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COOPERATIVE GAMES UNDER AUGMENTING SYSTEMS*

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Abstract. The goal of this paper is to develop a theoretical framework in order to analyze cooperative games in which only certain coalitions are allowed to form. We will axiomatize the structure of such allowable coalitions using the theory of *antimatroids*, a notion developed for combinatorially abstract sets. There have been previous models developed to confront the problem of unallowable coalitions. Games restricted by a communication graph were introduced by Myerson and Owen. We introduce a new combinatorial structure called *augmenting system*, which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. The main result of the paper is a direct formula of Shapley and Banzhaf values for games under augmenting systems restrictions.

Key words. cooperative game, Shapley value, Banzhaf value, set systems

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1. Introduction.

Cooperative games under combinatorial restrictions are cooperative games in which the players have restricted communication possibilities, which are defined by a combinatorial structure. The first model in which the restrictions are defined by the connected subgraphs of a graph is introduced by Myerson [11]. Since then, many other situations where players have communication restrictions have been studied in cooperative game theory. Contributions on graph-restricted games include Owen [12], Borm, Owen, and Tijs [3], and Hamiache [8]. In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph*. Another type of combinatorial structure introduced by Gilles, Owen, and van den Brink [7] is equivalent to a subclass of antimatroids. This line of research focuses on the possibilities of coalition formation determined by the positions of the players in the so-called *permission structure*. Sandholm et al. [14] analyze coalition formation in combinatorial problems.

In the present paper, we use the restricted cooperation model derived from a combinatorial structure called *augmenting system*. Section 2 introduces this structure, which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. Furthermore, this new set system includes the conjunctive and disjunctive systems derived from a permission structure. Section 3 introduces games under augmenting systems which generalize the ones studied on graphs and permission structures. Using the structural properties from these systems we will be able to express the dividends in terms of the original game. This result will be essential in section 4 to provide direct formulas to compute the Shapley and Banzhaf values for games under augmenting systems restrictions. In these formulas, these values are computed by means of the original game without having to calculate the restricted game and taking into account only the coalitions in the augmenting system. Finally, in section 5 we consider the potential and the Owen multilinear extension (MLE) for the restricted game. These results generalize, unify and simplify results of Owen [12],

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Gilles, Owen, and van den Brink [7], and Bilbao [2].

2. Augmenting systems. Antimatroids were introduced by Dilworth [5] as particular examples of semimodular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte, Lovász, and Schrader [10]). In this section, a general cooperation structure is introduced, which is a weakening of the antimatroid structure.

Let N be a finite set. A *set system* over N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathcal{F}$ are called *feasible*. We will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively.

DEFINITION 2.1. *A set system $(N, \mathcal{A})$ is an antimatroid if*

A1. $\emptyset \in \mathcal{A}$,

A2. *for $S, T \in \mathcal{A}$, we have $S \cup T \in \mathcal{A}$,*

A3. *for $S \in \mathcal{A}$ with $S \neq \emptyset$, there exists $i \in S$ such that $S \setminus i \in \mathcal{A}$.*

The definition of antimatroid implies the following *augmentation property*: If $S, T \in \mathcal{A}$ with $|T| > |S|$, then there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{A}$. We call a set system $(N, \mathcal{F})$ *normal* if $N = \bigcup_{S \in \mathcal{F}} S$. If $(N, \mathcal{A})$ is a normal antimatroid, then property A2 implies that $N \in \mathcal{A}$.

DEFINITION 2.2. *An augmenting system is a normal set system $(N, \mathcal{F})$ with the following properties:*

P1. $\emptyset \in \mathcal{F}$,

P2. *for $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, we have $S \cup T \in \mathcal{F}$,*

P3. *for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{F}$.*

Remark. It follows from the definition that normal antimatroids are always augmenting systems.

PROPOSITION 2.3. *An augmenting system $(N, \mathcal{F})$ is an antimatroid if and only if $\mathcal{F}$ is closed under union.*

Proof. The necessary condition follows from A2. Conversely, we only have to prove A3. Let $S \in \mathcal{F}$ with $S \neq \emptyset$. By property P3 there exists a chain of feasible subsets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{s-1} \subset S_s = S$$

such that $S_k \in \mathcal{F}$ and $|S_k| = k$ for $0 \leq k \leq s$. Hence there exists an element $i \in S$ such that $S \setminus i = S_{s-1} \in \mathcal{F}$. $\square$

Example. The following collections of subsets of $N = \{1, \ldots, n\}$, given by $\mathcal{F} = 2^N$ and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are the maximum augmenting system and a minimal augmenting system over N, respectively.

