Triangulations Without Pointed Spanning Trees
Extended Abstract [1]

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Abstract

Problem 50 in the Open Problems Project [2] asks whether any triangulation on a point set in the plane contains a pointed spanning tree as a subgraph. We provide a counterexample. As a consequence we show that there exist triangulations which require a linear number of edge flips to become Hamiltonian.

Key words: triangulation, spanning tree, pointed pseudo-triangulation, Hamiltonian cycle, edge flip

1. Introduction

Let S be a finite set of points in the plane in general position (no three points are on a common line), and let G be a straight-line graph (drawing in the plane) with vertex set S and edges E. A point $p \in S$ is *pointed* in G if there exists an angle less than π that contains all edges incident to p in G. The graph G is pointed if all its vertices are pointed.

A *triangulation* of S is a maximal planar straight-line graph on top of S. A *spanning tree* on S is a connected, acyclic graph with vertex set S. Several interesting relations between triangulations and spanning trees exist. For example it is well known that the Delaunay triangulation of S contains a minimum spanning tree of S as a subgraph. Another example is a result of Schnyder [3] who shows that every triangulation of a point set with three extreme vertices allows a partition of its interior edges into three trees.

In this note we disprove the following conjecture which was posed as Problem 10 at the First Gremo Workshop on Open Problems (Stels, Switzerland) in July 2003 (by Bettina Speckmann) and at the CCCG 2003 open-problem session (Halifax, Canada) in August 2003. It later on became Problem 50 in the Open Problems Project of the computational geometry community [2].

Conjecture 1 *Every triangulation of a set of points in the plane (in general position) contains a pointed spanning tree as a subgraph.*

This conjecture arose while proving substructure properties when investigating flips in pointed and non-pointed pseudo-triangulations [1]. *Pseudo-triangulations* are a generalization of triangulations. A *pseudo-triangle* is a planar polygon with exactly three interior angles less than π. A pseudo-triangulation of S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. Pseudo-triangulations have become a versatile data structure. Beside several applications

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in computational geometry, the rich combinatorial properties of pseudo-triangulations stimulated research, see e.g. [1] and references therein.

Obviously Conjecture 1 would be true if a triangulation always contained a Hamiltonian path or a pointed pseudo-triangulation as a subgraph. Several triangulations not containing these structures are known, but for each example it is still easy to find a pointed spanning tree as a subgraph. This observation supported the general belief that the conjecture should be true. However, in the next section we provide a (non-trivial) counterexample. In Section 3 we discuss some implications of this result, like a lower bound for the number of necessary edge flips to transform a given triangulation such that it contains a Hamiltonian cycle.

2. A Counterexample

Figure 1(a) shows the simplest example of a connected straight-line graph not containing a pointed spanning tree as a subgraph. We call this graph a *3-star* and it is a spanning tree which is not pointed at it's central point.

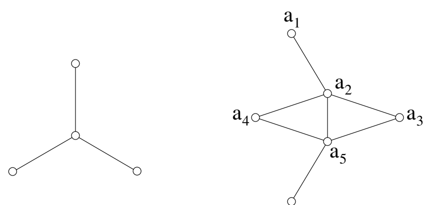

Fig. 1. Small connected straight-line graphs that do not contain a pointed spanning tree as a subgraph: (a) the 3-star (b) the bird graph

The graph on top of the points $a_1,..,a_6$ in Figure 1(b) is called the *bird graph.* It can be seen as a triangulated composition of two 3-stars. In the next lemma we show that the bird graph does not contain a pointed spanning tree either.

Lemma 2 *The bird graph does not contain a pointed spanning tree as a subgraph.*

Proof Assume that the bird graph contains a pointed spanning tree T as a subgraph. Because of connectivity, the edges a_1a_2 and a_5a_6 are in T. The edge a_2a_5 cannot be in T, as otherwise any edge incident to a_3 would violate the pointedness condition in either a_2 or a_5. Thus, either a_3 or a_4 has to be connected to both, a_2 and a_5. This prevents the other vertex, a_4 resp. a_3, to be connected anyhow. □

In a next step, we extend the bird graph by two additional points b_1, b_2, see Figure 2. Intuitively speaking b_1 and b_2, respectively, are connected by edges to each visible point of the bird graph. Moreover we add the edge b_1, b_2. We call the resulting full triangulation of the triangle b_1, b_2, a_1 with interior points $a_2, .., a_6$ the *bird cage graph.* The following lemma shows that it will play a crucial role in the construction of a triangulation not containing a pointed spanning tree.

