Controllability of non-scalar parabolic systems

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General objective:

Study some null controllability problems for **non-scalar parabolic systems**.

Non-scalar parabolic systems: arise in chemical reactions, when we model problems from the Biology and in a wide variety of physical situations.

In this course we will deal with non-scalar systems which in fact are **coupled parabolic scalar equations**. We do not present results relating to the controllability problems of systems which come from fluid mechanics as Stokes, Navier-Stokes, ...

GOAL:

1. Show the important differences between scalar and non-scalar problems.
2. Give necessary and sufficient conditions (**Kalman conditions**) which characterize the controllability properties of these systems.

We will only deal with

1. **Linear systems**
2. In general, “simple" Parabolic Systems: **Coupling Matrices of Constant Coefficients**.

Contents

1. Introduction

1. Introduction

Let us fix $T > 0$ and let H and U be two separable Hilbert spaces. Let us consider $T_0 \in (0, T)$ and the autonomous system:

$$(1) \qquad \begin{cases} y' = Ay + Bu & \text{on } (T_0, T), \\ y(T_0) = y_0 \in H. \end{cases}$$

A and B are "appropriate" operators, $y_0 \in H$ is the initial datum at $t = T_0$ and $u \in L^2(T_0, T; U)$ is the control (exerted by means of the operator B).

Assume the problem is well-posed: $\forall (y_0, u)$ there exists a unique weak solution $y \in C^0([T_0, T]; H)$ to (1) which depends continuously on the data.

Let us denote by $y(t; T_0, y_0, u) \in H$ the solution to the system and by $y(t; y_0, u) = y(t; 0, y_0, u)$.

1. Introduction

- **Exact Controllability:** System (1) is **exactly controllable** at time T if $\forall (y_0, y_1) \in H \times H$, there exists $u \in L^2(0,T;U)$ s.t. the solution y of (1) satisfies $y(T; y_0, u) = y_1$.
- **Controllability to trajectories:** System (1) is **controllable to trajectories** at time T if $\forall (y_0, \widehat{y}_0) \in H \times H$ and $\widehat{u} \in L^2(0,T;U)$, there exists $u \in L^2(0,T;U)$ s.t. the corresponding weak solution to (1) satisfies $y(T; y_0, u) = y(T; \widehat{y}_0, \widehat{u})$.
- **Null Controllability:** System (1) is **null controllable** at time T if $\forall y_0 \in H$ there exists $u \in L^2(0,T;U)$ s.t. $y(T; y_0, u) = 0$.
  Linear case: Controllability to trajectories and null controllability are equivalent.
- **Approximate Controllability:** System (1) is **approximately controllable** at time T if $\forall (y_0, y_1) \in H \times H$, and every $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ s.t.

$$\|y(T; y_0, u) - y_1\|_H \leq \varepsilon.$$

Remark

For the non-autonomous system

$$(2) \qquad y' = A(t)y + B(t)u \quad \text{in } (0,T),$$

it is possible to give stronger definitions of controllability: It will be said that equation (2) is **totally exactly controllable** on $(0,T)$ if $\forall T_0, T_1 \in (0,T)$, with $T_0 < T_1$, and $\forall (y_0, y_1) \in H \times H$ there exists $u \in L^2(T_0,T_1;H)$ such that the solution to (2) in (T_0,T_1) satisfies $y(T_1;T_0,y_0,u) = y_1$.

Following the previous definition we can also define the concepts for equation (2): **totally exactly controllable to trajectories** on $(0,T)$, **totally null controllable** on $(0,T)$ and **totally approximately controllable** on $(0,T)$. In the autonomous case the different concepts of controllability at time T and total controllability on $(0,T)$ coincide.

Remark

Problems (1) and (2) are **linear**. Then, System (1) (resp. System (2)) is **null controllable** at time T (resp., **totally null controllable** on $(0, T)$) **if and only if** the system is **exactly controllable to the trajectories** at time T (**totally exactly controllable to trajectories** on $(0, T)$). ■

Remark

We will deal with parabolic problems. So, due to the regularizing effect of these problems, it is well-known that the exact controllability result fails. Therefore, in this course we will study **null** or **approximate controllability** results for the system under consideration.

1. Introduction

In this course we are going to deal with **time-dependent second order elliptic** operators. Thus, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2 and let us fix $T > 0$.

Notation: $Q_T = \Omega \times (0, T)$, $\Sigma_T = \partial\Omega \times (0, T)$ and, for $\mathcal{O} \subseteq \Omega$ or $\mathcal{O} \subseteq \partial\Omega$, $1_{\mathcal{O}}$ denotes the characteristic function of the set $\mathcal{O}$.

Let $L(t)$ be the operator given by:

$$(3) \qquad \boxed{L(t)y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x,t)\frac{\partial y}{\partial x_j}\right) + D(x,t)\cdot\nabla y + c(x,t)y.}$$

The coefficients of L satisfy

$$(4) \qquad \begin{cases} \alpha_{ij} \in W^{1,\infty}(Q_T)\ (1 \leq i,j \leq N),\ D \in L^\infty(Q_T;\mathbb{R}^N),\ c \in L^\infty(Q_T), \\ \alpha_{ij}(x,t) = \alpha_{ji}(x,t) \quad \forall (x,t) \in Q_T, \end{cases}$$

and the **uniform elliptic condition**: there exists $a_0 > 0$ such that

$$(5) \qquad \sum_{i,j=1}^{N} \alpha_{ij}(x,t)\xi_i\xi_j \geq a_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \forall (x,t) \in Q_T.$$

2. The parabolic scalar case

2. The parabolic scalar case

Let $\omega \subseteq \Omega$ be an open subset, $\Gamma_0 \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$.