Example. In a communication graph $G = (N, E)$, the set system $(N, \mathcal{F})$ given by $\mathcal{F} = \{S \subseteq N : (S, E(S))$ is a connected subgraph of $G\}$ is an augmenting system.

Example. Gilles, Owen, and van den Brink [7] showed that the feasible coalitions system $(N, \mathcal{F})$ derived from the conjunctive or disjunctive approach contains the empty set and the ground set N and that it is closed under union. Algaba et al. [1] showed that the coalitions systems derived from the conjunctive and disjunctive approach were identified to *poset antimatroids* and *antimatroids with the path property*, respectively. Thus, these coalitions systems are augmenting systems.

Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [6].

DEFINITION 2.4. *A set system $(N, \mathcal{G})$ is a convex geometry if it satisfies the following properties:*

C1. $\emptyset \in \mathcal{G}$,

C2. *for* $S, T \in \mathcal{G}$*, we have* $S \cap T \in \mathcal{G}$*,*

C3. *for* $S \in \mathcal{G}$ *with* $S \neq N$*, there exists* $i \in N \setminus S$ *such that* $S \cup i \in \mathcal{G}$*.*

Proposition 2.5. *An augmenting system* $(N, \mathcal{F})$ *is a convex geometry if and only if* $\mathcal{F}$ *is closed under intersection and* $N \in \mathcal{F}$*.*

Proof. The necessary conditions follow from properties C2 and C3. To prove sufficiency, note that $(N, \mathcal{F})$ satisfies C1 and C2, i.e., it is a closure system over N. Moreover, $(N, \mathcal{F})$ satisfies property P3 and $N \in \mathcal{F}$. Then for every $S \in \mathcal{F}$ with $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathcal{F}$. □

Definition 2.6. *Let* $(N, \mathcal{F})$ *be an augmenting system. For a feasible coalition* $S \in \mathcal{F}$*, we define the set* $S^* = \{i \in N \setminus S : S \cup i \in \mathcal{F}\}$ *of augmentations of* S *and the set* $S^+ = S \cup S^* = \{i \in N : S \cup i \in \mathcal{F}\}$*.*

Proposition 2.7. *Let* $(N, \mathcal{F})$ *be an augmenting system. Then the interval* $[S, S^+]_{\mathcal{F}} = \{C \in \mathcal{F} : S \subseteq C \subseteq S^+\}$ *is a Boolean algebra for every nonempty* $S \in \mathcal{F}$*.*

Proof. It is suffices to show that $[S, S^+]_{\mathcal{F}} = \{C \subseteq N : S \subseteq C \subseteq S^+\}$, i.e., for every $C \subseteq N$ such that $S \subseteq C \subseteq S^+$ we have $C \in \mathcal{F}$. If $S^* = \emptyset$, then $[S, S^+]_{\mathcal{F}} = \{S\}$. Otherwise, $S^* = \{i_1, \ldots, i_p\}$ and $S \subseteq C \subseteq S^+$ implies $C = S \cup \{i_1, \ldots, i_q\}$ for some $1 \leq q \leq p$. We prove that $C \in \mathcal{F}$ by induction on q. For $q = 1$ we know that $S \cup \{i_1\} \in \mathcal{F}$. Assume $S \cup \{i_1, \ldots, i_k\} \in \mathcal{F}$. Since $S \cup \{i_{k+1}\} \in \mathcal{F}$ and $(S \cup \{i_1, \ldots, i_k\}) \cap (S \cup \{i_{k+1}\}) = S \neq \emptyset$, property P2 yields $S \cup \{i_1, \ldots, i_k, i_{k+1}\} \in \mathcal{F}$. □

Let $(N, \mathcal{F})$ be a set system and let $S \subseteq N$ be a subset. A feasible subset $C \in \mathcal{F}$ with $C \subseteq S$ is called a *basis* of S if $C \cup i \notin \mathcal{F}$ for all $i \in S \setminus C$. The maximal nonempty feasible subsets of S are called *components* of S. Clearly, every component of S is a basis of S. However, the converse is not true, as the following example shows.

Example. If $N = \{1, 2, 3\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, N\}$, then $C = \{1\}$ is a basis of N, but the only component of N is the ground set N.

Observe that if $(N, \mathcal{A})$ is an antimatroid, then any subset $S \subseteq N$ has a unique basis given by the following operator $\text{int}(S) = \bigcup \{C \in \mathcal{A} : C \subseteq S\}$. This feasible set is also the unique component of S.

