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# Stabilization of oscillations in a phase transition model

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In this paper we analyze a model presenting formation of microstructure depending on the parameters and the initial data. In particular we investigate how the presence of stochastic perturbations affects this phenomenon in its asymptotic behavior. Two different sufficient conditions are provided in order to prevent the formation of microstructure: the first one for Stratonovich's noise while the second for Itô's noise. The main contribution of the paper is that these conditions are independent of the initial values unlike in the deterministic model. Thus, we can interpret our results as some kind of stabilization produced by both types of noise.

**Keywords:** Stabilisation; Stratonovich Noise; Phase Transitions

## 1. Introduction

The main objective of this paper is to analyze the effects produced by different types of noise on the asymptotic dynamics of the following evolution equation

$$\begin{cases} v_t = -\varepsilon^2 v_{xxxx} + \frac{1}{2} W''(v_x) v_{xx} - \frac{\mu}{\varepsilon^2} (v - u), & (x, t) \in I \times (0, T), \\ v(x, 0) = v_0(x), & \text{in } I, \\ v = v_{xx} = 0, & \text{on } \partial I, \end{cases} \quad (1)$$

where  $I = (0, 1)$ , the function  $W(p) = (p^2 - 1)^2$  is the so called double well potential,  $\varepsilon > 0$  is a small parameter,  $\mu \geq 0$ , and  $v_0(x)$  and  $u(x)$  are given functions on  $I$ .

As we will explain in the next section, this problem is well posed and the existence and regularity of solution can be proved by the same arguments used in [12].

Equation (1) is the  $L^2$ -gradient dynamics associated to the following functional arising in nonlinear elasticity (see [22]):

$$F_{\varepsilon, \mu}(v, u) = \frac{1}{2} \varepsilon^2 \int_I v_{xx}^2 dx + \frac{1}{2} \int_I W(v_x) dx + \frac{\mu}{2\varepsilon^2} \int_I (v - u)^2 dx. \quad (2)$$

It can be considered as a simple model for microstructure formation (in the sense of wrinkles, see [20] for a detailed discussion of this topic) in the context of phase transition. In general, any structure on a scale between the atomic and the macroscopic is regarded as a microstructure.

In this case, it is natural to expect that for small values of  $\varepsilon$ , the function  $v$  must be close to  $u$  in  $L^2$  norm, while its derivative must be close to the minima of the potential  $W$ , that is  $\pm 1$ . These two tendencies can be in hard competition producing an oscillating behavior of the first derivative (microstructure formation). This phenomenon should depend on the choice of the given function  $u$  and on the parameter  $\mu$ . One expects microstructure formation (wrinkles, see [20] once more) in the regions where  $|u_x| \ll 1$ , while no microstructure should be observed in the region where  $|u_x| \simeq 1$ .

Moreover, the third term suggests that  $\|v - u\|_{L^2(I)} = O(\varepsilon)$  from which we expect that the amplitude of the oscillations is of order  $O(\varepsilon)$ . On the other hand, if  $v^{\varepsilon, \mu}$  denotes a minimizer to (2) and  $J$  is an interval in which  $v_x^{\varepsilon, \mu}$  oscillates, that is,  $v_x^{\varepsilon, \mu}$  is "far" from  $u_x$ , we expect that  $J$  has  $\varepsilon$ -dependent measure that goes to zero as  $\varepsilon \rightarrow 0$  in order to prevent the blow up of the third term of the functional for  $\varepsilon \rightarrow 0$ . Actually, the wave length will result to be of order  $O(\varepsilon)$  (see [11]).

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Furthermore, in [11] (see also [15] for a generalization of the model) the precedent conjectures about the properties of the minimizer of (2) have been rigorously proved. In particular the author provided a critical value of  $\mu$  and conditions on  $u$  in order to have presence or absence of microstructure in a subinterval of  $I$ . It is worth noting that two different definitions of microstructures have been considered in [11]: one involving the first derivatives and another one related to the second derivatives of minimizers.

The first definition involving the first derivative is the following:

**Definition 1** *The minimizers  $v^{\varepsilon,\mu}$  of (2) do not present microstructure in  $J \subset I$  if*

$$\|v^{\varepsilon,\mu} - u\|_{W^{1,2}(J)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

*The minimizers  $v^{\varepsilon,\mu}$  of (2) do present microstructure in  $J \subset I$  if there exists a positive constant  $C$  such that:*

$$\|v^{\varepsilon,\mu} - u\|_{W^{1,2}(J)} > C, \quad \text{for all } 0 < \varepsilon \ll 1,$$

*and the number of oscillations of  $v_x^{\varepsilon,\mu}$  in  $J$  goes to infinity as  $\varepsilon \rightarrow 0$ , where  $W^{1,2}(J) = \{v \in L^2(J) : v_x \in L^2(J)\}$ .*

The second definition involves the second derivative of the minimizers and it is based on the number of change of sign of  $w_{xx}^{\varepsilon,\mu} = v_{xx}^{\varepsilon,\mu} - u_{xx}$ :