We consider the **linear** problems for the **operator** $L(t)$:

$$(6) \quad \boxed{\begin{cases} \partial_t y + L(t)y = v1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}}$$

$$(7) \quad \boxed{\begin{cases} \partial_t y + L(t)y = 0 & \text{in } Q_T, \\ y = h1_{\Gamma_0} \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}}$$

In (6) and (7), $y(x,t)$ is the state, y_0 is the initial datum and v and h are the control functions (which are localized in ω -**distributed control**- or on Γ_0 -**boundary control**-).

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Question: Functional spaces for y_0, v and h?

CONTROL SPACES:

- **Distributed control problem**: We can take $L^2(Q_T)$ as control space and $L^2(\Omega)$ as initial datum space. The problem is well-posed: $\forall y_0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$ there exists a unique weak solution to (6) $y \in C^0([0,T]; L^2(\Omega))$ which depends continuously on the data.

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- **Boundary control problem**:
  1. If in (3), $D \equiv 0$ in Q_T, we can take $L^2(\Sigma_T)$ as control space and $H^{-1}(\Omega)$ as initial datum space. Again, the problem is well-posed: $\forall y_0 \in H^{-1}(\Omega)$ and $h \in L^2(\Sigma_T)$ there exists a unique weak solution to (7) $y \in C^0([0,T]; H^{-1}(\Omega))$ which depends continuously on the data. **Solution defined by transposition**.

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  2. In the general case, we can take $L^2(\Omega)$ as initial datum space and

$$X(\Gamma_0) = \{h : h = H|_{\Sigma_T} \text{ with } H \in L^2(0,T;H_0^1(\widetilde{\Omega})),\ H_t \in L^2(0,T;H^{-1}(\widetilde{\Omega}))\},$$

  as control space, where $\widetilde{\Omega}$ is an open set s.t. $\Omega \subset \widetilde{\Omega}$, $\partial\Omega \cap \widetilde{\Omega} \subset\subset \Gamma_0$ and $\widetilde{\Omega} \setminus \overline{\Omega} \neq \emptyset$. The problem is well-posed and the solution depends continuously on the data.

2. The parabolic scalar case

Theorem

Let us fix $T > 0$. The following conditions are equivalent

1. *For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\omega \subset \Omega$, nonempty open subset, and any coefficients α_{ij} $(1 \le i,j \le N)$, D and c, satisfying (4) and (5), System (6) is null controllable in $L^2(\Omega)$ at time $T > 0$ with* ***distributed controls*** *$v \in L^2(Q_T)$.*
2. *For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\Gamma_0 \subset \partial\Omega$, nonempty relative open subset, and any coefficients α_{ij} $(1 \le i,j \le N)$, D and c, satisfying (4) and (5), System (7) is null controllable in $L^2(\Omega)$ at time $T > 0$ with* ***boundary controls*** *$h \in L^2(0,T;H^{1/2}(\partial\Omega))$.*

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Proof: We will use in a fundamental way that the problem under consideration is **scalar** (in fact, same number of equations and controls). We follow some ideas from [BODART,G.-B.,PÉREZ-GARCÍA] Comm. PDE (2004) and [G.-B.,PÉREZ-GARCÍA] Asymp. Anal. (2006). $\cdots$

Remark (Regularizing effect)

The previous proof shows that if the distributed and boundary null controllability results for Systems (6) and (7) are valid with controls in $L^2(Q_T)$ and $L^2(0,T;H^{1/2}(\partial\Omega))$, then the previous systems are null controllable with controls in $L^\infty(Q_T)$ and $L^\infty(\Sigma_T)$ (and even better for regular coefficients). ■

Remark

In the proof of Theorem 1 we have strongly used that the operator $\partial_t + L(t)$ is scalar. We will see that the previous equivalence is not valid for non-scalar parabolic operators. ■

2. The parabolic scalar case

From now on, we will concentrate on the distributed control problem (6). Let us introduce the **adjoint problem**

$$(8) \qquad \begin{cases} -\partial_t \varphi + L^*(t)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_T & \text{in } \Omega, \end{cases}$$

where $\varphi_T \in L^2(\Omega)$ is given and $L^*(t)$ is the operator given by

$$L^*(t)\varphi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x,t)\frac{\partial \varphi}{\partial x_j}\right) - \nabla \cdot (D\varphi) + c(x,t)\varphi \text{ a.e. in } Q_T.$$

This problem is also well-posed and the solution depends continuously on φ_T: there exists a constant $\widetilde{C} > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ System (8) has only **one solution** $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ and it satisfies

$$\|\varphi\|_{L^2(0,T;H_0^1(\Omega))} + \|\varphi\|_{C^0([0,T];L^2(\Omega))} \leq \widetilde{C}\|\varphi_T\|_{L^2(\Omega)}.$$

Theorem (Observability Inequality)

Under the previous assumptions, System (6) *is null controllable at time* $T > 0$ *if and only if there exists a constant* $C > 0$ *s.t.*

$$(9) \qquad \|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \le C \iint_{\omega\times(0,T)} |\varphi|^2 dxdt, \quad \forall \varphi_T \in L^2(\Omega),$$

where φ *is the solution of* (8) *associated to* φ_T*.*

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Remark

The **Observability Inequality** (9) in particular implies a better result: If (9) holds then, $\forall y_0 \in L^2(\Omega)$ there is a distributed control $v \in L^2(Q_T)$ s.t.

$$\|v\|^2_{L^2(Q_T)} \leq C\|y_0\|^2_{L^2(\Omega)} \quad \text{and} \quad y(\cdot,T) = 0,$$

being y the solution to (6) corresponding to y_0 and $C > 0$ the constant in (9).

2. The parabolic scalar case

1. The one-dimensional case: The moment method

We follow [FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971).

