Metamodeling abduction

Ángel NEPOMUCENO-FERNÁNDEZ, Fernando SOLER-TOSCANO

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ABSTRACT: Abduction can be intended as a special kind of deductive consequence. In fact a general trend is to consider it as a backward deduction with some additional conditions. However, there can be more than one kind of deduction, so that any definition of abduction must take that into account. From a logical perspective the problem is precisely the formalization of conditions when the deductive consequence is fixed. In this paper, we adopt Makinson's method to define new consequence relations, hence abduction is defined as a reverse relation corresponding to each one of such relations.

Keywords: abduction, explicative relations, pivotal consequences, structural rules.

1. Supra-classical consequence relations

1.1. Classical consequence

As it is well known, the classical consequence relation is defined from a formal language and is characterized by several structural rules. We are following in part Makinson (2003) to present requisites that a deductive consequence relation must accomplish, so we consider a propositional formal language L —to abbreviate L also represents the set of all its formulae— that contains a set of propositional variables, the falsehood $\bot$, and, at least, the connective $\rightarrow$, the material implication. $x \in L$ represents that x is a formula of L and $A \subseteq L$ that A is a set of formulae of L. A valuation ν: $L \mapsto \{0, 1\}$ is a function that assigns one of the truth values to every propositional variable and is in accordance with truth tables for molecular formulae constructed using the corresponding connectives. Given a set of formulae $A \subseteq L$ and a valuation ν, if $\nu(a) = 1$ for every $a \in A$, then $\nu(A) = 1$ will be written. Whatever the valuation ν may be, according to that, $\nu(\emptyset) = 1$.

In general, a consequence relation is defined as a subset of $\wp(L) \times L$. Since formulae of L may represent certain propositions that are relevant in determined inferential contexts, it is a relation between sets of formulae and formulae that represents a relation between sets of propositions and propositions. Concretely, in the case of the *classical consequence*, given $A \subseteq L$ and $x \in L$, one says that x is classical consequence of A if and only if —"iff", from now on— for all valuations ν, if $\nu(A) = 1$, then $\nu(x) = 1$, in symbols $A \quad x$. If $\emptyset \quad x$, then $\nu(x) = 1$ for every valuation ν and, following the classical point of view, we shall say that in this case x is a tautology but when $x \quad z$ for all $z \in L$, then x is a contradiction.

When A has only one element, that is to say, when $A = \{a\}$, we shall write $a \quad x$ instead of $\{a\} \quad x$. Similarly, if $A, B \subseteq L$ and $x \in L$, our usual notation will be $A, B \quad x$ or $A \cup B \quad x$. Another way of defining consequence is to take it as an operation

Cn: $\wp(L) \mapsto L$, though both representations are interchangeable: given the relation and any set $A \subseteq L$, $Cn(A) = \{x \in L \mid A \quad x\}$, and, reciprocally, given *Cn*, is defined by the condition $A \quad x$ iff $x \in Cn(A)$. To simplify, the format of relation is definitively adopted through the paper. The classical consequence accomplishes conditions that are known as *structural rules*. These are the following ones, for a, b, $x \in L$ and $A, B \subseteq L$:

- *Reflexivity*: $A \quad a$, for all $a \in A$
- *Cut*: if $A \quad b$ for all $b \in B$ and $A \cup B \quad x$, then $A \quad x$
- *Monotony*: If $A \quad x$ and $A \subseteq B$, then $B \quad x$

So, the classical relation is a *closure* relation, since it verifies the indicated conditions. Other important characteristics of that relation are

- *Compactness*: If $A \quad x$, then there is a finite $A' \subseteq A$ such that $A' \quad x$
- *Uniform substitution*: For the usual uniform substitution σ —defined as $\sigma : L \mapsto L$—, if $A \quad x$, then $\sigma(A) \quad \sigma(x)$, where $\sigma(A) = \{\sigma(a) \mid a \in A\}$

Taking the classical closure relation other ones can be defined. This will be used to tackle the problem of abduction from a logical point of view.

