar{D}t-\alpha\int_0^t z^*(\theta_s\theta_{-t}\omega)ds} \\ &= \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\sigma_0\} e^{-\bar{D}t-\alpha\int_{-t}^0 z^*(\theta_s\omega)ds}\end{aligned}$$

and

$$\begin{aligned}\kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) &\leq \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\kappa_0\} e^{-\left(\tilde{D}-\frac{mS^0}{a+S^0}\right)t-\alpha\int_0^t z^*(\theta_s\theta_{-t}\omega)ds} \\ &= \sup_{(\sigma_0,\kappa_0)\in D(\theta_{-t}\omega)} \{\kappa_0\} e^{-\left(\tilde{D}-\frac{mS^0}{a+S^0}\right)t-\alpha\int_{-t}^0 z^*(\theta_s\omega)ds}.\end{aligned}$$

Therefore, thanks to the temperedness of $D(\omega)$ and (11), there exists $T_D(\omega)$ such that

$$\sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) \leq \varepsilon \quad \text{and} \quad \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) \leq \varepsilon$$

for all $\varepsilon > 0$, $u_0 \in D(\theta_{-t}\omega)$, when $t > T_D(\omega)$.

On the other hand, from (9)-(10) we always have

$$\frac{d(\sigma+\kappa)}{dt} \geq -(\bar{D}+\alpha z^*)(\sigma+\kappa),$$

thus

$$(\sigma+\kappa)(t) \geq (\sigma+\kappa)(0)e^{-\bar{D}t-\alpha\int_0^t z^*(\theta_s\omega)ds}$$

which tends to zero when t goes to infinty since $\bar{D} \geq 0$.

Hence, $\sigma+\kappa \geq 0$ iff $\sigma \geq -\kappa$, thus

$$\sigma(t;\theta_{-t}\omega,\sigma_0(\theta_{-t}\omega)) \geq -\varepsilon$$

for all $\varepsilon > 0$, $u_0 \in D(\theta_{-t}\omega)$, when $t > T_D(\omega)$.

We define

$$B^2_\varepsilon(\omega) := \{(\sigma,\kappa) \in \mathscr{X} \,:\, -\varepsilon \leq \sigma \leq \varepsilon, 0 \leq \kappa \leq \varepsilon\},$$

then $B^2_\varepsilon(\omega)$ is positively invariant according to Lemma 2 and absorbing in $\mathscr{X}$.

In conclusion, considering

$$B_\varepsilon(\omega) = B^1_\varepsilon(\omega) \cup B^2_\varepsilon(\omega) = B^2_\varepsilon(\omega),$$

it follows directly from Proposition 1 that the random dynamical system generated by the system (12)-(13) possesses a unique random attractor given by

$$\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_\varepsilon(\omega), \quad \text{for all}\, \varepsilon > 0,$$

thus

$$\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega} = \{(0,0)\}.$$

3.4 Existence of the random attractor for the stochastic chemostat system

We have proved that the system (9)-(10) has a unique global solution $u(t;\omega,u_0)$ which remains in $\mathscr{X}$ for all $u_0 \in \mathscr{X}$ and generates the RDS φ_u.

Now, we define a mapping

$$T : \Omega \times \mathscr{X} \longrightarrow \mathscr{X}$$

as follows

$$T(\omega,\zeta) = T(\omega,(\zeta_1,\zeta_2)) = \begin{pmatrix} T_1(\omega,\zeta_1) \\ T_2(\omega,\zeta_2) \end{pmatrix} = \begin{pmatrix} (\zeta_1 - S^0)e^{\alpha z^*(\omega)} \\ \zeta_2 e^{\alpha z^*(\omega)} \end{pmatrix}$$

whose inverse is given by

$$T^{-1}(\omega,\zeta) = \begin{pmatrix} S^0 + \zeta_1 e^{-\alpha z^*(\omega)} \\ \zeta_2 e^{-\alpha z^*(\omega)} \end{pmatrix}.$$