Consider the boundary null controllability problem for the classical one-dimensional heat equation in $(0, \pi)$ (for simplicity):

$$(10) \qquad \begin{cases} y_t - y_{xx} = 0 & \text{in } Q_T = (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$. The problem is **well-posed** and the solution (defined by transposition) depends continuously on the data y_0 and v. The operator $-\partial_{xx}$ on $(0,\pi)$ with homogenous Dirichlet boundary conditions admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0,\pi)$$

which is a Hilbert basis of $L^2(0,\pi)$. In the sequel, we will use the notation

$$y_k = (y, \phi_k)_{L^2(0,\pi)}, \quad \forall y \in L^2(0,\pi).$$

The idea of the **moment method** is simple: Given $y_0 \in H^{-1}(0,\pi)$, $\varphi_T \in H_0^1(0,\pi)$ and $v \in L^2(0,T)$, then

$$\langle y(\cdot,T), \varphi_T \rangle - \langle y_0, \varphi(\cdot,0) \rangle = \int_0^T v(t)\varphi_x(0,t)\,dt.$$

where y is the solution to (10) and φ is the solution to the **adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0,1\} \times (0,T), \quad \varphi(\cdot,T) = \varphi_T & \text{in } (0,\pi). \end{cases}$$

The idea of the **moment method** is simple: Given $y_0 \in H^{-1}(0,\pi)$, $\varphi_T \in H_0^1(0,\pi)$ and $v \in L^2(0,T)$, then

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Property

$v \in L^2(0,\pi)$ is a **null control** for system (10) (i.e., $v \in L^2(0,T)$ is a control s.t. the solution y to (10) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$) if and only if

$$-\langle y_0,\varphi(\cdot,0)\rangle = \int_0^T v(t)\varphi_x(0,t)\,dt, \quad \forall \varphi_T \in H_0^1(0,\pi).$$

Given $y_0 \in H^{-1}(0,\pi)$, there exists a control $v \in L^2(0,T)$ such that the solution y to (10) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$ if and only if there exists $v \in L^2(0,T)$ satisfying

$$-\langle y_0, e^{-\lambda_k T}\phi_k\rangle = \int_0^T v(t) e^{-\lambda_k (T-t)} \phi_{k,x}(0)\, dt, \quad \forall k \geq 1,$$

i.e., if and only if $v \in L^2(0,T)$ and

$$\int_0^T e^{-\lambda_k (T-t)} v(t)\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_k T} y_{0,k} \equiv c_k \quad \forall k \geq 1.$$

This problem is called a **moment problem**. We have the following result:

Given $y_0 \in H^{-1}(0,\pi)$, there exists a control $v \in L^2(0,T)$ such that the solution y to (10) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$ if and only if there exists $v \in L^2(0,T)$ satisfying

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$$\int_0^T e^{-\lambda_k(T-t)}v(t)\,dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}}e^{-\lambda_k T}y_{0,k} \equiv c_k \quad \forall k \geq 1.$$

This problem is called a **moment problem**. We have the following result:

Theorem

*For any $y_0 \in H^{-1}(0,\pi)$ and $T > 0$, there exists $v \in L^2(0,T)$ solution to the previous **moment problem**. That is, v is a **null control** for equation (10).*

Proof: Biorthogonal Families: ([FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971)). There exists a family $\{p_k\}_{k\geq 1} \subset L^2(0,T)$ satisfying

1. $\displaystyle\int_0^T e^{-\lambda_k t} p_l(t)\,dt = \delta_{kl}, \quad \forall k,l \geq 1.$
2. $\forall \varepsilon > 0,\ \exists C(\varepsilon,T) > 0$ s.t. $\|p_k\|_{L^2(0,T)} \leq C(\varepsilon,T)e^{\varepsilon\lambda_k}$.

The control is obtained as a linear combination of $\{p_k\}_{k\geq 1}$, that is,

$$\boxed{v(T-s) = \sum_{k\geq 1} c_k\, p_k(s) = -\sqrt{\frac{\pi}{2}} \sum_{k\geq 1} \frac{1}{k} e^{-\lambda_k T} y_{0,k}\, p_k(s)}$$

and the previous bounds are used to prove that this combination converges in $L^2(0,T)$. ■

Two ingredients:

Existence and **bounds** of a biorthogonal family to real exponentials.

Remark

Theorem 2.2 is a consequence of the existence of a biorthogonal family in $L^2(0,T)$ to the sequence $\{e^{-\lambda_k t}\}_{k\geq 1}$ ($\lambda_k = k^2$), which satisfies appropriate **bounds**. In fact, in ([FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971)) the authors prove a general result on existence of a **biorthogonal family** in $L^2(0,T)$ to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ which satisfies appropriate **bounds** for sequences $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ such that

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty \quad \text{and} \quad |\Lambda_k - \Lambda_l| \geq \rho |k-l|, \quad \forall k, l \geq 1.$$

for a constant $\rho > 0$. ■

Consequence:

The previous result is valid for any nonempty bounded interval (a,b) and for any second order operator **self-adjoint elliptic operator**

$$Ly = -\left(\alpha(x)y_x\right)_x + c(x)y,$$

with $\alpha \in C^1([a,b])$ and $\alpha > 0$ in (a,b), and $c \in C^0([a,b])$. **Then**, if we apply Theorem 1, we also get a **distributed controllability** result for the problem

$$\begin{cases} y_t + Ly = v1_\omega & \text{in } Q_T = (a,b)\times(0,T), \\ y(a,\cdot) = 0, \quad y(b,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (a,b), \end{cases}$$

with $y_0 \in L^2(0,\pi)$ and $\omega \subseteq (a,b)$, a nonempty open subset.