1.2. New consequence relations

Following the way initialized in Makinson (2003), some ways to obtain new relations could be described, so we consider other consequence relations: the so called *pivotal-assumptions*, *pivotal-valuations* and *pivotal-rules* consequences.

Let be the classical consequence relation. For $K \subseteq L$, $A \subseteq L$ and $x \in L$, we say that x is consequence (in the sense of) of A modulo K iff x is consequence of $K \cup A$, in symbols $A \quad_K x$ iff $A \cup K \quad x$. This is a pivotal-assumption consequence, which may capture inferences in which some premises are assumed though they are not ever explicit. Of course, $\quad_K$ is a supraclassical relation, that is to say, it accomplishes reflexivity, cut and monotony —and compactness, in this case. However, though the classical consequence relation is closed under substitution, not all supraclassical consequence relations are closed under substitution. For $\quad_K$, $A \quad_K x$, but there can be a substitution σ such that $\sigma(A) \;/_K\; \sigma(x)$. In fact —given a language L— there is no supraclassical consequence relation that is closed under substitution except to the classical itself —and the total $\wp(L) \times L$—, as indicated in Makinson (2003).

Let be the classical consequence relation, V the class of all valuations, and consider $W \subset V$, then a pivotal-valuation consequence $\quad_W$ is such that for $A \subset L$ and $x \in L$, $A \quad_W x$ iff for all $\nu \in W$, if $\nu(A) = 1$, then $\nu(x) = 1$. This relation is not closed under substitution. In fact, from the supraclassical consequence relation $\quad_K$, it is possible to find $W \subset V$ such that $\quad_K = \quad_W$. When $K = \varnothing$ and $W = V$, such result is trivial. Let K and W be such that $K \subset L$, $K \neq \varnothing$ and $W = \{\nu \in V \mid \nu(K) = 1\}$, then for every $x \in L$,

$A \quad _K x$ iff $K \cup A \quad x$ iff for every $\nu \in V$, if $\nu(K \cup A) = 1$, then $\nu(x) = 1$, but this holds for every valuation of W, so the former is equivalent to $A \quad _W x$. Finally, given a supraclassical consequence relation $\quad _W$, if it is compact, then there can be $K \subseteq L$ such that $\quad _K = \quad _W$.

Finally, consider that instead of adding assumptions or valuations, a set of rules is taken, which is a relation defined in L, that is to say, the set of rules R has elements of the form (a, x), where $a, x \in L$, so $R \subset L^2$, or, which is the same, $R \in \wp(L^2)$. For $A \subset L$, $R(A) = \{y \in L \mid$ there is $x \in A$ and $(x, y) \in R\}$. $A \subseteq L$ is closed under a set R of rules iff for all $x \in L$, if $x \in A$ and $(x, y) \in R$, then $y \in A$, in other words, iff $R(A) \subseteq A$. Now let be the classical consequence relation, a pivotal-rules consequence relation, or a consequence relation modulo a set of rules R, is defined as follows, for $A \subset L$ and $x \in L$: $A \quad _R x$ iff $x \in A'$ for every A' such that $A \subseteq A'$, and $R(A') \subseteq A'$ —$x \in A'$, for every superset A' of A such that it is closed under the corresponding supraclassical consequence operation and the set of rules R. Pivotal-rules consequences, like previous pivotal consequences, have the mentioned structural rules and may verify compactness.

2. *Explicative relations*

The slogan "no abduction without deduction" suggests a clear way to obtain solutions of abductive problems by developing the corresponding work that starts with a deductive consequence so considered. However, according to Kapitan's thesis —as summarized in (Hintikka 1999, p. 91)— abduction should be seen as an inferential process though it is neither deduction nor induction, therefore the formal method to tackle it would be specific. So, an abductive logic will be defined, with universality pretensions in a similar way than in Beziau (2005), by means of a relation too, though this will not be a subset of $\wp(L) \times L$ but a subset of $\wp(L) \times \wp(L)$, or an operation C^*: $\wp(L) \mapsto \wp(\wp(L))$.