We know that $v(t) = (S(t),x(t))$ and $u(t) = (\sigma(t),\kappa(t))$ are related by (7)-(8). Since T is a homeomorphism, thanks to Lemma 1 we obtain a conjugated RDS given by

$$\begin{aligned}
\varphi_v(t,\omega)v_0 &:= T^{-1}(\theta_t\omega,\varphi_u(t,\omega)T(\omega,v_0)) \\
&= T^{-1}\left(\theta_t\omega,\varphi_u(t,\omega)\begin{pmatrix} (S(0)-S^0)e^{\alpha z^*(\omega)} \\ x(0)e^{\alpha z^*(\omega)} \end{pmatrix}\right) \\
&= T^{-1}(\theta_t\omega,\varphi_u(t,\omega)u_0) \\
&= T^{-1}(\theta_t\omega,u(t;\omega,u_0)) \\
&= \begin{pmatrix} S^0 + \sigma(t)e^{-\alpha z^*(\theta_t\omega)} \\ \kappa(t)e^{-\alpha z^*(\theta_t\omega)} \end{pmatrix} \\
&= v(t;\omega,v_0)
\end{aligned}$$

which means that φ_v is an RDS for our original stochastic system (5)-(6).

Moreover, the global random attractor of the random system (9)-(10)

$$\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega} = \{(0,0)\}$$

becomes

$$\widetilde{\mathscr{A}} = \{\widetilde{A}(\omega)\}_{\omega\in\Omega} = \{(S^0,0)\},$$

the global random attractor of the stochastic system (5)-(6).

3.5 Numerical simulations and final comments

To confirm the results above, in this section we show some numerical simulations for (3)-(4). We use the Euler-Maruyama method [15] considering an initial value $(S_0, x_0) = (5, 10)$, $S^0 = 1$, $D = 3$, $a = 0.6$, $m = 3$ and the following numerical scheme:

$$S_j = S_{j-1} + f(x_{j-1}, S_{j-1})\Delta t + g(x_{j-1}, S_{j-1}) \cdot (W(\tau_j) - W(\tau_{j-1})),$$
$$x_j = x_{j-1} + \widetilde{f}(x_{j-1}, S_{j-1})\Delta t + \widetilde{g}(x_{j-1}, S_{j-1}) \cdot (W(\tau_j) - W(\tau_{j-1})),$$

where we define functions f, g, $\widetilde{f}$ and $\widetilde{g}$ as

$$f(x_{j-1}, S_{j-1}) = \left[(S^0 - S_{j-1})D - \frac{mS_{j-1}x_{j-1}}{a + S_{j-1}}\right],$$
$$g(x_{j-1}, S_{j-1}) = \alpha(S^0 - S_{j-1}),$$
$$\widetilde{f}(x_{j-1}, S_{j-1}) = x_{j-1}\left(\frac{mS_{j-1}}{a + S_{j-1}} - D\right),$$
$$\widetilde{g}(x_{j-1}, S_{j-1}) = \alpha x_{j-1},$$

and

$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} dW_k,$$

where R is a nonnegative integer number and dW_k are $\mathcal{N}(0,1)-$distributed independent random variables which can be generated numerically by pseudorandom number generators.

From now on, the red lines in the pictures represent the stochastic solutions of system (3)-(4) and the blue ones the deterministic solutions of the same system.

By the previous sections, we know that system (3)-(4) possesses a random attractor given by $\tilde{\mathscr{A}} = \{(S^0, 0)\}$ as long as (11) is satisfied. For the following different values of α we obtain the following values of $\tilde{\lambda}$:

(a) **Case $\alpha = 0.1$:**

$$\widetilde{\lambda} := \frac{\widetilde{D}a}{m - \widetilde{D}} = 359.4 \geq 1 = S^0.$$

(b) **Case** $\alpha = 0.5$**:**

$$\widetilde{\lambda} := \frac{\widetilde{D}a}{m - \widetilde{D}} = 13.8 \geq 1 = S^0.$$

(c) **Case** $\alpha = 1$**:**

$$\widetilde{\lambda} := \frac{\widetilde{D}a}{m - \widetilde{D}} = 3 \geq 1 = S^0.$$

(d) **Case** $\alpha = 1.5$**:**

$$\widetilde{\lambda} := \frac{\widetilde{D}a}{m - \widetilde{D}} = 1 \geq 1 = S^0.$$

Summing up, in all the above cases $\widetilde{\lambda} \geq S^0$ and $D \geq \frac{\alpha^2}{2}$ hold, hence the solutions of system (3)-(4) for the previous values of the parameters go to $(S^0, 0) = (1, 0)$, the random attractor.

The following pictures show what we expected from the theory and numerical computing and we also can observe what happens when the intensity of noise increases.