Proposition 2.8. *Let* $(N, \mathcal{F})$ *be an augmenting system and let* $S \subseteq N$ *be a subset. Then a nonempty feasible subset* $C \subseteq S$ *is a basis of* S *if and only if* C *is a component of* S*.*

Proof. Let $C \in \mathcal{F}$ be a basis of S and suppose C is not a component of S, i.e., there exists $D \in \mathcal{F}$ such that $C \subset D \subseteq S$. Then because of P3 there exists $i \in D \setminus C \subseteq S \setminus C$ such that $C \cup i \in \mathcal{F}$, which is a contradiction. □

We denote by $C_{\mathcal{F}}(S)$ the set of the components of a subset $S \subseteq N$. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. This set will play a role in the concept of a game restricted by an augmenting system.

Proposition 2.9. *A set system* $(N, \mathcal{F})$ *satisfies property* P2 *if and only if for any* $S \subseteq N$ *with* $C_{\mathcal{F}}(S) \neq \emptyset$*, the components of* S *form a partition of a subset of* S*.*

Proof. We suppose that $(N, \mathcal{F})$ satisfies P2 and let S_1, S_2 be components of S. If $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \in \mathcal{F}$ and we have that $S_i \subset S_1 \cup S_2 \subseteq S$ for $i \in \{1, 2\}$. This contradicts the fact that S_1 and S_2 are components of S. Conversely, assume for any S with $C_{\mathcal{F}}(S) \neq \emptyset$ that its components form a partition of a subset of S. Suppose that $(N, \mathcal{F})$ does not satisfy P2. Then there are $A, B \in \mathcal{F}$, with $A \cap B \neq \emptyset$ and $A \cup B \notin \mathcal{F}$. Hence there must be a component $C_1 \in C_{\mathcal{F}}(A \cup B)$ with $A \subseteq C_1$ and a component $C_2 \in C_{\mathcal{F}}(A \cup B)$ with $B \subseteq C_2$ such that $C_1 \neq C_2$. This contradicts the fact that the components of $A \cup B$ are disjoint. □

Let $N = \{1, \ldots, n\}$ be a set of players with $n > 2$ and we consider a subset S of starting players. If $i \in S$, then the set $\{i\}$ is feasible. Each starting player i looks for a player $k \notin S$ to generate a new feasible coalition $\{i,k\}$. These coalitions with cardinality 2 search for new players, which agree to join one by one. If we assume that common elements of two feasible coalitions are intermediaries between the two coalitions in order to establish the feasibility of its union, we obtain an augmenting system $(N,\mathcal{F})$. Since the individual players $k \notin S$ are not feasible, the family $\mathcal{F}$ is not generated by the connected subgraphs of a graph. Moreover, if players $i,j \in S$, then $\{i\},\{j\} \in S$ and $\{i,j\} \notin S$ and hence $(N,\mathcal{F})$ is not an antimatroid.

Example. Let $N = \{1,2,3,4\}$ and we consider $S_1 = \{1,2,4\}$ and $S_2 = \{1,4\}$. By using the above coalition formation model we can obtain the following augmenting systems, represented in Figure 1.

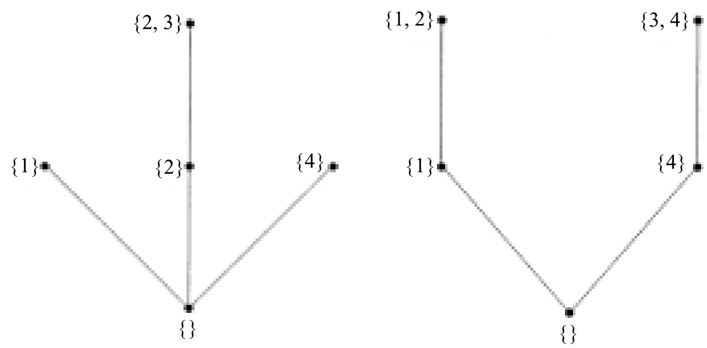

FIG. 1.

The sets of maximal feasible coalitions are partitions of the players into disjoint coalitions, that is, the coalition structures $CS_1 = \{\{1\},\{4\},\{2,3\}\}$ and $CS_2 = \{\{1,2\},\{3,4\}\}$. Coalition structure generation has been studied by Sandholm et al. [14].

Example. Let us consider $N = \{1,2,3,4\}$ and

$$\mathcal{F} = \{\varnothing,\{1\},\{4\},\{1,2\},\{3,4\},\{1,2,3\},\{2,3,4\},N\}.$$

Since $\{1,2,3\}$ and $\{2,3,4\}$ are feasible, property P2 implies that the grand coalition N is a feasible set; see Figure 2.