In scientific literature the concept of "abduction" is used to refer a variety of explanatory processes and, taking it as an inference, there are various styles according to requirements for that, as studied in Aliseda (1997) and Aliseda (2006). First we shall define a new relation in order to capture a weak form of abduction —the most basic—, then methods for other forms will be able to be developed.

Given the propositional formal language L, let be the classical relation. $\quad _\alpha$ is defined in $\wp(L)$, that is to say, $\quad _\alpha \subseteq \wp(L) \times \wp(L)$, which for $A, B \subset L$ must accomplish the condition $A \quad _\alpha B$ iff $B \quad a$, for all $a \in A$. This could be read as "the set A is explained by B" or "the set B explains A". Since this notion has no restriction about the extension of A and B, $\varnothing \quad _\alpha B$ —$B \quad x$ for all $x \in \varnothing$—, which represents that the empty set of formulae is (vacuously) explained by any set of formulae. When $A \quad _\alpha \varnothing$, every element of A has the characteristic that has been called "tautological" in terms of classical logic. On the other hand, if $L \quad_{\alpha} A$, then every element of A is a "contradiction", since $A \quad x$ for all $x \in L$.

The relation $\quad_{\alpha}$ has some structural rules that are justified from basic ones of the corresponding . For any A, B, $C \subset L$, such basic rules are:

- *Reflexivity*: $A \quad_{\alpha} A$. Each set of formulae explains itself. Given $\quad_{\alpha}$, since $A \quad a$, for all $a \in A$, $A \quad_{\alpha} A$.
- *Right cut*: if $C \quad_{\alpha} A \cup B$ and $B \quad_{\alpha} A$, then $C \quad_{\alpha} A$. Given $\quad_{\alpha}$, suppose that $C \quad_{\alpha} A \cup B$ and $B \quad_{\alpha} A$, by definition $A, B \quad c$ and $A \quad b$, for all $c \in C$ and $b \in B$ respectively, then by cut of , $A \quad c$ for all $c \in C$, so that $C \quad_{\alpha} A$.
- *Right monotony*: if $A \quad_{\alpha} B$, then $A \quad_{\alpha} B \cup C$. When a set is explicative, its supersets are also explicative. Suppose that $A \quad_{\alpha} B$, by definition $B \quad a$ for all $a \in A$ and, by monotony of , $B \cup C \quad a$ for all $a \in A$, so that $A \quad_{\alpha} B \cup C$.

Another important property derives from compactness of the basic classical relation. Let   be the classical relation that verifies compactness and $\quad_{\alpha}$ the abductive relation defined as above, for A, $B \subset L$, if $A \quad_{\alpha} B$ and A is finite —in symbols $|A| < \omega_0$—, then there exists $B' \subseteq B$ such that $|B'| < \omega_0$ and $A \quad_{\alpha} B'$. Suppose that $A \quad_{\alpha} B$ and $|A| < \omega_0$, then by definition $B \quad a$ for all $a \in A$, so there is $B_a \subseteq B$ for every $a \in A$ such that $|B_a| < \omega_0$ and $B_a \quad a$. Then define $B' = \bigcup_{a \in A} B_a$, but $B' \quad a$ for every $a \in A$ and $|B'| < \omega_0$, because of which $A \quad_{\alpha} B'$.