**Definition 2** *Let  $f : (a, b) \subset I \rightarrow \mathbb{R}$  be a continuous function and let  $\mathcal{I}$  be the set of all finite partitions  $\{x_i\}_{i=0}^{N+1}$  of  $(a, b)$ ,  $x_0 := a < x_1 < \dots < x_N < x_{N+1} := b$  such that for any  $i \in \{0, \dots, N\}$  either  $f \geq 0$  in  $(x_i, x_{i+1})$  or  $f \leq 0$  in  $(x_i, x_{i+1})$ . We define*

$$\mathcal{N}(f; (a, b)) := \inf_{\mathcal{I}} \{x_i\},$$

$$\mathcal{N}^{\varepsilon,\mu} = \inf \mathcal{N}(w_{xx}^{\varepsilon,\mu}; (a, b)),$$

*where the inf is taken over all the minimizers  $v^{\varepsilon,\mu} = w^{\varepsilon,\mu} + u$  of (2).*

*It is said that  $v^{\varepsilon,\mu}$  oscillates around  $u$  in  $(a, b)$  if*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{N}^{\varepsilon,\mu} = +\infty.$$

It is natural to expect that the asymptotic behavior of the solution of (1) resembles the properties of minimizers of (2).

In particular, according to the results in [11] we expect that, if  $\mu$  is sufficiently large, then the asymptotic dynamics is very simple: the global attractor  $\mathcal{A}_{\varepsilon,\mu}$  (see [25]) reduces to the trivial set  $\{v = u\}$ , moreover, the critical value for the parameter  $\mu$  should be very similar to that found in [11] in the variational framework.

The case  $\mu = 0$  has been studied in several ways: in [3] the authors studied the global dynamics, and they found three different time scales with peculiar dynamical behaviors. The papers [12]–[14] are devoted to the third time scale, of order  $O(\varepsilon^2)$ , in which the system experiments finite dimensional dynamics. In particular, in [14] the existence of a global and of an exponential attractor ([19]) is proved while in [9],[8] the stability of the model with respect to autonomous and stochastic perturbations has been tested.

In this paper we consider the case  $\mu > 0$  and we study the asymptotic behavior of the solution of (1) according to its dependence on the parameter  $\mu$ .

In [11] it has been proved that the phenomenon of microstructure formation not only depends on  $\mu$  but also on the choice of the function  $u$ . For simplicity, we consider only the case  $u = 0$ . However this is a nontrivial and interesting case since  $u_x = 0$  and for any  $x \in I$  the distance between  $u_x(x)$  and  $\{-1, 1\}$  is maximal. Moreover,  $u_x = 0$  corresponds to the maximum of the non convex region of the potential  $W(\cdot)$ . Then, for this choice, the competition between the second and third term of (2) becomes harder.

The paper is organized as follows. In Section 2 we consider the deterministic case and obtain a condition for the absence of microstructure in the asymptotic behavior according to Definition 3. Then we analyze how the presence of stochastic noise affects this phenomenon and modifies the results of the deterministic context. For this reason, in Section 3, we first recall some classical results about stochastic processes and Brownian motions which will be helpful in our analysis. Although in the modeling of stochastic phenomena there are many possibilities for the choice of the appropriate random or stochastic term in the model, we will consider in this paper two different kinds of noise which have been used very often in the literature, and have also provided a nice controversy between the suitability of one or the other, and somehow can be considered as canonical ones. We will not aim at going deeper into this controversy, but only show that different types of interpretation of the noise can produce different stabilization results. To this end, we will consider a linear multiplicative noise in the Stratonovich sense (Section 4) while the second interpretation will be in the Itô sense (Section 5). The main point which is worth highlighting is that when we consider the Itô interpretation, the sufficient conditions obtained are in general better, but in some situations this interpretation is not the most appropriate to model the problem and the Stratonovich formulation is used instead. Still this interpretation yields some kind of stabilization on the deterministic situation as the value of the parameter  $\mu$  ensuring the absence of microstructure formation is independent of the initial value and the intensity of the noise.

## 2. Deterministic case

In this section we consider the deterministic equation (1) with  $u = 0$  and study the problem of presence of microstructure in the asymptotic behavior.

The equation reads

$$\begin{cases} v_t = -\varepsilon^2 v_{xxxx} + \frac{1}{2} W''(v_x) v_{xx} - \frac{\mu}{\varepsilon^2} v, & (x, t) \in I \times (0, T), \\ v(x, 0) = v_0(x), & \text{in } I, \\ v = v_{xx} = 0, & \text{on } \partial I. \end{cases} \quad (3)$$

In this dynamical context we will consider the following definition for the absence of microstructure.