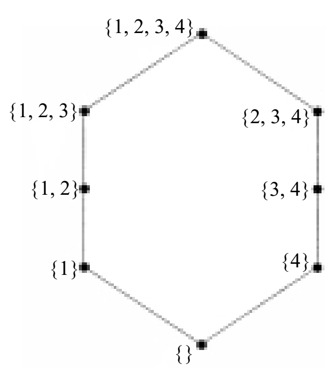

FIG. 2.

Example. The set system given by $N = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\varnothing, \{1\}, \{4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, N\}$$

is an augmenting system. Since $\{1, 4\} \notin \mathcal{F}$, the system $(N, \mathcal{F})$ represented in Figure 3 is not an antimatroid.

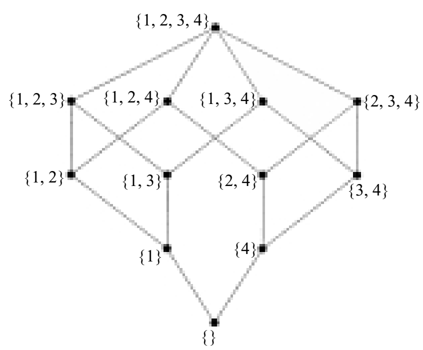

FIG. 3.

3. Games restricted by augmenting systems.

DEFINITION 3.1. *Let $v : 2^N \to \mathbb{R}$ be a cooperative game and let $(N, \mathcal{F})$ be an augmenting system. The restricted game $v^{\mathcal{F}} : 2^N \to \mathbb{R}$ is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

Remark. If $(N, \mathcal{F})$ is the augmenting system given by the connected subgraphs of a graph $G = (N, E)$, then the game $\left(N, v^{\mathcal{F}}\right)$ is a graph-restricted game which is studied by Myerson [11] and Owen [12].

If $S \in \mathcal{F}$, then $v^{\mathcal{F}}(S) = v(S)$. Let us denote by Γ^N the vector space of all cooperative games (N, v), i.e., functions $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Every cooperative game (N, v) is uniquely determined by the collection of its values $\{v(S) : S \subseteq N,\ S \neq \emptyset\}$. Then Γ^N will be identified with $\mathbb{R}^{2^n - 1}$. For any $S \subseteq N,\ S \neq \emptyset$, we define the *unanimity* game

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Every game is a unique linear combination of unanimity games (cf. Shapley [15]),

$$v = \sum_{S \subseteq N} d_S u_S, \quad \text{where } d_S = \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T).$$

We shall call d_S the *dividend* of S in the game v. Owen [12] showed the following property: *The unanimity games u_S, where S is connected in the graph G, form a basis of the graph-restricted games.*

Let $(N, \mathcal{F})$ be the system of connected subgraphs of a graph $G = (N, E)$. Hamiache [8] proved a formula for computing the dividends in the game $v^{\mathcal{F}}$ by using the values in the original game v. Next, we extend Hamiache's formula and Owen's property to the case when $(N,\mathcal{F})$ is an augmenting system.

PROPOSITION 3.2. *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game. Then the restricted game $\left(N,v^{\mathcal{F}}\right)$ satisfies $v^{\mathcal{F}}=\sum_{C\in\mathcal{F}}d_C u_C$, where the dividend*

$$d_C=\sum_{\{S\in\mathcal{F}\,:\,S\subseteq C\subseteq S^+\}}(-1)^{|C|-|S|}v(S)$$

for every nonempty $C\in\mathcal{F}$ and $d_C=0$ otherwise.

Proof. The game $v^{\mathcal{F}}$ satisfies for every $C\subseteq N$

$$v^{\mathcal{F}}(C)=\sum_{T\subseteq N}d_T u_T(C)=\sum_{T\subseteq C}d_T,$$

where d_T the dividend of T in the game $v^{\mathcal{F}}$. Then, the Möbius inversion formula implies (see Stanley [16]) that

$$d_C=\sum_{T\subseteq C}(-1)^{|C|-|T|}v^{\mathcal{F}}(T).$$

It follows from $v^{\mathcal{F}}(\emptyset)=0$ that $d_\emptyset=0$. So we may assume that $C\neq\emptyset$. The definition of $v^{\mathcal{F}}$ implies that

$$\begin{aligned}d_C&=\sum_{T\subseteq C}(-1)^{|C|-|T|}\left(\sum_{S\in C_{\mathcal{F}}(T)}v(S)\right)\\&=\sum_{\{S\in\mathcal{F}\,:\,S\subseteq C\}}\left(\sum_{\{T\subseteq C\,:\,S\in C_{\mathcal{F}}(T)\}}(-1)^{|C|-|T|}\right)v(S).\end{aligned}$$

Let $S\in\mathcal{F}$ with $S\subseteq C$. We first show that

$$\{T\subseteq C:S\in C_{\mathcal{F}}(T)\}=\left\{T\subseteq C:T\setminus S\subseteq C\setminus S^+\right\}.$$

We take $T\subseteq C$. If $S\in C_{\mathcal{F}}(T)$, then by Proposition 2.8, S is a basis of T and hence the set of its augmentations S^* satisfies $S^*\cap T=\emptyset$. Then for each $i\in T\setminus S$ we have $i\in C$ and $i\notin S\cup S^*=S^+$.