2. The parabolic scalar case

2. General case: Carleman Inequalities

We follow [Fursikov,Imanuvilov] 1996 and [Imanuvilov,Yamamoto] 2003.

We will consider the following parabolic equation:

$$(11) \qquad \begin{cases} -\partial_t z + L_0(t) z = F_0 + \displaystyle\sum_{i=1}^{N} \frac{\partial F_i}{\partial x_i} & \text{in } Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, T) = z_T & \text{in } \Omega, \end{cases}$$

with $z_T \in L^2(\Omega)$, $F_i \in L^2(Q_T)$, $i = 0, 1, \ldots, N$, and $L_0(t)$ the self-adjoint parabolic operator given by

$$L_0(t) y = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x,t) \frac{\partial y}{\partial x_j} \right)$$

with coefficients α_{ij} satisfying (4) and (5).

Lemma

Let $\mathcal{B} \subset \Omega$ be a nonempty open subset and $d \in \mathbb{R}$. Then, $\exists\ \beta_0 \in C^2(\overline{\Omega})$ (positive and only depending on Ω and $\mathcal{B}$) and $\widetilde{C}_0, \widetilde{\sigma}_0 > 0$ (only depending on Ω, $\mathcal{B}$ and d) s.t. for every $z_T \in L^2(\Omega)$, the solution z to (11) satisfies

(12)

$$\begin{cases} \mathcal{I}(d,z) \leq \widetilde{C}_0 \left(s^d \iint_{\mathcal{B}\times(0,T)} e^{-2s\beta} \gamma(t)^d |z|^2 \right. \\ \left. + s^{d-3} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-3}|F_0|^2 + s^{d-1} \sum_{i=1}^{N} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-1}|F_i|^2 \right), \end{cases}$$

$\forall s \geq \widetilde{s}_0 = \widetilde{\sigma}_0\,(T+T^2)$; $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\beta(x,t) = \beta_0(x)/t(T-t)}$

and $\boxed{\mathcal{I}(d,z) \equiv s^{d-2} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-2}|\nabla z|^2 + s^d \iint_{Q_T} e^{-2s\beta}\gamma(t)^d|z|^2}$.

Lemma

When $F_i \equiv 0$ *for* $1 \le i \le N$, $\exists\ \widetilde{C}_1$ *and* $\widetilde{\sigma}_1$ *(which only depend on* Ω, $\mathcal{B}$ *and* d*)* *s.t.,* $\forall z_T \in L^2(\Omega)$, *the solution* z *to* (11) *satisfies*

(13)

$$\mathcal{I}_1(d,z) \le \widetilde{C}_1 \left(s^d \iint_{\mathcal{B}\times(0,T)} e^{-2s\beta}\gamma(t)^d |z|^2 + s^{d-3} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-3} |F_0|^2 \right),$$

for all $s \ge \widetilde{s}_1 = \widetilde{\sigma}_1 \left(T + T^2\right)$ *where*

$$\mathcal{I}_1(d,z) \equiv s^{d-4} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-4} \left(|\partial_t z|^2 + |\Delta z|^2\right) + \mathcal{I}(d,z)\,.$$

Proof: See [FURSIKOV,IMANUVILOV] 1996; [IMANUVILOV,YAMAMOTO] (2003) and [FERNÁNDEZ-CARA,GUERRERO] SICON (2006).

Corollary

There exists a positive constant $C_0 = C_0(\Omega, \omega)$ *such that for every* $\varphi_T \in L^2(\Omega)$ *and* φ *the corresponding solution to* (8), *the* ***observability inequality*** (9) *holds with*

$$C = \exp\left(C_0\left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + \|D\|_\infty^2\right)\right).$$

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$$C = \exp\left(C_0\left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + \|D\|_\infty^2\right)\right).$$

Proof: We follow [FERNÁNDEZ-CARA,ZUAZUA] Ann. IHP (2000) and [DOUBOVA,FERNÁNDEZ-CARA,MG-B,ZUAZUA] SICON (2002).

The Carleman inequality (12) applied to problem (8) implies ($\mathcal{B} \equiv \omega$, $d = 3$)

$$s \iint_{Q_T} e^{-2s\beta}\gamma(t)|\nabla\varphi|^2 + s^3 \iint_{Q_T} e^{-2s\beta}\gamma(t)^3|\varphi|^2$$
$$\leq \widetilde{C}_0\left(s^3 \iint_{\omega\times(0,T)} e^{-2s\beta}\gamma(t)^3|\varphi|^2\right.$$
$$\left. + \|c\|_\infty^2 \iint_{Q_T} e^{-2s\beta}|\varphi|^2 + s^2\|D\|_\infty^2 \iint_{Q_T} e^{-2s\beta}\gamma(t)^2|\varphi|^2\right).$$

As a consequence we can prove that for
$s \geq C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ ($C_1 = C_1(\Omega, \omega)$) one has

$$[s\gamma(t)]^3 - \widetilde{C}_0\|c\|_\infty^2 - \widetilde{C}_0[s\gamma(t)]\|D\|_\infty^2 \geq \frac{1}{2}[s\gamma(t)]^3 .$$

Consequently, for $s = C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ that

$$\iint_{Q_T} e^{-2s\beta} t^{-3}(T-t)^{-3}|\varphi|^2 \leq \widetilde{C}_1 \iint_{\omega\times(0,T)} e^{-2s\beta} t^{-3}(T-t)^{-3}|\varphi|^2$$

and therefore

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2 \leq e^{C(1+1/T+\|c\|_\infty^{2/3}+\|D\|_\infty^2)} \iint_{\omega\times(0,T)} |\varphi|^2.$$

This last inequality combined with **energy estimates** implies (9) and the proof is complete. ■

Corollary

Let us fix $T > 0$, $\Omega \subset \mathbb{R}^N$, $\omega \subseteq \Omega$ and $\Gamma_0 \subseteq \partial\Omega$ (arbitrary) as before. Then, there exist positive constants $C_0 = C_0(\Omega, \omega)$ and $\widehat{C}_0 = \widehat{C}_0(\Omega, \Gamma_0)$ s.t.