**Definition 3** *The solution of (3),  $v$ , does not present microstructure in the asymptotic behavior if*

$$\lim_{t \rightarrow \infty} \|v(t)\|_{\infty} = 0.$$

We consider the following spaces  $H = L^2(I)$ ,  $V = H^2(I)$  and the linear self-adjoint operator  $A = \frac{\partial^4}{\partial x^4}$  together with its domain

$$D(A) = \{v \in H^4(I) : v = v_{xx} = 0, \text{ on } \partial I\}.$$

From classical results about evolution equations (see [24] or Theorems 3.1 and 3.3 on [25]) it is straightforward to see that this problem is well posed and, in fact, we have the following result:

**Theorem 1** *Problem (3) admits a unique solution  $v$  such that*

$$\begin{aligned} \text{if } v_0 \in H &\Rightarrow v \in C^0([0, T], H) \cap L^2(0, T; V), \quad \forall T > 0, \\ \text{if } v_0 \in V &\Rightarrow v \in C^0([0, T], V) \cap L^2(0, T; D(A)), \quad \forall T > 0. \end{aligned}$$

We start our analysis by observing that the energy is decreasing. Indeed, multiplying the equation by  $v_t$  and integrating on  $I$  we obtain

$$\begin{aligned} \|v_t\|^2 &= -\frac{\varepsilon^2}{2} \frac{d}{dt} \|v_{xx}\|^2 - \frac{1}{2} \frac{d}{dt} \int_I W(v_x) dx - \frac{\mu}{2\varepsilon^2} \frac{d}{dt} \|v\|^2 \\ &= -\frac{d}{dt} F_{\varepsilon, \mu}(v(x, t), 0), \end{aligned}$$

where we use  $\|\cdot\|$  to denote the  $L^2$  norm. Consequently,

$$F_{\varepsilon, \mu}(v(x, t), 0) \leq F_{\varepsilon, \mu}(v(x, 0), 0) := F_0.$$

From the previous inequality we obtain the following a priori estimates:

$$\|v_{xx}\| \leq \frac{\sqrt{2F_0}}{\varepsilon}, \quad \|v\| \leq \varepsilon \sqrt{\frac{2F_0}{\mu}}. \quad (4)$$

Then, using our boundary conditions and integrating by parts we have:

$$\int_I v v_{xx} dx = [v v_x]_0^1 - \int_I v_x^2 dx,$$

from which

$$\int_I v_x^2 dx = \left| \int_I v v_{xx} dx \right| \leq \int_I |v| |v_{xx}| dx \leq \|v\| \|v_{xx}\| \leq \frac{1}{2} \{\|v\|^2 + \|v_{xx}\|^2\},$$

and, on account of (4),

$$\|v_x\|^2 \leq \frac{1}{2} \{\|v\|^2 + \|v_{xx}\|^2\} \quad (5)$$

$$\leq F_0 (\varepsilon^{-2} + \varepsilon^2 \mu^{-1}). \quad (6)$$

Finally, using the Gagliardo - Nirenberg inequality (see [4] pag. 233) for  $u \in H^1$ ,  $p = \infty$ ,  $q = r = 2$ ,  $N = 1$ ,  $a = \frac{1}{2}$ , which reads

$$\|u\|_{\infty} \leq C \|u\|^{1/2} \|u\|_{H^1}^{1/2}, \quad \text{for } u \in H^1, \quad (7)$$

and, particularizing for  $u = v_x$ , we obtain

$$\|v_x\|_{\infty} \leq C \|v_x\|^{1/2} \|v_x\|_{H^1}^{1/2},$$

and using the above estimate we conclude

$$\|v_x\|_\infty \leq \tilde{C}(\varepsilon, \mu). \quad (8)$$

In the following lines we study the asymptotic behavior of the  $L^2$  norm of  $v$  and  $v_x$  respectively, in particular we are interested in the absence of oscillations. While it is natural to expect that  $\|v(t)\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $t \rightarrow \infty$  in both cases of microstructure presence and absence, we will check its  $L^\infty$  norm in order to detect oscillations.

Multiplying the equation by  $v$  and integrating over  $I$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \varepsilon^2 \|v_{xx}\|^2 = \frac{1}{2} \int_I W''(v_x) v_{xx} v dx - \frac{\mu}{\varepsilon^2} \|v\|^2.$$

Then, thanks to the Hölder inequality and the fact that  $a^2 + b^2 \geq 2ab$ ,

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \varepsilon^2 \|v_{xx}\|^2 \leq \frac{1}{16\varepsilon^2} \int_I [W''(v_x)]^2 v^2 dx + \varepsilon^2 \|v_{xx}\|^2 - \frac{\mu}{\varepsilon^2} \|v\|^2.$$