Such structural rules are basic and express the essential characteristics of the explicative relation $\quad_{\alpha}$. Though each of them has been defined taking into account the underlying consequence relation ' , they are primitive because no one of them is derived from another, but some derived rules can be obtained. For X, Y, Z, $Q \subset L$, and separating antecedent and consequent by a line when necessary, the following rules are proved as derived

- *Isotonicity*,

$$\frac{X \quad_{\alpha} Y, Z \quad_{\alpha} Q}{X \cup Z \quad_{\alpha} Y \cup Q} :$$

1. $X \quad_{\alpha} Y$, premise
2. $Z \quad_{\alpha} Q$, premise
3. $X \cup Z \quad_{\alpha} X \cup Z$, reflexivity
4. $X \cup Z \quad_{\alpha} X \cup Z \cup Y \cup Q$, right monotony (3)
5. $X \quad_{\alpha} Y \cup Z \cup Q$, right monotony (1)
6. $X \cup Z \quad_{\alpha} Y \cup Z \cup Q$, right cut (4, 5)
7. $Z \quad_{\alpha} Y \cup Q$, right monotony (2)
8. $X \cup Z \quad_{\alpha} Y \cup Q$, right cut (6,7)

- *Thinning*,

$$\frac{X \cup Z \quad_{\alpha} Y}{X \quad_{\alpha} Y}:$$

1. $X \cup Z \quad_{\alpha} Y$, premise
2. $X \quad_{\alpha} X$, reflexivity
3. $X \quad_{\alpha} X \cup Y \cup Z$, right monotony (2)
4. $X \quad_{\alpha} Y$, right cut (1,3)

- *Unity of explanation,*

$$\frac{X \quad_{\alpha} Y, Z \quad_{\alpha} Y}{X \cup Z \quad_{\alpha} Y}:$$

1. $X \quad_{\alpha} Y$, premise
2. $Z \quad_{\alpha} Y$, premise
3. $X \cup Z \quad_{\alpha} Y \cup Y$, isotonicity (1,2),
4. $X \cup Z \quad_{\alpha} Y$, since $Y \cup Y = Y$ for all $Y \subseteq L$.

- A form of *left transitivity,*

$$\frac{X \quad_{\alpha} Y, Z \quad_{\alpha} X}{X \cup Z \quad_{\alpha} Y}:$$

1. $X \quad_{\alpha} Y$, premise
2. $Z \quad_{\alpha} X$, premise
3. $Z \quad_{\alpha} Y \cup X$, right monotony (2)
4. $Z \quad_{\alpha} Y$, right cut (2,3)
5. $X \cup Z \quad_{\alpha} Y$, isotonicity (4,5)

- *Reflexivity of parts of explanations*, $X \quad_{\alpha} X \cup Y$:
  1. $X \quad_{\alpha} X$, reflexivity
  2. $X \quad_{\alpha} X \cup Y$, right monotony (1)

- *Explication of* $\varnothing$, $\varnothing \quad_{\alpha} X$:
  1. $\varnothing \quad_{\alpha} \varnothing$, reflexivity
  2. $\varnothing \quad_{\alpha} \varnothing \cup X$, right monotony (1)
  3. $\varnothing \quad_{\alpha} X$, since $\varnothing \cup X = X$ for all $X \subseteq L$

Since there can be other consequence relations, new explicative consequences can be defined if we considerer pivotal-assumptions, pivotal-valuations and pivotal-rules consequences. So, given $K \subset L$,

$$A \quad_{\alpha, K} B \text{ iff } B \quad_{K} a \text{ for all } a \in A.$$

Given the set of valuations W,

$$A \quad {}_{\alpha, W} B \text{ iff } B \quad {}_{W}\, a \text{ for all } a \in A.$$

Finally, for a set of rules R,

$$A \quad {}_{\alpha, R} B \text{ iff } B \quad {}_{R}\, a \text{ for all } a \in A.$$

Since basic rules of $\quad_{\alpha}$ depend on the structural rules of $\quad$, and $\quad_{K}$, $\quad_{W}$ and $\quad_{R}$ share such structural rules, the explicative relations $\quad_{\alpha, K}$, $\quad_{\alpha, W}$ and $\quad_{\alpha, R}$ also have the basic rules of $\quad_{\alpha}$. Besides, the studied derived rules also can be proved.