Conversely, let $T\subseteq C$ be a set such that $T\setminus S\subseteq C\setminus S^+$. Then for each $i\in T\setminus S$ we have $i\notin S^+$ and hence $S\cup i\notin\mathcal{F}$. Thus, the feasible set S is a basis of T and we conclude that $S\in C_{\mathcal{F}}(T)$.

Therefore, the coefficients of d_C satisfy

$$\begin{aligned}\sum_{\{T\subseteq C\,:\,S\in C_{\mathcal{F}}(T)\}}(-1)^{|C|-|T|}&=\sum_{\{T\subseteq C\,:\,S\subseteq T,\,T\setminus S\subseteq C\setminus S^+\}}(-1)^{|C|-|T|}\\&=(-1)^{|C|-|S|}\left(\sum_{R\subseteq C\setminus S^+}(-1)^{-|R|}\right).\end{aligned}$$

Next, we compute

$$\sum_{R\subseteq C\setminus S^+}(-1)^{-|R|}=\sum_{R\subseteq C\setminus S^+}(-1)^{|R|}=(1-1)^{\left|C\setminus S^+\right|}=\begin{cases}1 & \text{if } C\setminus S^+=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, $C \setminus S^{+} = \emptyset \Leftrightarrow C \subseteq S^{+}$, and hence

$$\begin{aligned} d_C &= \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C,\ C \setminus S^{+} = \emptyset\}} (-1)^{|C|-|S|} v(S) \\ &= \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C \subseteq S^{+}\}} (-1)^{|C|-|S|} v(S). \end{aligned}$$

To complete the proof we observe that Proposition 2.7 implies that the set $C \in \mathcal{F}$. Otherwise $C \setminus S^{+} \neq \emptyset$, and so $d_C = 0$ for all $C \notin \mathcal{F}$. □

4. The Shapley and Banzhaf values. Let (N, v) be a game and let $(N, \mathcal{F})$ be an augmenting system. The *Shapley value* for player i in the restricted game $v^{\mathcal{F}}$ is given by

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{(s-1)!(n-s)!}{n!} \left[v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)\right],$$

where $n = |N|$ and $s = |S|$. This value is an average of the *marginal contributions* $v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)$ of a player i to all coalitions $S \in 2^N \setminus \{\emptyset\}$. In this value, the sets S of different size get different weight. The *Banzhaf value* for player i in the restricted game $v^{\mathcal{F}}$ is given by

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{1}{2^{n-1}} \left[v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)\right]$$

for all $i \in N$. If the number of players is n, then the function that measures the worst case running time for computing these indices is in $O(n2^n)$ (see Deng and Papadimitriou [4]). Moreover, to obtain the restricted game $v^{\mathcal{F}}$ we need to compute the set of the components $C_{\mathcal{F}}(S)$ of every subset $S \subseteq N$. Then it is necessary to consider all the feasible subsets of S, and hence the time complexity is $O(t)$, where

$$t = \sum_{s=0}^{n} \binom{n}{s} 2^s = 3^n.$$

The Shapley and Banzhaf values are linear mappings with respect to the characteristic function, and the images of the unanimity games are, respectively (cf. Owen [12]),

$$\Phi_i(N, u_S) = \begin{cases} 1/|S| & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i'(N, u_S) = \begin{cases} 1/2^{|S \setminus i|} & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of dividends d_S in game $v^{\mathcal{F}}$, we have that

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{d_S}{|S|}, \tag{1}$$

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{d_S}{2^{|S \setminus i|}}.$$

In the next theorem, two *explicit formulas,* in terms of v, for the Shapley and Banzhaf values of the players in the restricted game $v^{\mathcal{F}}$ are proved. These formulas generalize the results obtained by Bilbao [2] for games restricted by convex geometries.