If we fix

$$\mu_C = \frac{1}{16} \max_I [W''(v_x)]^2,$$

then we obtain

$$\frac{d}{dt} \|v\|^2 \leq -\frac{2}{\varepsilon^2} (\mu - \mu_C) \|v\|^2.$$

By the Gronwall Lemma we have that

$$\|v(t)\| \leq \|v_0\| e^{-\frac{t}{\varepsilon^2} (\mu - \mu_C)},$$

and, if  $\mu > \mu_C$ , then the asymptotic dynamics is described by

$$\lim_{t \rightarrow \infty} \|v(t)\| \leq 0.$$

This means that the global attractor  $\mathcal{A}_{\mu, \varepsilon} = \{v = 0\}$ , moreover the value of  $\mu_C$  is very similar to that one appearing in [11]. We note that, using the estimate (8), the critical value  $\mu_C$  is always well defined and finite, and can be estimated by using the initial datum. Since we are interested in the effect of stochastic perturbations, in this context we are not interested in finding the exact value of  $\mu_C$  and of the subsequent critical values. The expression of the critical values are obtained in [11], using fine techniques, for the variational problem.

A similar result is true for the  $L^2$  norm of  $v_x$ , in fact if we multiply equation (3) by  $v_{xx}$  and integrate over  $I$  we obtain:

$$\frac{1}{2} \frac{d}{dt} \|v_x\|^2 = -\varepsilon^2 \|v_{xxx}\|^2 + \frac{1}{2} \int_I W'(v_x) v_{xxx} dx - \frac{\mu}{2\varepsilon^2} \|v_x\|^2$$

The term involving the potential can be estimated in the following way:

$$\begin{aligned} \frac{1}{2} \int_I W'(v_x) v_{xxx} dx &= 2 \int_I v_x (v_x^2 - 1)^2 v_{xxx} dx \\ &\leq \varepsilon^2 \|v_{xxx}\|^2 + \frac{1}{\varepsilon^2} \int_I v_x^2 (v_x^2 - 1)^2 dx \\ &\leq \varepsilon^2 \|v_{xxx}\|^2 + \frac{\tilde{\mu}_C}{2\varepsilon^2} \|v_x\|^2, \end{aligned}$$

where we have set

$$\tilde{\mu}_C = \max_I 2(v_x^2 - 1)^2 = \max_I 2W(v_x).$$

Again the critical value  $\tilde{\mu}_C$  is well defined due to the estimate (8) and can be estimated by using the initial datum. It is worth mentioning that the corresponding critical values for the variational problem do not depend on  $\varepsilon$  (see [11]), although this is not straightforward to prove (we just want to observe now that it is possible to obtain an upper bound of their values by using the above a priori estimates). However, our intention in providing these details of the deterministic case is just to have an introduction on the topic, but our real interest is to analyze the effects produced by noise. In this respect, we will see how the addition of noise will simplify this task and critical values of the parameter  $\mu$  can be explicitly provided in a much easier way. This is why we will refer to these effects as stabilization produced by the noise.

From the previous inequalities we obtain:

$$\frac{d}{dt} \|v_x\|^2 \leq -\frac{1}{\varepsilon^2} (\mu - \tilde{\mu}_C) \|v_x\|^2,$$

and thanks again to Gronwall's lemma we conclude:

$$\|v_x(t)\| \leq \|v_x(0)\| e^{-\frac{t}{2\epsilon^2}(\mu - \tilde{\mu}_C)}.$$

If  $\mu > \tilde{\mu}_C$ , then the asymptotic dynamics is described by the following inequality

$$\lim_{t \rightarrow \infty} \|v_x(t)\| \leq 0.$$

By the above inequality we obtain the same results for the  $L^\infty$  norms if  $\mu > \max\{\mu_C, \tilde{\mu}_C\}$ :

$$\lim_{t \rightarrow \infty} \|v(t)\|_\infty \leq 0,$$

which means that the limit behavior is zero with no oscillations. We can summarize these results in the following theorem.

**Theorem 2** *If  $\mu > \max\{\mu_C, \tilde{\mu}_C\}$ , the solution of (3) does not exhibit microstructure in the asymptotic behavior.*

We remark that if  $\mu < \min\{\mu_C, \tilde{\mu}_C\}$  the asymptotic dynamics could be very complicated.

### 3. Some preliminary definitions and basic results on Ornstein-Uhlenbeck processes

As we mentioned in the introduction, we are not interested in complex noise terms, but only aim to show how different simple noises can produce different effects on the deterministic system. For this reason, and for simplicity in our analysis, the noisy terms in our perturbed models will be based on a real and standard Wiener process or Brownian motion  $\beta_t(\cdot)$  defined on a filtered and complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  where the filtration satisfies the standard assumptions (see [1] for more details). However, as our approach will be based on the techniques from the field of random dynamical systems, as we will transform our stochastic equations into conjugated random differential equations, we will consider the canonical representation for the Wiener process being defined also two-sided, i.e. for  $t \in \mathbb{R}$ . In other words, we will consider its canonical representation. To this end, we identify the probability space with the canonical space of continuous functions  $\Omega = C_0(\mathbb{R})$ , i.e. every event  $\omega \in \Omega$  is a continuous functions  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\omega(0) = 0$ , and we define the shift  $\theta$  by  $\theta_t(\omega)(\cdot) = \omega(t + \cdot) - \omega(t)$ . Moreover, we can identify  $\beta_t(\omega) = \omega(t)$  for every  $\omega \in \Omega$ .