1. $\forall y_0 \in L^2(\Omega)$ *there is a control* $v \in L^2(\Omega)$ *which satisfies*

$$\|v\|^2_{L^2(Q_T)} \leq e^{C_0(1+1/T+\|c\|_\infty^{2/3}+\|D\|^2_\infty)} \|y_0\|^2_{L^2(\Omega)},$$

   and $y(\cdot, T) = 0$ *in* Ω, *(y is the solution to* (6) *associated to* y_0 *and* v*).*

2. $\forall y_0 \in L^2(\Omega)$ *there is a control* $h \in L^2(0, T; H^{1/2}(\Omega))$ *which satisfies*

$$\|h\|^2_{L^2(0,T;H^{1/2}(\Omega))} \leq e^{\widehat{C}_0(1+1/T+\|c\|_\infty^{2/3}+\|D\|^2_\infty)} \|y_0\|^2_{L^2(\Omega)},$$

   and $y(\cdot, T) = 0$ *in* Ω, *(y is the solution to* (7) *associated to* y_0 *and* v *and, in fact,* $y \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$*).*

Remark

It is important to point out that the **boundary null controllability** result for problem (7), when the coefficient D of $L(t)$ (see (3)) is regular enough, can be obtained from an appropriate boundary Carleman inequality for problem (11) with $F_i \equiv 0$, $1 \leq i \leq N$. This Carleman inequality is like (13) for an appropriate weight function $\widetilde{\beta}_0 \in C^2(\overline{\Omega})$ (which depends only on Ω and Γ_0) instead of β_0 and with the local term

$$s^{d-2} \iint_{\Gamma_0 \times (0,T)} e^{-2s \frac{\widetilde{\beta}_0}{t(T-t)}} \gamma(t)^{d-2} \left| \frac{\partial z}{\partial n} \right|^2$$

instead of the integral over $\mathcal{B} \times (0,T)$ in the right hand side of (13) (z is the solution to (11) associated to $z_T \in L^2(\Omega)$). ■

2. The parabolic scalar case

3. Final comments in the scalar case

1. The null controllability property for the N-dimensional case was solved independently by G. Lebeau and L. Robbiano (for the heat equation) and by A. Fursikov and O. Imanuvilov (for a general parabolic equation). With a different approach, Lebeau-Robbiano obtained the distributed null controllability result for System (6)

$$\begin{cases} \partial_t y + L_0 y = v 1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

when L_0 is a self-adjoint elliptic operator independent of t. For more details, see [LEBEAU,ROBBIANO] Comm. P.D.E. (1995).

2. Until now, we have only dealt with the **null controllability** problem for a scalar parabolic system with distributed and boundary controls. For the corresponding **approximate controllability** we can obtain similar results:

Approximate controllability

Proposition (Distributed control)

*System (6) is **approximately controllable** at time $T > 0$ if and only if the **adjoint problem** (8) satisfies the **unique continuation property**: "If φ is a solution to (8) and $\varphi = 0$ in $\omega \times (0, T)$, then $\varphi \equiv 0$ in Q_T".* ■

Remark (Boundary control)

In the case of System (7) we can get a similar result. In this case the **unique continuation property** for System (8) is: "If φ is a solution to (8) and $\partial_n \varphi = 0$ on $\Gamma_0 \times (0, T)$, then $\varphi \equiv 0$ in Q_T".

Theorem

*System (6) (resp. System (7)) is **approximately controllable** at time $T > 0$, for any ω and $T > 0$ (resp., for any Γ_0 and T).* ■

Remark

The **distributed controllability** result for System (6) **is equivalent** to the **boundary controllability** result for System (7).

Summarizing:

- System (6) and system (7) are approximately controllable and exactly controllable to trajectories at time T.
- The controllability properties of both systems are equivalent. ■

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3. Finite-dimensional systems

3. Finite-dimensional systems

Let us consider the autonomous linear system

(14) $$y' = Ay + Bu \quad \text{on } [0,T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0,T;\mathbb{C}^m)$ is the control.

Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

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Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

Let us define (*controllability matrix*)

$$[A \,|\, B] = (B\,,\, AB\,,\, A^2B\,,\, \cdots\,,\, A^{n-1}B) \in \mathcal{L}(\mathbb{C}^{nm};\mathbb{C}^n).$$

On the other hand, let $\{\theta_l\}_{1\le l\le \hat{p}} \subset \mathbb{C}$ be the set of distinct eigenvalues of A^*. For $l : 1 \le l \le \hat{p}$, we denote by m_l the geometric multiplicity of θ_l. The sequence $\{w_{l,j}\}_{1\le j\le m_l}$ will denote a basis of the eigenspace associated to θ_l.