3. Abduction and explicative relations

3.1 Explicative solutions of abductive problems

In general, for $A \subset L$ and $y \in L$, $<A, y>$ represents an *abductive problem* with respect to a consequence relation $\quad$. Then, the set of solutions S is defined as

$$S = \{x \in L \mid \{y\} \quad_{\alpha} A \cup \{x\}\}.$$

In terms of explicative relation, S is the set of solutions iff $\{y\} \quad_{\alpha} A \cup S$, that is to say, an abductive problem can be seen as the problem of explicating by means of a consequence relation y from A and a set S that must be obtained. When the consequence relation is a pivotal consequence, then the set of solutions S could be defined taking into account a set of sentences K, a set of valuations W or a set of rules R, and it would be verified that

- $\{y\} \quad_{\alpha, K} A \cup S$,
- $\{y\} \quad_{\alpha, W} A \cup S$,
- $\{y\} \quad_{\alpha, R} A \cup S$,

according to the kind of pivotal consequence taken to define the explicative relation.

By using the same terminology than in Aliseda (1997) and Aliseda (2006), consider the following sets: T, F, $S \subset L$, called "theory", "facts" —initially, F has only one element— and "solutions", respectively. From the pivotal assumption relation $\quad_{T}$, let $\quad_{\alpha, T}$ be the corresponding explicative relation, then a general abductive problem can be tackled in terms of explicative relations, so $<T, F>$ represents a *general abductive problem* with respect to the consequence relation $\quad_{T}$. Abductive problems, as presented in Aliseda (1997) and Aliseda (2006) are general abductive problems in which F is unitary (in Kakas *et al* (1993) it is the same with respect to the general case, when no additional restriction is imposed to S). Given a such general abductive problem, $<T, F>$, the following conditions characterize different types of solutions:

- *Plain*:
  1. $F \quad_{\alpha, T} S$,
- *Consistent*:
  1. $F \quad_{\alpha, T} S$,

2. $L \ /_{\alpha, T} S$,

- *Explicative*:
  1. $F \quad _{\alpha, T} S$,
  2. $F \ /_{\alpha, \varnothing} S$ or $F \ /_{\alpha, S} \varnothing$
- *Minimal*:
  1. $F \quad _{\alpha, T} S$,
  2. $F \ /_{\alpha, T} S'$, for all $S' \subset S$

3.2 Consistent explicative abduction

In Accordance with the given classification, by combinations of requisites, for a general abductive problem $<T, S>$, a consistent explicative abduction must be defined accomplishing

1. $F \quad _{\alpha, T} S$,
2. $L \ /_{\alpha, T} S$,
3. $F \ /_{\alpha, \varnothing} S$ or $F \ /_{\alpha, S} \varnothing$

Then the set S of solutions should be defined as

$$S = \{x \in L \mid F \quad _{\alpha, T} \{x\} \ \& \ x \ \ /_{T} \perp \ \& \ x \ \ /_{T} \ y \}$$

In fact this is one of the most interesting forms of abduction, since it represents a current inferential pattern of scientific argumentation. Methodology of scientific knowledge suggests that given a theory T, which can be seen as a set of sentences of a specialized language, and a fact that could be explained in terms of T, this fact must be represented as a sentence of such language with some syntactical relation with T, since such fact has to be related with the set of facts captured by T. Taking that into account, a general definition of abductive problem should include a syntactical prerequisite. So, given $R \subset \wp(L) \times \wp(L)$, which verifies certain syntactical requisites, for example that for every $<X, Y> \in R$, if $y \in Y$, then there must be $x \in X$ such that x and y share propositional variables, and given a theory $T \subset L$, $<T, F>$ is a general abductive problem iff $<T, F> \in R$. Then, in general, the set of solutions should be defined as follows

$$S = \{x \in L \mid <T, \{x\}> \in R \ \& \ F \quad _{\alpha, T} \{x\} \ \& \ x \ \ /_{T} \perp \ \& \ x \ \ /_{T} \ y \}$$

Some characterizations of abduction give criteria to restrict the set of solutions: using integrity constraints, specifying criteria of preference, crucial literals, etc., —see Kakas *et al.* 1993, p. 4. In order to capture some of restricted forms of abduction, pivotal consequences could be took, for example if a set $I \subset L$ of integrity constraints, then a useful explicative consistent relation should be $\quad _{\alpha, T}$ with the additional condition that $I \quad _{\alpha, T}$, since it must be verified that $S \quad _{T} I$. Restrictions about valuations or inference rules could be captured by means of $\quad _{\alpha, W}$ and $\quad _{\alpha, R}$, respectively.