Now we introduce a special stationary stochastic process known as Ornstein-Uhlenbeck process which will be crucial for our analysis.

Let us consider the one-dimensional stochastic differential equation

$$dz = -\lambda z dt + d\beta_t \tag{9}$$

for some  $\lambda > 0$ . This equation possesses a special stationary solution known as the stationary Ornstein-Uhlenbeck (OU) process and we will first describe now its representation and basic properties.

In Caraballo et al. [10] the next result is proved.

**Lemma 1** *Let  $\lambda$  be a positive number. There exists a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant subset  $\overline{\Omega} \in \mathcal{F}$  of  $\Omega = C_0(\mathbb{R}, \mathbb{R})$  of full measure such that*

$$\lim_{t \rightarrow \pm\infty} \frac{|\omega(t)|}{t} = 0 \quad \text{for } \omega \in \overline{\Omega},$$

and, for such  $\omega \in \overline{\Omega}$ , the random variable given by

$$z^*(\omega) := -\lambda \int_{-\infty}^0 e^{\lambda\tau} \omega(\tau) d\tau$$

is well defined. Moreover, for  $\omega \in \overline{\Omega}$ , the mapping

$$\begin{aligned} (t, \omega) \rightarrow z^*(\theta_t \omega) &= -\lambda \int_{-\infty}^0 e^{\lambda\tau} \theta_t \omega(\tau) d\tau \\ &= -\lambda \int_{-\infty}^0 e^{\lambda\tau} \omega(t + \tau) d\tau + \omega(t) \end{aligned} \tag{10}$$

is a stationary solution of (9) with continuous trajectories. In addition, for  $\omega \in \overline{\Omega}$

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{|z^*(\theta_t \omega)|}{|t|} &= 0, & \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_\tau \omega) d\tau &= 0, \\ \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_\tau \omega)| d\tau &= \mathbb{E}|z^*| < \infty, \end{aligned} \tag{11}$$

where  $\mathbb{E}$  denotes the expectation.

- Remark 1**
1. In the sequel, as we will be using the (OU) process for our computations, we will always refer to this probability space  $\overline{\Omega}$  although we may denote it without the bar.
  2. This (OU) process plays a crucial role in the theory of random dynamical systems in order to construct some homeomorphisms between the solutions of stochastic differential equations and random differential equations. We will not aim at giving more details on this but will exploit this technique in order to analyze the asymptotic behavior of our models.

## 4. Stabilization with Stratonovich noise.

We consider the following stochastic perturbation of equation (3) by multiplicative noise in the sense of Stratonovich:

$$\begin{cases} v_t = -\varepsilon^2 v_{xxxx} + \frac{1}{2}[W'(v_x)]_x - \frac{\mu}{\varepsilon^2}(v) + \frac{\sigma}{\varepsilon^2}v \circ d\beta_t, & (x, t) \in I \times (0, T), \\ v(x, 0) = v_0(x), & \text{in } I, \\ v = v_{xx} = 0, & \text{on } \partial I, \end{cases} \quad (12)$$

where  $\beta_t(\cdot)$  is the one dimensional Wiener process introduced in Section 3. This equation appears when we admit that the parameter  $\mu$  can be affected by some white noise, for instance, instead of  $\mu$  we consider  $\mu - \sigma\beta_t$  and, in this way, (3) becomes (12) (the sign preceding  $\sigma$  is not relevant at all for our analysis). We would like to point out that this choice is suggested by a large literature (see for example [2] and [26]) in which Stratonovich's noise is regarded as very much appropriate for applications in many real systems.

In order to justify the well-posedness of the problem we first have to rewrite the equation in its equivalent Itô's formulation (see Section 5 for more details). Thanks to the initial and boundary conditions as in the deterministic case, we can rewrite the problem in a variational formulation for which the theory of stochastic partial differential equations developed, for instance, by Pardoux [21] and by Da Prato and Zabczyk [18] can be applied. However, we will exploit the techniques of the theory of random dynamical systems to analyze the problem because the linear structure of the noisy term allows us to do it. The advantage of this technique is that, to solve problem(12), we can perform a change of variable, by means of an appropriate homeomorphism involving the (OU) process, and transform our stochastic problem into a random one for which the deterministic tools can be used. As we are not interested in this paper in proving the existence of random attractors for our model, we will not recall more details about the theory of random dynamical systems. The interested reader can obtain more information in the published literature on such topic (see e.g. [17]).

Thus, to transform the above stochastic different equation into a random differential equation, we consider the following change of variables:

$$u(t, \omega) = T^{-1}(\theta_t \omega)v(t, \omega),$$

where

$$T(\omega) = e^{\frac{\sigma}{\varepsilon^2}z^*(\omega)},$$

and where  $z^*$  is the (OU) process introduced in Section 3. For simplicity and whenever there is no possibility of confusion, we will simply write  $u$  or  $u(t)$  instead of  $u(t, \omega)$ .