3. Finite-dimensional systems

The following classical result can be found in

R. KALMAN, Y.-CH. HO, K. NARENDRA, *Controllability of linear dynamical systems*, 1963.

and gives a complete answer to the problem of controllability of finite dimensional autonomous linear systems:

Theorem

Under the previous assumptions, the following conditions are equivalent

1. *System* (14) *is exactly controllable at time* T, *for every* $T > 0$.
2. *There exists* $T > 0$ *such that system* (14) *is exactly controllable at time* T.
3. $\operatorname{rank}\,[A \mid B] = n$ *or* $\ker[A \mid B]^* = \{0\}$ (*Kalman rank condition*).
4. *Hautus test:* $\operatorname{rank}\begin{pmatrix} A^* - \theta_l I_n \\ B^* \end{pmatrix} = n, \quad \forall l : 1 \le l \le \hat{p}$.
5. $\operatorname{rank}\,[B^* w_{l,1}\,,\, B^* w_{l,2}\,,\, \cdots\,,\, B^* w_{l,m_l}] = m_l$, *for every* $l : 1 \le l \le \hat{p}$. ∎

3. Finite-dimensional systems

Remark

1. The four controllability concepts (exact, exact to trajectories, null and approximate controllability) for System (15) are equivalent (**finite-dimensional space**).
2. Observe that $\{B^* w_{l,1}, B^* w_{l,2}, \dots, B^* w_{l,m_l}\} \subset \mathbb{C}^m$. Condition 5 in Theorem 4 says this set is linearly independent for any $l : 1 \le l \le \hat{p}$. In particular, $\boxed{m_l \le m}$ $\forall l : 1 \le l \le \hat{p}$.
3. Given the o.d.s. (**adjoint problem**)

$$-\varphi' = A^*\varphi \quad \text{in } [0,T], \quad \varphi(T) = \varphi_T \in \mathbb{C}^n,$$

it is not difficult to prove the following result: *"System* (14) *is* ***exactly controllable*** *at time T* ***if and only if*** the following property for the **adjoint problem** holds (**unique continuation property**)

If $B^*\varphi(\cdot) = 0$ on $[0,T]$, then $\varphi_T \equiv 0$."

3. Finite-dimensional systems

Consider now the case of time dependent matrices:

(15) $$x' = A(t)x + B(t)u \quad \text{on } [0,T],$$

where $A \in C^{n-2}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in C^{n-1}([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ are given and $u \in L^2(0,T;\mathbb{R}^m)$ is a control.
Let us define

$$\begin{cases} B_0(t) = B(t), \\ B_i(t) = A(t)B_{i-1}(t) - \dfrac{d}{dt}B_{i-1}(t), \end{cases}$$

$(1 \le i \le n-1)$ and, we introduce the **Kalman matrix** denoted (as in the autonomous case) by $[A \,|\, B] \in C^0([0,T];\mathcal{L}(\mathbb{R}^{nm};\mathbb{R}^n))$ and given by:

$$[A \,|\, B](t) = (B_0(t)\,,\, B_1(t)\,,\, \cdots\,,\, B_{n-1}(t))\,.$$

(When A and B are constant matrices, this matrix coincides with the controllability matrix).

3. Finite-dimensional systems

$$x' = A(t)x + B(t)u \quad \text{on } [0, T], \tag{15}$$

With the previous notation, one has:

Theorem (Silverman-Meadows)

Under the previous assumptions, one has:

1. *If there exists* $t_0 \in [0, T]$ *such that* $\text{rank}\,[A \mid B](t_0) = n$, *then System* (15) *is exactly controllable at time* T.
2. *System* (15) *is totally exactly controllable on* $(0, T)$ *if and only if there exists* E*, a dense subset of* $(0, T)$*, such that* $\text{rank}\,[A \mid B](t) = n$ *for every* $t \in E$. ■

In the particular case in which A and B are constant matrices, the exact controllability of System (15) is equivalent to the **Kalman rank condition**.

3. Finite-dimensional systems

Remark

The first item in Theorem 3.1 gives a **sufficient** condition for the controllability of System (15) on $(0, T)$ but, in this time-dependent case, this condition is not **necessary** (see [CORON], *Control and Nonlinearity*, 2007). Nevertheless, when A and B are analytic on $(0, T)$ this condition is also necessary.
Again, the four controllability concepts for System (15) are equivalent but, in this case the positive controllability result depends on the final observation time $T > 0$.

3. Finite-dimensional systems

Goal

We have a complete characterization of the controllability results for finite-dimensional linear ordinary differential systems (a Kalman condition). Is it possible to obtain similar results for Partial Differentials Systems? We will focus on coupled linear parabolic systems.

What are the possible generalizations to Systems of Parabolic Equations?

4. Two simple examples

1. Distributed null controllability of a linear reaction-diffusion system

Let us consider the 2×2 linear reaction-diffusion system

$$(16) \qquad \begin{cases} y_t - D\Delta y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v 1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \qquad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Here Ω, ω and T are as before, $y_0 \in L^2(\Omega; \mathbb{R}^2)$, $v \in L^2(Q_T)$ is the control, and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad (A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

One has

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Here Ω, ω and T are as before, $y_0 \in L^2(\Omega; \mathbb{R}^2)$, $v \in L^2(Q_T)$ is the control, and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad \left(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

One has

Theorem

System (16) *is exactly controllable to trajectories at time* T *if and only if*

$$\det [A \mid B] := \det [B\,,\ AB] \neq 0 \iff a_{21} \neq 0.$$

Proof: $\boxed{\Longrightarrow}$: If $a_{21} = 0$, then y_2 is independent of v.

$\boxed{\Longleftarrow}$: The controllability result for system (16) is equivalent to the observability inequality: $\exists C > 0$ such that

$$\|\varphi_1(\cdot,0)\|_{L^2}^2 + \|\varphi_2(\cdot,0)\|_{L^2}^2 \le C \iint_{\omega\times(0,T)} |\varphi_1(x,t)|^2 \, dx \, dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega;\mathbb{R}^2)$ of the adjoint problem:

$$(17) \qquad \begin{cases} -\varphi_t - D\Delta\varphi = A^*\varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot,T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

It is a consequence of the global Carleman inequality (13) for $L_0 = -d_i\Delta$ $(i = 1,2)$.