THEOREM 4.1. *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game. Then*

$$\Phi_i\left(N,v^{\mathcal{F}}\right)=\sum_{\{T\in\mathcal{F}\,:\,i\in T\}}\frac{(t-1)!\,t^*!}{t^+!}v(T)-\sum_{\{T\in\mathcal{F}\,:\,i\in T^*\}}\frac{t!\,(t^*-1)!}{t^+!}v(T),$$

$$\beta_i'\left(N,v^{\mathcal{F}}\right)=\sum_{\{T\in\mathcal{F}\,:\,i\in T\}}\frac{1}{2^{t^+-1}}v(T)-\sum_{\{T\in\mathcal{F}\,:\,i\in T^*\}}\frac{1}{2^{t^+-1}}v(T),$$

where $t=|T|$, $t^*=|T^*|$, and $t^+=|T^+|$.

Proof. By Proposition 3.2, we know that $d_S=0$ unless $S\in\mathcal{F}$. We use the formula (1) and Proposition 3.2 for computing

$$\begin{aligned}\Phi_i\left(N,v^{\mathcal{F}}\right)&=\sum_{\{S\in\mathcal{F}\,:\,i\in S\}}\frac{d_S}{|S|}\\&=\sum_{\{S\in\mathcal{F}\,:\,i\in S\}}\frac{1}{|S|}\left[\sum_{\{T\in\mathcal{F}\,:\,T\subseteq S\subseteq T^+\}}(-1)^{|S|-|T|}v(T)\right].\end{aligned}$$

Reversing the order of summation and denoting $s=|S|$ and $t=|T|$, we obtain

$$\begin{aligned}\Phi_i\left(N,v^{\mathcal{F}}\right)&=\sum_{T\in\mathcal{F}}\left[\sum_{\{S\in\mathcal{F}\,:\,i\in S,\,T\subseteq S\subseteq T^+\}}\frac{(-1)^{s-t}}{s}\right]v(T)\\&=\sum_{T\in\mathcal{F}}c_i(T)v(T),\end{aligned}$$

where

$$c_i(T)=\sum_{\{S\in\mathcal{F}\,:\,T\cup i\subseteq S\subseteq T^+\}}\frac{(-1)^{s-t}}{s}.$$

First, we suppose $i\in T$. By Proposition 2.7 the interval $[T,T^+]$ is a Boolean algebra and hence the summation index is $\{S\subseteq N: T\subseteq S\subseteq T^+\}$. Now we consider $S=T\cup R$, where $R=S\setminus T$, $r=|R|$, and $t^*=|T^*|$. Then

$$\begin{aligned}c_i(T)&=\sum_{R\subseteq T^*}\frac{(-1)^r}{t+r}=\sum_{r=0}^{t^*}\binom{t^*}{r}\frac{(-1)^r}{t+r}\\&=\sum_{r=0}^{t^*}\binom{t^*}{r}(-1)^r\int_0^1 x^{t+r-1}\,dx\\&=\int_0^1 x^{t-1}\sum_{r=0}^{t^*}\binom{t^*}{r}(-x)^r\,dx\\&=\int_0^1 x^{t-1}(1-x)^{t^*}\,dx\\&=\frac{(t-1)!\,t^*!}{t^+!}.\end{aligned}$$

Next, assume that $i \notin T$; hence the index is $\{S \in \mathcal{F} : T \cup i \subseteq S \subseteq T^+\}$. Then $i \in T^+ \setminus T$ and hence $i \in T^*$. Now the previous result yields (note that $[T \cup i, T^+]$ is a Boolean algebra)

$$c_i(T) = -\sum_{\{S \subseteq N \,:\, T \cup i \subseteq S \subseteq T^+\}} \frac{(-1)^{s-(t+1)}}{s} = -\frac{t!(t^* - 1)!}{t^+!}.$$

Inserting the coefficients, we have

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{T \in \mathcal{F} \,:\, i \in T\}} \frac{(t-1)!\, t^*!}{t^+!} v(T) - \sum_{\{T \in \mathcal{F} \,:\, i \in T^*\}} \frac{(t)!(t^*-1)!}{t^+!} v(T). \tag{2}$$

The proof of the formula of the Banzhaf value is similar. The only difference is that the coefficients are

$$c_i(T) = \sum_{r=0}^{t^*} \binom{t^*}{r} (-1)^r \left(\frac{1}{2}\right)^{t+r-1} = \left(\frac{1}{2}\right)^{t^+-1} \quad \text{if } i \in T,$$

$$c_i(T) = -\left(\frac{1}{2}\right)^{t^+-1} \quad \text{if } i \in T^*. \qquad \square$$

Remark. Notice that if $\mathcal{F} = 2^N$, then $T^* = N \setminus T$ and $T^+ = N$ for every $T \in \mathcal{F}$. Thus, the formulas obtained in the above theorem are equal to the classical Shapley and Banzhaf values for the game v. Moreover, equation (2) is equal to the equation of Shapley [15].