Using the above change of variables we can rewrite the stochastic equation (12) as the following random partial differential equation:

$$u_t + \varepsilon^2 u_{xxxx} = \frac{1}{2}T^{-1}(\theta_t \omega)W'[T(\theta_t \omega)u_x]_x + H(\theta_t \omega)u, \quad (13)$$

where the last term is given by

$$H(\theta_t \omega) = \frac{\sigma}{\varepsilon^2}z^*(\theta_t \omega) - \frac{\mu}{\varepsilon^2}.$$

Notice that this equation is a deterministic partial differential equation depending on the random parameter  $\omega$ , and for that reason, it can be analyzed by the deterministic techniques.

Multiplying now the previous equation by  $u$  and integrating in  $I$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \varepsilon^2 \|u_{xx}\|^2 + \frac{1}{2} T^{-1}(\theta_t \omega) \int_I W'[T(\theta_t \omega)u_x] u_x dx = H(\theta_t \omega) \|u\|^2.$$

In details,

$$\begin{aligned} \frac{1}{2} T^{-1}(\theta_t \omega) \int_I W'[T(\theta_t \omega)u_x] u_x dx &= 2 \int_I u_x^2 [T(\theta_t \omega)^2 u_x^2 - 1] dx \\ &= 2T(\theta_t \omega)^2 \|u_x\|_4^4 - 2\|u_x\|^2, \end{aligned}$$

whence

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \varepsilon^2 \|u_{xx}\|^2 + 2T(\theta_t \omega)^2 \|u_x\|_4^4 = H(\theta_t \omega) \|u\|^2 + 2\|u_x\|^2.$$

Then, disregarding the positive term  $2T(\theta_t\omega)^2\|u_x\|_4^4$ ,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &\leq \left\{H(\theta_t\omega) + \frac{1}{\varepsilon^2}\right\}\|u\|^2 \\ &= \left\{\frac{\sigma}{\varepsilon^2}z^*(\theta_t\omega) - \frac{(\mu-1)}{\varepsilon^2}\right\}\|u\|^2,\end{aligned}$$

where we have also used the following interpolating inequality

$$\|u_x\|^2 \leq \frac{1}{2}\left\{\frac{1}{\varepsilon^2}\|u\|^2 + \varepsilon^2\|u_{xx}\|^2\right\},$$

which is deduced in a similar way as it was proved (5).

Then, the Gronwall Lemma yields

$$\begin{aligned}\|u(t)\| &\leq \|u(0)\| \exp\left(\int_0^t \left\{\frac{\sigma}{\varepsilon^2}z^*(\theta_s\omega) - \frac{(\mu-1)}{\varepsilon^2}\right\} ds\right) \\ &= \|u(0)\| \exp\left(\frac{t}{\varepsilon^2}\left\{\frac{\sigma}{t}\int_0^t z^*(\theta_s\omega) ds - (\mu-1)\right\}\right).\end{aligned}$$

Thus, using the properties (11) of  $z^*$ , we deduce that  $\lim_{t\rightarrow+\infty} \frac{1}{t}\int_0^t z^*(\theta_\tau\omega) d\tau = 0$  for all  $\omega \in \bar{\Omega}$ . Then, if  $\mu > 1$ , there exists  $T(\omega) > 0$  such that

$$\frac{1}{t}\int_0^t z^*(\theta_\tau\omega) d\tau < \frac{\mu-1}{2\sigma}, \text{ for all } t \geq T(\omega).$$

Therefore,

$$\|u(t)\| \leq \|u(0)\| \exp\left(-\frac{t}{\varepsilon^2}\frac{(\mu-1)}{2}\right), \text{ for all } t \geq T(\omega),$$

what implies that, if  $\mu > 1$ , independently of the initial datum, the solution  $u = 0$  is asymptotically stable with probability one:

$$\lim_{t\rightarrow\infty} \|u(t)\| = 0, \mathbb{P} - a.s. \text{ (i.e. for all } \omega \in \bar{\Omega}\text{)}.$$

We note that the condition  $\mu > 1$  is better than any estimate that we can find of the constant  $\mu_C$  from the deterministic case, by using the initial datum and the a priori estimates. Moreover, the previous condition is uniform with respect to the initial data. A similar result is true for the  $L^2$  norm of  $u_x$ . Indeed, multiplying the equation by  $u_{xx}$  and integrating over  $I$ , we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \varepsilon^2\|u_{xxx}\|^2 + 6T(\theta_t\omega)^2\|u_x u_{xx}\|^2 = 2\|u_{xx}\|^2 + H(\theta_t\omega)\|u_x\|^2.$$