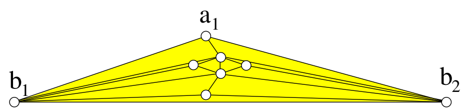

Fig. 2. The bird cage graph: any connected, pointed, spanning subgraph contains at least one interior edge incident to b_1 or b_2, respectively.

Lemma 3 *Any connected, pointed, spanning subgraph of the bird cage graph contains at least one interior edge incident to b_1 or b_2, respectively.*

Proof Let A be a connected, pointed, spanning subgraph of the bird cage graph. By Lemma 2, the subgraph B of A induced by the points $a_1, .., a_6$ does not contain a pointed spanning tree. That is, B consists of at least two components. Because of connectivity, A has to include edges that connect these components. For this, at least one interior edge incident to b_1 or b_2, respectively, has to be in A. □

We are now ready to prove the main theorem of this note.

Theorem 4 *There exist triangulations on top of a point set in the plane in general position that do not contain a pointed spanning tree as a subgraph.*

Proof As indicated in Figure 3 we connect three points a, b, c pairwise by bird cage graphs (shaded triangles), such that for each bird cage graph the

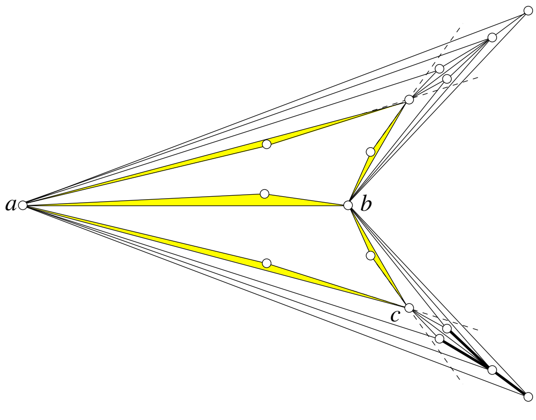

Fig. 3. A bird wing graph containing five bird cage graphs.

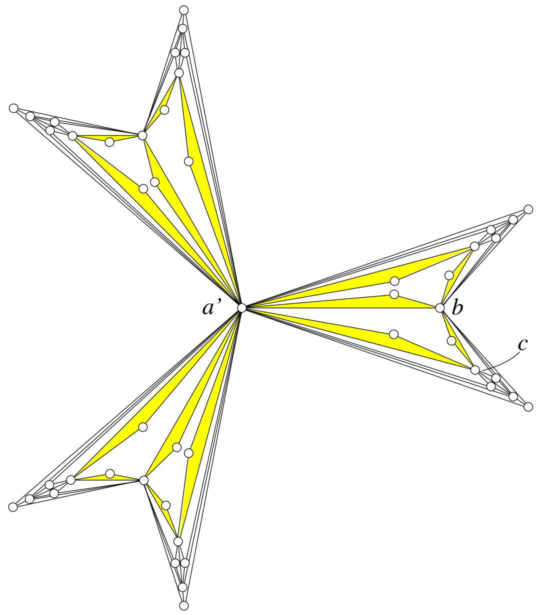

Fig. 4. A graph consisting of three bird wing graphs.

connected vertices correspond to b_1 and b_2, respectively. Furthermore, we add four points near point c. This four points form a three star and are connected to a, b and c as shown in the figure. Next the whole construction, except the bird cage graph connecting a to b, is mirrored along the line a,b. We call the resulting graph a bird wing graph, cf. Figure 3. Note that all edges incident to c form three wedges in an obvious way. If we group the edges of a wedge togehter the resulting graph corresponds to a 3-star with c as its center. The same holds for b and its incident edges. To complete our construction we finally form another 3-star like graph with center a' by joining three bird wing graphs at their a-vertices, see Figure 4. Let us denote the resulting graph by $\mathcal{G}$ and its vertex set by $\mathcal{S}$.