Let us consider a set system $(N, \mathcal{F})$. An element i of a feasible set $S \in \mathcal{F}$ is an *extreme point* of S if $S \setminus i \in \mathcal{F}$. The set of extreme points of S is denoted by $\mathrm{ex}(S)$. The formulas for computing the Shapley and Banzhaf values of the players in the restricted game $v^{\mathcal{F}}$ can be further simplified when the player is an extreme point of every feasible coalition. Before doing so, we will need a lemma.

Lemma 4.2. *Let $(N, \mathcal{F})$ be an augmenting system. If $i \in \mathrm{ex}(S)$ for all $S \in \mathcal{F}$ which contains i with $S \neq \{i\}$, then $(S \setminus i)^+ = S^+$.*

Proof. Note first that $i \in (S \setminus i)^+$ and $i \in S^+$. For every $j \in (S \setminus i)^+$ with $j \neq i$, we have $(S \setminus i) \cup j \in \mathcal{F}$. Then $((S \setminus i) \cup j) \cap S = S \setminus i \neq \emptyset$ implies $((S \setminus i) \cup j) \cup S = S \cup j \in \mathcal{F}$ and hence $j \in S^+$. Conversely, for every $j \in S^+$, $j \neq i$, we know that $S \cup j \in \mathcal{F}$. Since $i \in S \subseteq S \cup j$, the assumption implies that $i \in \mathrm{ex}(S \cup j)$. Then $(S \cup j) \setminus i = (S \setminus i) \cup j \in \mathcal{F}$ and thus $j \in (S \setminus i)^+$. $\square$

Theorem 4.3. *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game such that $v(i) = 0$ for all $i \in N$. If $i \in \mathrm{ex}\,(S)$ for all $S \in \mathcal{F}$ that contains i, then*

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \in \mathcal{F} \,:\, i \in S,\ |S|>1\}} \frac{(s-1)!\, s^*!}{s^+!} \left[v\left(S\right) - v(S \setminus i)\right],$$

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \in \mathcal{F} \,:\, i \in S,\ |S|>1\}} \frac{1}{2^{s^+-1}} \left[v(S) - v(S \setminus i)\right],$$

where $s = |S|$, $s^* = |S^*|$, and $s^+ = |S^+|$.

Proof. We remark first that if i satisfies the hypothesis, then

$$\{S \in \mathcal{F} : i \in S,\ |S| > 1\} = \{S \in \mathcal{F} : i \in \mathrm{ex}\,(S)\,,\ |S| > 1\}\,.$$

Taking $T = S \setminus i$ we obtain $\{T \in \mathcal{F} : i \in T^*\} = \{S \setminus i : S \in \mathcal{F},\ i \in \operatorname{ex}(S)\}$. Next, we apply Theorem 4.1 and therefore, by Lemma 4.2,

$$\begin{aligned}
\Phi_i\left(N, v^{\mathcal{F}}\right) &= \sum_{\{S\in\mathcal{F}\,:\, i\in S\}} \frac{(s-1)!\, s^*!}{s^+!} v(S) - \sum_{\{T\in\mathcal{F}\,:\, i\in T^*\}} \frac{t!\,(t^*-1)!}{t^+!} v(T) \\
&= \sum_{\{S\in\mathcal{F}\,:\, i\in \operatorname{ex}(S),\ |S|>1\}} \frac{(s-1)!\, s^*!}{s^+!} v(S) \\
&\quad - \sum_{\{S\in\mathcal{F}\,:\, i\in \operatorname{ex}(S),\ |S|>1\}} \frac{(s-1)!\, s^*!}{s^+!} v(S\setminus i) \\
&= \sum_{\{S\in\mathcal{F}\,:\, i\in S,\ |S|>1\}} \frac{(s-1)!\, s^*!}{s^+!} \left[v\left(S\right) - v(S\setminus i)\right]
\end{aligned}$$

(note that $v(i) = 0$ for all $i \in N$). The result for the Banzhaf value follows similarly. □

Remark. Let $(N, \mathcal{F})$ be an augmenting system that is a convex geometry. Then for every $i \in \operatorname{ex}(N)$ we have $S \setminus i = (N \setminus i) \cap S \in \mathcal{F}$ for all $S \in \mathcal{F}$ such that $i \in S$. Hence, if $i \in \operatorname{ex}(N)$, then $i \in \operatorname{ex}(S)$ for all $S \in \mathcal{F}$ with $i \in S$.