Using again our boundary conditions and integrating by parts, we have

$$\int_I u_x u_{xxx} dx = [u_x u_{xx}]_0^1 - \int_I u_{xx}^2 dx,$$

and, consequently,

$$\|u_{xx}\|^2 = \int_I u_{xx}^2 dx = \left|\int_I u_x u_{xxx} dx\right| \leq \int_I |u_x| |u_{xxx}| dx \leq \|u_x\| \|u_{xxx}\| \leq \frac{1}{2}\left\{\frac{1}{\varepsilon^2}\|u_x\|^2 + \varepsilon^2\|u_{xxx}\|^2\right\}.$$

Then, disregarding the positive term  $6T(\theta_t\omega)^2\|u_x u_{xx}\|^2$ , we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2 &\leq \left\{H(\theta_t\omega) + \frac{1}{\varepsilon^2}\right\}\|u_x\|^2 \\ &\leq \left(\frac{\sigma}{\varepsilon^2}z^*(\theta_t\omega) - \frac{(\mu-1)}{\varepsilon^2}\right)\|u_x\|^2,\end{aligned}$$

and thanks again to the Gronwall lemma

$$\begin{aligned}\|u_x\| &\leq \|u_x(0)\| \exp\left(\int_0^t \left\{\frac{\sigma}{\varepsilon^2}z^*(\theta_s\omega) - \frac{(\mu-1)}{\varepsilon^2}\right\} ds\right) \\ &= \|u_x(0)\| \exp\left(\frac{t}{\varepsilon^2}\left\{\frac{\sigma}{t}\int_0^t z^*(\theta_s\omega) ds - (\mu-1)\right\}\right).\end{aligned}$$

Using again the ergodic properties of the (OU) described in (11), we obtain the same conclusion as for  $\|u\|$ , i.e.

$$\lim_{t\rightarrow\infty} \|u_x(t, \omega)\| = 0, \mathbb{P} - a.s.$$

Thus, using inequality (7) we obtain that no oscillations are observed in the asymptotic behavior:

$$\lim_{t \rightarrow \infty} \|u(t)\|_{\infty} = 0, \mathbb{P} - a.s.$$

**Theorem 3** For any  $\sigma$  and  $\mu > 1$ , the solution of (12) does not present microstructure in the asymptotic behavior.

**Proof.** Observe that the solution  $v$  of (12) is given by

$$v(t) = T(\theta_t \omega) u(t).$$

Consequently,

$$\begin{aligned} \|v(t)\| &= T(\theta_t \omega) \|u(t)\| \\ &\leq \|u(0)\| \exp \left( \frac{t}{\varepsilon^2} \left\{ \frac{\sigma}{t} \int_0^t z^*(\theta_s \omega) ds + \frac{\sigma z^*(\theta_t \omega)}{t} - (\mu - 1) \right\} \right). \end{aligned}$$

Notice that, thanks to the ergodic properties (11) of  $z^*$ , the two first terms in the previous exponential go to zero as  $t$  goes to  $+\infty$ , that is,

$$\lim_{t \rightarrow +\infty} \left( \frac{\sigma}{t} \int_0^t z^*(\theta_s \omega) ds + \frac{\sigma z^*(\theta_t \omega)}{t} \right) = 0, \text{ for all } \omega \in \bar{\Omega}.$$

Then, if  $\mu > 1$ , there exists  $T(\omega) > 0$  such that

$$\frac{\sigma}{t} \int_0^t z^*(\theta_\tau \omega) d\tau + \frac{\sigma z^*(\theta_t \omega)}{t} < \frac{\mu - 1}{2}, \text{ for all } t \geq T(\omega),$$

and we conclude that

$$\lim_{t \rightarrow \infty} \|v(t)\| = 0, \mathbb{P} - a.s.$$

Analogous arguments imply also that

$$\lim_{t \rightarrow \infty} \|v_x(t)\| = 0, \mathbb{P} - a.s.$$

and

$$\lim_{t \rightarrow \infty} \|v(t)\|_{\infty} = 0, \mathbb{P} - a.s.$$

as required.

## 5. Stabilization with Itô noise.

In this section we consider a stochastic perturbation in the Ito sense and discuss the difference with the Stratonovich case. As it is already well documented in the literature (see [5, 6, 7]), very simple forms for Itô's noise can produce a stabilization effect provided the intensity of the noise is large enough. The consequence on our study will be that for such kind of noise the absence of oscillation may take place for any positive value of the parameter  $\mu$ , independently of the initial values as well.