Coming back to the adjoint problem for system (17), if we apply to $z = \varphi_1$ and $z = \varphi_2$ inequality (13) in $\mathcal{B} = \omega_0 \subset\subset \omega$ with $d = 3$. After some computations we get

$$\mathcal{I}_1(3, \varphi_1) + \mathcal{I}_1(3, \varphi_2) \le \widetilde{C}_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \ge \widetilde{s}_2 = \widetilde{\sigma}_2(\Omega, \omega_0)(T + T^2)$.

We now use the first equation in (17), $\boxed{a_{21}\varphi_2 = -(\varphi_{1,t} + \Delta\varphi_1 + a_{11}\varphi_1)}$, to prove ($\varepsilon > 0$): $\cdots$ ([DE TERESA], Comm. PDE, (2000))

$$s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} |\varphi_2|^2 \le \varepsilon \mathcal{I}_1(3, \varphi_2) + \frac{C}{\varepsilon} s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_1|^2.$$

$\forall s \ge \widetilde{s}_2 = \widetilde{\sigma}_2(\Omega, \omega_0)(T + T^2)$.

From the two previous inequalities (global Carleman estimate)

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \le C_2 s^7 \iint_{\omega\times(0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_1|^2,$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$. Combining this inequality and energy estimates for system (17) we deduce the desired observability inequality.

Remark

- System (16) is always controllable if we exert a control in each equation (**two controls**).
- The controllability result for system (16) is **independent** of the diffusion matrix D. We will see that the situation is more intricate if in the system a general control vector $B \in \mathbb{R}^2$ is considered.
- The same result can be obtained for the distributed approximate controllability at time T. Therefore, **approximate** and **null controllability** are equivalent concepts (distributed case).
- The proof of the sufficient part of Theorem 4.1 is still valid when $A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^2))$ under the assumption: There exist an open subset $\omega_0 \subset\subset \omega$ and $T_0, T_1 \in (0, T)$, with $T_0 < T_1$ s.t.

$$\boxed{a_{21}(x,t) \geq a_0 > 0} \text{ or } \boxed{-a_{21}(x,t) \geq a_0 > 0} \quad \text{in } \omega_0 \times (T_0, T_1). \qquad \blacksquare$$

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4. Two simple examples

2. Boundary null controllability of a linear reaction-diffusion system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(18)\qquad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T = (0,\pi)\times(0,T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \ \boxed{(d_1 \neq d_2)}, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Question

Are the controllability properties of system (18) independent of d_1 and d_2???
NO.

As before, system (18) is null controllable at time T if and only if the observability inequality

$$\|\varphi_1(\cdot,0)\|^2_{H^1_0(0,\pi)} + \|\varphi_2(\cdot,0)\|^2_{H^1_0(0,\pi)} \leq C \int_0^T |\varphi_{1,x}(0,t)|^2 \, dt,$$

holds. Again φ is the solution associated to $\varphi_0 \in H^1_0(0,\pi;\mathbb{R}^2)$ of the adjoint problem:

$$(19) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q_T, \\ \varphi|_{x=0} = \varphi|_{x=\pi} = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

Let us see that, in general, this inequality fails (**even if** $a_{21} = 1 \neq 0$**!!!!!**).

A necessary condition:

Proposition

Assume that system (18) *is null controllable at time* T. *Then* ($\lambda_k = k^2$),

$$d_1\lambda_k \neq d_2\lambda_j, \quad \forall k, j \geq 1 \quad (\Longleftrightarrow \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

Proof: By contradiction, assume that $d_1\lambda_k = d_2\lambda_j$ for some k, j and take $K = \max\{k, j\}$. The idea is transforming system (19) into an o.d.s. Recall that λ_k and ϕ_k are the eigenvalues and normalized eigenfunctions of $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

Idea: Take $\varphi_0 \in X_K = \{\varphi_0 = \sum_{\ell=1}^{K} a_\ell \phi_\ell : a_\ell \in \mathbb{R}^2\} \subset H_0^1(0, \pi; \mathbb{R}^2)$.

Consider also

$$B_K = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathbb{R}^{2K}, \quad \left(B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad \text{and}$$

$$\mathcal{L}_K^* = \text{diag}\,(-\lambda_1 D + A^*, -\lambda_2 D + A^*, \cdots, -\lambda_K D + A^*) \in \mathcal{L}(\mathbb{R}^{2K}).$$

Taking in (19) arbitrary initial data $\varphi_{0,K} = \sum_{\ell=1}^{K} a_\ell \phi_\ell \in H_0^1(0,\pi;\mathbb{R}^2)$ where $a_\ell \in \mathbb{R}^2$, it is not difficult to see that system (19) is equivalent to the o.d. system

(20) $$-Z' = \mathcal{L}_K^* Z \quad \text{on } [0,T], \quad Z(0) = Z_0 \in \mathbb{R}^{2K}.$$

From the observability inequality for system (19) we deduce the unique continuation property for the solutions to (20):

$$\boxed{B_K^* Z(\cdot) = 0 \quad \text{in } (0,T) \Longrightarrow Z \equiv 0.}$$

In particular system

$$Y' = \mathcal{L}_K Y + B_K v \quad \text{on } [0,T], \quad Y(0) = Y_0 \in \mathbb{R}^{2K}.$$

is exactly controllable at time T. Then $\boxed{\operatorname{rank}\,[\mathcal{L}_K \mid B_K] = 2K}$.

We deduce that $\mathcal{L}_K^*$ cannot have eigenvalues with **geometric multiplicity** 2 or greater.