A question arises about methods to obtain the set of solutions, for example, in the sense of consistent explicative abduction, namely whether there is some calculus that could be defined. A precise answer depends on the consequence relation that is been used. As we said above, L has at least a symbol for material implication, that is to say $\to$. The studied classical relation accomplishes the operative rule called *modus ponens*: for $A \subset L$ and $x, y \in L$, $K \subset L$, a set of valuations W and a set of rules R,

$$\frac{A \quad x \to y,\ A \quad x}{A \quad y}\ .$$

Such relation also verifies the so called *deduction theorem*, which must indicate the prototypical style of inference (some interesting reflections about that in van Benthem 1993):

$$\frac{A, x \quad y}{A \quad x \to y}\ .$$

Reciprocally, let us see how it is verified

$$\frac{A \quad x \to y}{A, x \quad y}\ :$$

1. $A \quad x \to y$, premise
2. $A, x \quad x$, reflexivity
3. $A, x \quad x \to y$, monotony
4. $A, x \quad y$, *modus ponens* 2, 3.

On the other hand, the classical consequence relation   has the following operative rules, the $\perp$-*rules*, for $A \subset L$ and $x \in L$:

- First $\perp$-rule: $A \quad x$ iff $A, x \to \perp \quad \perp$.
- Second $\perp$-rule (combining with $\to$):

$$\frac{A, y \quad x;\ A, y \quad z}{A, (x \to \perp) \to y \quad z}\ .$$

By working with classical consequence relation, semantic tableaux method has been used in Aliseda (1997), Aliseda (2006) and Reyes *et al* (2006) and dual resolution is the calculus for abduction introduced Soler *et al* (2006). On the other hand, $_K$, $_W$ and $_R$, defined as supraclassical consequence relations, verify *modus ponens* and deduction theorem, and such first and second $\perp$-rules could be settled. In general, the following proposition allows to apply any deductive calculus,

It is sufficient condition to obtain solutions of general abductive problems by means of a deductive calculus that:

- *the corresponding consequence relation verifies modus ponens, deduction theorem and* $\perp$*-rules,*
- *the calculus must be sound and complete.*

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Ángel NEPOMUCENO is lecturer in Logic and Philosophy of Science at University of Seville. He is responsible researcher the projects "Logic and language: discourse ambiguity and context elaboration" and "Computational tools for implementing logical and linguistic strategies for treating discourse ambiguity". He works mainly in formal logic. He has taken part in numerous congresses and published nine books and more than sixty articles in national and international publications.

ADDRESS: Ángel Nepomuceno Fernández, Departamento de Filosofía, Lógica y Filosofía de la Ciencia, Universidad de Sevilla, Calle Camilo José Cela, s/n, 41018, Sevilla. E-mail: nepomuce@us.es.

Fernando SOLER TOSCANO is assistant professor at the Department of Philosophy, Logic and Philosophy of Science at the University of Seville. He takes part of the research projects "Logic and language: discourse ambiguity and context elaboration" and "Computational tools for implementing logical and linguistic strategies for treating discourse ambiguity". He also cooperates with Indisys within the framework of the CENIT project ATLANTIDA, leaded by Boeing Research and Technologies Europe.

ADDRESS: Fernando Soler Toscano, Departamento de Filosofía, Lógica y Filosofía de la Ciencia, Universidad de Sevilla, Calle Camilo José Cela, s/n, 41018, Sevilla. E-mail: fsoler@us.es.