Example. Let $K_{1,n-1}$ be a star on n vertices and let 1 be the center of star. The augmenting system of the connected subgraphs of $K_{1,n-1}$ is given by $\mathcal{F} = \{S \subseteq N : 1 \in S \text{ or } |S| = 1\}$. Then $\operatorname{ex}(N) = \{2, \dots, n\}$, and for all $S \in \mathcal{F}$ such that $|S| > 1$, we infer that $1 \in S$, $S^* = N \setminus S$, and $S^+ = N$. Moreover, the set $\{S \in \mathcal{F} : 1 \in S^*,\ |S| > 1\} = \emptyset$. Using these properties, the following results can be derived from Theorems 4.1 and 4.3:

1. If (N, v) is a game such that $v(i) = 0$ for all $i \in N$, then

$$\Phi_1\left(N, v^{\mathcal{F}}\right) = \sum_{\{S\in\mathcal{F}\,:\, 1\in S,\ |S|>1\}} \frac{(s-1)!(n-s)!}{n!} v(S).$$

2. If (N, v) is a game such that $v(i) = 0$ for all $i \in N$, then

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S\in\mathcal{F}\,:\, i\in S,\ |S|>1\}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S\setminus i)\right]$$

for all $i \in \{2, \dots, n\}$.

Remark. The time complexity of the direct formulas showed in Theorems 4.1 and 4.3 is polynomial in the cardinality $|\mathcal{F}|$.

Example. Let us consider an augmenting system $(N, \mathcal{F})$ such that the family of its maximal elements is a coalition structure $CS = \{T_1, \dots, T_p\}$. Then the number of feasible elements is

$$|\mathcal{F}| = |T_1| + \cdots + |T_p| + 1 = |N| + 1,$$

and hence $|\mathcal{F}|$ is polynomial in $|N|$. For instance, the augmenting systems represented in Figure 1 satisfy $|\mathcal{F}| = 5$.

Example. Let $(N, \mathcal{F})$ be an augmenting system with exactly two maximal chains. Then $|\mathcal{F}| = 2\left(|N| - 1\right) + 2 = 2\,|N|$, and hence $|\mathcal{F}|$ is polynomial in $|N|$. For instance, the augmenting system represented in Figure 2 satisfies $|\mathcal{F}| = 8$.

5. The potential and the MLE. The potential function for cooperative games was defined by Hart and Mas-Colell [9]. Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* (S, v) is obtained by restricting v to 2^S. Let Γ denote the set of all games. The *potential* is a function $P : \Gamma \to \mathbb{R}$ which assigns to each game (N, v) a real number $P(N, v)$ and satisfies the following recursive equations:

$$P(\emptyset, v) = 0, \quad P(S, v) = \frac{1}{|S|}\left[v(S) + \sum_{i \in S} P(S \setminus i, v)\right]$$

for all nonempty $S \subseteq N$. Then the *marginal contribution* of i coincides with its Shapley value $P(N, v) - P(N \setminus i, v) = \Phi_i(N, v)$ for all $i \in N$. Moreover, there are two explicit formulas for the potential:

$$P(N, v) = \sum_{S \subseteq N} \frac{d_S}{|S|}, \quad P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S),$$

where $s = |S|$ and $n = |N|$. The explicit formula for the potential of $v^{\mathcal{F}}$ can be obtained by a method similar to the one that is used in Theorem 4.1.

THEOREM 5.1. *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Then*

$$P\left(N, v^{\mathcal{F}}\right) = \sum_{S \in \mathcal{F}} \frac{(s-1)!\, s^*!}{s^+!} v(S),$$

where $s = |S|$, $s^* = |S^*|$ and $s^+ = |S^+|$.

The *MLE* of the game (N, v) is the function of n real variables (see Owen [13]) $f(v)(q_1, \dots, q_n) = \sum_{S \subseteq N} \prod_{j \in S} q_j \; d_S$, where d_S is the dividend of S in the game (N, v). Owen showed that

$$\Phi_i(N, v) = \int_0^1 \frac{\partial f(v)}{\partial q_i}(t, \dots, t)\; dt,$$
$$\beta_i'(N, v) = \frac{\partial f(v)}{\partial q_i}\left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

PROPOSITION 5.2. *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Then the MLE of $v^{\mathcal{F}}$ is given by*

$$f\left(v^{\mathcal{F}}\right)(q_1, \dots, q_n) = \sum_{S \in \mathcal{F}} \prod_{j \in S} q_j \left(\sum_{\{T \in \mathcal{F} : T \subseteq S \subseteq T^+\}} (-1)^{|S|-|T|} v(T)\right).$$

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