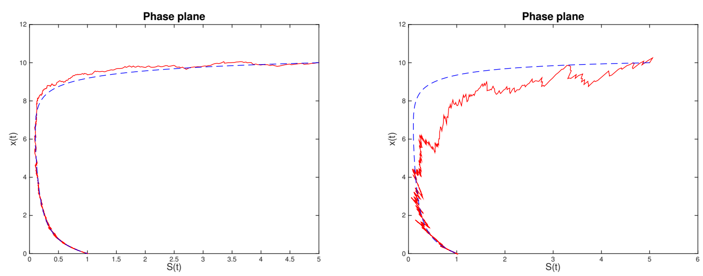

Fig. 1 $\alpha = 0.1$ on the left and $\alpha = 0.5$ on the right

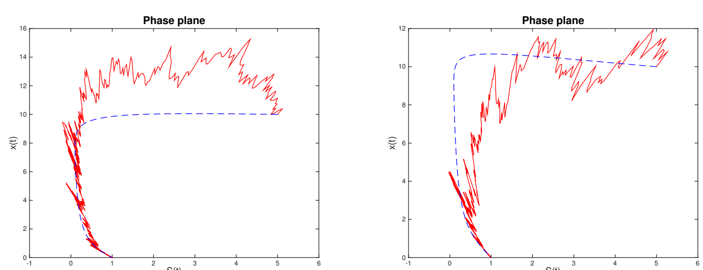

Fig. 2 $\alpha = 1$ on the left and $\alpha = 1.5$ on the right

However, the next pictures show what happens if $\widetilde{\lambda} < S^0$ holds true. In this case $D = 1.5$ instead of $D = 3$ as in the previous cases.

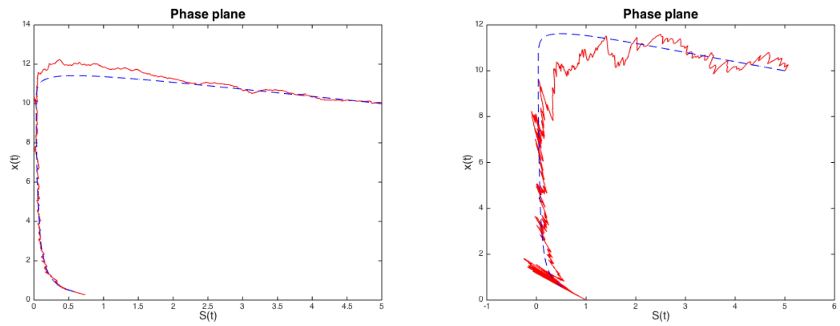

Fig. 3 $\alpha = 0.1$ on the left and $\alpha = 0.5$ on the right

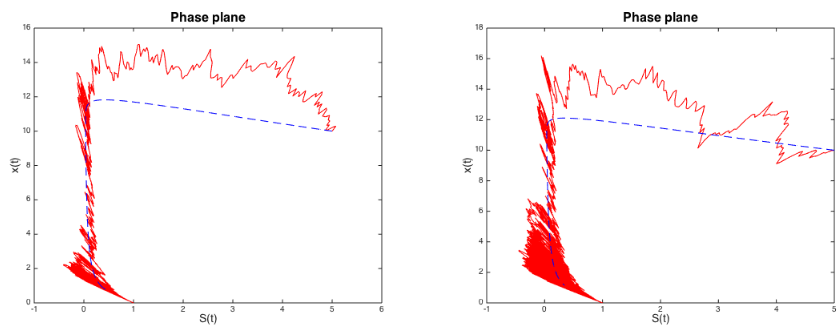

Fig. 4 $\alpha = 0.7$ on the left and $\alpha = 0.9$ on the right

Remark 3. We would like to mention that the fact that the substrate S (or its corresponding σ) may take negative values does not produce any mathematical inconsistence in our analysis, in other words, our mathematical analysis is accurate to handle the mathematical problem. However, from a biological point of view, this may reflect some troubles and suggests that either the fact of perturbing the dilution rate with an additive noise may not be a realistic situation, or that we should try to use a some kind of switching system to model our real chemostat in such a way that when the dilution may be negative we use a different equation to model the system. This will lead us to a different analysis in some subsequent papers by considering a different kind of randomness or stochasticity in this parameter or designing a different model for our problem.

On the other hand, it could also be considered a noisy term in each equation of the deterministic model in the same fashion as in the paper by Imhof and Walcher [16], which ensures the positivity of both the nutrient and biomass, although does not preserve the wash out equilibrium from the deterministic to the stochastic model. We are currently interested on this kind of chemostat models and we will analyze them in future papers.

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