Let us then consider the following Itô equation which appears when we assume that the parameter  $\mu$  is now affected by an Itô white noise (notice that this is only a matter of modeling, and it is the modeler, at light of some other physical considerations, who makes the decision of choosing one or another interpretation; we, as mathematicians, only wish to show that different effects can appear for those alternatives.

$$\begin{cases} dv = \left[ -\varepsilon^2 v_{xxxx} + \frac{1}{2} [W'(v_x)]_x - \frac{\mu}{\varepsilon^2} v \right] dt + \frac{\sigma}{\varepsilon^2} v d\beta_t, & (x, t) \in I \times (0, T), \\ v(x, 0) = v_0(x), & \text{in } I, \\ v = v_{xx} = 0, & \text{on } \partial I, \end{cases}$$

which can be rewritten in its equivalent Stratonovich form (see [1]):

$$\begin{cases} dv = \left[ -\varepsilon^2 v_{xxxx} + \frac{1}{2} [W'(v_x)]_x - \left( \frac{\mu}{\varepsilon^2} + \frac{\sigma^2}{2\varepsilon^2} \right) v \right] dt + \frac{\sigma}{\varepsilon^2} v \circ d\beta_t, & (x, t) \in I \times (0, T), \\ v(x, 0) = v_0(x), & \text{in } I, \\ v = v_{xx} = 0, & \text{on } \partial I. \end{cases} \quad (14)$$



The same reasons argued in the case of Stratonovich noise ensure the well-posedness of this problem and, in order to solve it, we consider again the same change of variables as in the previous section:

$$u(t) = T^{-1}(\theta_t \omega) v(t),$$

where

$$T(\omega) = e^{\frac{\sigma}{\varepsilon^2} z^*(\omega)},$$

and  $z^*$  is the (OU) process from Section 3. In this case, our transformed equation reads

$$u_t + \varepsilon^2 u_{xxxx} = \frac{1}{2} T^{-1}(\theta_t \omega) W' [T(\theta_t \omega) u_x]_x + L(\theta_t \omega) u, \quad (15)$$

where the last term is given by

$$L(\theta_t \omega) = \frac{\sigma}{\varepsilon^2} z^*(\theta_t \omega) - \frac{\mu}{\varepsilon^2} - \frac{\sigma^2}{2\varepsilon^2}.$$

Then, as in the previous section, we obtain

$$\begin{aligned} \|u(t)\| &\leq \|u(0)\| \exp \left( \int_0^t \left\{ \frac{\sigma}{\varepsilon^2} z^*(\theta_s \omega) - \frac{(\mu-1)}{\varepsilon^2} - \frac{\sigma^2}{2\varepsilon^2} \right\} ds \right) \\ &= \|u(0)\| \exp \left( \frac{t}{\varepsilon^2} \left\{ \frac{\sigma}{t} \int_0^t z^*(\theta_s \omega) ds - \left( \frac{\sigma^2}{2} - 1 \right) - \mu \right\} \right). \end{aligned}$$

Taking into account once again the ergodic properties (11), we can deduce that, if  $\frac{\sigma^2}{2} \geq 1$ , then for any  $\mu > 0$ , it holds

$$\lim_{t \rightarrow +\infty} \|u(t)\| \leq 0, \quad \mathbb{P} - a.s.$$

By the same hypotheses on  $\sigma$  and  $\mu$  we obtain similar results for the  $L^2$  norm of  $u_x$  and  $L^\infty$  norm of  $u$ . Consequently, by using again the arguments in the proof of Theorem 3, we can easily prove the following result.

**Theorem 4** *The solution of equation (14) does not present microstructure in the asymptotic behavior provided that the values of  $\sigma$  and  $\mu$  are such that either  $|\sigma| \geq \sqrt{2}$ ,  $\mu > 0$  or  $|\sigma| < \sqrt{2}$ ,  $\mu > 1 - \frac{\sigma^2}{2}$ .*

## 6. Conclusion

In this article we have considered two different kind of stochastic perturbations of an evolution equation presenting microstructure formation. As we have already mentioned, the choice of the type of noise corresponds to the modeler and is based on the particular properties of the problem and the environment in which the phenomenon is taking place. Needless to say that there are many possibilities for the choice of the noise (bounded real noise in some of the parameters, white noise in some of the parameters or the vector field, additive or multiplicative noise in the vector field, Brownian motion, fractional Brownian motions, noise with jumps, etc). We only wish to show that the choice of the type of noise can be determining on the asymptotic behavior of the phenomenon, and we chose some simple situations to illustrate this.

In the case of a Stratonovich noise we obtain a condition for preventing oscillations in the asymptotic behavior which is uniform with respect the initial data. The condition only depends on  $\mu$ , that is  $\mu > 1$ , and is independent of the intensity of the noise  $\sigma$ . We remark that for the unperturbed case, an optimal bound for the parameter  $\mu$  for the absence of microstructure is not available, while its available bound is larger than 1. In the case of a stochastic perturbation in the Itô sense, we obtain a better result. In particular, the suppression of oscillations in the asymptotic behavior is ensured for large enough values of  $\sigma$ , that is  $\sigma^2 > 2$ , while it is independent of both the initial data and the values of  $\mu > 0$ .

Another interesting problem related to the study of this model is the case in which the equilibrium  $u = 0$  is destabilized, that is oscillations persist even for large values of the parameter  $\mu$ . The problem of destabilization still remains open and it will be addressed in some future research.

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