Let G be any planar straight-line graph drawing on top of $\mathcal{S}$ which contains $\mathcal{G}$ as a subgraph. Note that G might be a complete triangulation of $\mathcal{S}$. Assume that there exists a pointed spanning tree PST as a subgraph of G. By similar argumentation as for the 3-star, PST does not contain any edge incident to a' in at least one of the three bird wing graphs. W.l.o.g. let b be a vertex of this bird wing graph. Applying Lemma 3, PST has to contain at least one edge in the interior of the bird cage graph between a' and b incident to b. From the property of the 3-star we can find a bird cage graph B incident to b such that PST does not contain any edge incident to b in B. W.l.o.g. let c be the vertex at the other end of B. By Lemma 3, PST contains at least one edge incident to c in B. The same holds for the bird cage graph between a' and c. Moreover, the four additional points near point c cannot be connected by any edge to either a', b, or c without destroying pointedness at one of these points. Since PST is connected, the three bold edges in Figure 3 have to be in PST. However, the resulting graph is not pointed any more, as the three bold edges form a 3-star. Thus, there exists no pointed spanning tree as a subgraph for G. Note that all arguments work for any planar straight-line graph containing $\mathcal{G}$ as a subgraph, because our argumentation is solely based on the interior of fully triangulated areas. □

3. Some Implications

The example in the proof of Theorem 4 is constructed on top of a point set S with 124 points. Adding more points to S in the outer face of $\mathcal{G}$ and completing the extended graph with edges to a full triangulation still gives a triangulation that does not contain a pointed spanning tree. Thus, Theorem 4 also holds if we put additional restrictions on the size of the convex hull of the underlying point set.

As a more interesting consequence we get a lower bound on the minimum number of edge flips that might be necessary to transform a triangulation such that it contains a pointed spanning tree as a subgraph. To this end, we combine a linear number of disjoint copies of the 124-point example. After completing the graph on the resulting point set to a full triangulation, at least one flip has to be executed in each copy.

Corollary 5 *There exist triangulations on top of an n-point set in the plane in general position that require $\Omega(n)$ edge flips to contain a pointed spanning tree as a subgraph.*

From the Delaunay flip algorithm it follows that a quadratic number of flips is always sufficient to obtain a pointed spanning tree as a subgraph. So far no better upper bound on this flip distance is known.

Of particular interest is the investigation of Hamiltonicity of triangulations. A triangulation is called Hamiltonian if it contains a Hamiltonian cycle. Let T be a non-Hamiltonian triangulation. What is the minimum number of edge flips that is always sufficient to come from T to a Hamiltonian triangulation T'? Note that Hamiltonicity implies the existence of a pointed spanning tree, whereas the reverse is not true in general. Therefore, we conclude from Corollary 5:

Corollary 6 *There exist non-Hamiltonian triangulations on top of an n-point set in the plane in general position that require $\Omega(n)$ edge flips to become Hamiltonian.*

The last statement can also be shown in a more direct way. Let a point set S with $|S| = n$ be given such that the convex hull of S contains 3 points. Then a triangulation on top of S has $2n - 5$ triangles. We place one additional point into each of these triangles and connect it by edges to the corners of the triangle. The set A of inserted points is independent, meaning that there is no edge between any two points of A in the resulting triangulation T on top of $S \cup A$. Assume there exists a sequence δ of vertices forming a Hamiltonian cycle. Between any two vertices of A in δ there must be at least one vertex of S because A is an independent set. Since $|A| > |S|$, T cannot be Hamiltonian, i.e., δ cannot exist. If we want T to become Hamiltonian we must perform a sequence of flips which reduces $|A|$. Any flip connects at most two elements of A and therefore reduces $|A|$ by at most 1. Hence, a linear number of flips is necessary because $|A|$ is about twice the cardinality of S. □

4. Open Problems

There are several related open questions. First, is there a smaller (in the number of points) counterexample to Conjecture 1? Moreover, how fast can we decide whether a given triangulation contains a pointed spanning tree as a subgraph? And if the answer is positive, how fast can we compute this tree? Regarding flipping, what are tight bounds on the required number of edge flips to transform a triangulation such that it contains a pointed spanning tree or a Hamiltonian cycle, respectively?

Acknowledgments

The innocent looking conjecture that any triangulation contains a pointed spanning tree as a subgraph has fascinated several people. On countless (unsuccessful) attempts to prove it true, several colleagues generously shared their inspiring ideas with us. We want to thank Franz Aurenhammer, Sergei Bespamyatnikh, Prosenjit Bose, Stefan Langerman, Pat Morin, Bettina Speckmann, Csaba D. Tóth, and David R. Wood for enjoyable discussions.

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