But $\theta = -d_1\lambda_k = -d_2\lambda_j$ is an eigenvalue of $\mathcal{L}_K^*$ with two linearly independent eigenvectors $V_1, V_2 \in \mathbb{R}^{2K}$ given by:

$$\begin{cases} V_1 = (V_{1,\ell})_{1\le\ell\le K}, \quad V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_{1,\ell} = 0 \quad \forall \ell \neq k, \\ V_2 = (V_{2,\ell})_{1\le\ell\le K}, \quad V_{2,j} = \begin{pmatrix} \frac{1}{\lambda_j(d_1-d_2)} \\ 0 \end{pmatrix} \text{ and } V_{2,\ell} = 0 \quad \forall \ell \neq j. \blacksquare \end{cases}$$

The result has been proved in [FERNÁNDEZ-CARA,G.-B.,DE TERESA], J. Funct. Anal. (2010).

Conclusion: **First difference with scalar problems**

distributed controllability $\not\equiv$ **boundary controllability**.

Even if System (16) is very close to System (18), their controllability properties are strongly different:

- System (16) (distributed control): We have obtained a complete characterization of the null controllability property (and even, a distributed Carleman estimate for the **adjoint problem** (17)).
- System (18) (boundary control): The system is not null controllable if $d_1\lambda_k = d_2\lambda_j$ for some $k, j \geq 1$.

The same non-scalar parabolic problem can be controlled to zero with distributed controls supported on an interval ω and, however, the ***null controllability result*** *fails when the control acts on a part of the boundary.*

$$(18) \qquad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \; d_1, d_2 > 0, \; d_1 \neq d_2, \; A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark

- Again, System (18) is always null controllable at time T if we exert two independent controls at the same point. In this case, equivalence between distributed and boundary controllability (as in the **scalar case**; see Theorem 1).
- If $d_1 \neq d_2$, one has: "*System* (18) *is approximately controllable at time T* $\iff \boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}}$". ■

$$(19) \qquad \begin{cases} -\varphi_t = D\varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0,\pi\}\times(0,T), \quad \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ d_1, d_2 > 0, \ d_1 \neq d_2, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Boundary approximate controllability

"*System* (18) *is approximately controllable at time* $T \iff \boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}}$".

What does this condition mean???: The eigenvalues of the operator $\mathcal{R}^*\Phi = D\Phi_{xx} + A^*\Phi$ are

$$\left\{ -\frac{k^2}{d_1} \right\}_{k\geq 1} \cup \left\{ -\frac{i^2}{d_2} \right\}_{i\geq 1}.$$

Then, $\boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}} \iff$ the eigenvalues of $\mathcal{R}^*$ are **simple**.

$$(18)\quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T, \\ y|_{x=0} = B v, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Second difference with scalar problems

Null controllability: Assume $\sqrt{d_1/d_2} \notin \mathbb{Q}$. Is System (18) null controllable at time T? i.e., are approximate controllability and null controllability equivalent for System (18)?

The answer is **negative**. In [LUCA,DE TERESA] (in preparation), the authors provide an example of matrix D satisfying $\sqrt{d_1/d_2} \notin \mathbb{Q}$ (and therefore, the system is approximately controllable at every positive time T) and such that System (18) is not null controllable at any time $T > 0$. Then, for System (18),

$$\textbf{approximate controllability} \quad \not\equiv \quad \textbf{null controllability}.$$

(See also [AMMAR-KHODJA,BENABDALLAH,DUPAIX,KOSTINE], ESAIM:COCV (2005) for some abstract non-scalar parabolic systems).

Observe that in this case, the elliptic operator in the **adjoint system**, $-\mathcal{R}^*$,

$$(19) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} - A^*\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0,\pi\} \times (0,T), \quad \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi), \end{cases}$$

has a sequence of simple positive real eigenvalues $\left\{\frac{\ell^2}{d_1}\right\}_{\ell\geq 1} \cup \left\{\frac{i^2}{d_2}\right\}_{i\geq 1} = \{\Lambda_k\}_{k\geq 1}$. Then, we could apply the **moment method** for obtaining the null controllability result (see Remark 7). One has

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty,$$

and we will see that this condition assures the existence of a biorthogonal family $\{p_k\}_{k\geq 1}$ to the family $\{e^{-\Lambda_k t}\}_{k\geq 1}$. However the "separability condition"

$$|\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1 \quad (\rho > 0)$$

fails and this condition is strongly connected with the bounds of the L^2-norm of the biorthogonal family $\{p_k\}_{k\geq 1}$.

5. Cascade system. Distributed controls

5. Cascade system. Distributed controls

We consider the linear parabolic system

$$\begin{cases} \partial_t y_1 + L_0^1(t) y_1 + \sum_{j=1}^n C_{1j} \cdot \nabla y_j + \sum_{j=1}^n a_{1j} y_j = v 1_\omega & \text{in } Q_T = \Omega \times (0,T), \\ \partial_t y_2 + L_0^2(t) y_2 + \sum_{j=1}^n C_{2j} \cdot \nabla y_j + \sum_{j=1}^n a_{2j} y_j = 0 & \text{in } Q_T, \\ \dots & \\ \partial_t y_n + L_0^n(t) y_n + \sum_{j=1}^n C_{nj} \cdot \nabla y_j + \sum_{j=1}^n a_{nj} y_j = 0 & \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T = \partial\Omega \times (0,T), \quad y_i(\cdot, 0) = y_0^i \text{ in } \Omega, & 1 \le i \le n, \end{cases}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q_T)$, $C_{ij} = C_{ij}(x,t) \in L^\infty(Q_T; \mathbb{R}^N)$ $(1 \le i,j \le n)$, $y_0^i \in L^2(\Omega)$ $(1 \le i \le n)$ and $L_0^k(t)$ is, for every $1 \le k \le n$, the second order operator $\boxed{L_0^k(t) y = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\alpha_{ij}^k(x,t) \frac{\partial y}{\partial x_j} \right)}$ where α_{ij}^k satisfy (4) and (5) for every k.

5. Cascade system. Distributed controls

Objective

Controllability properties of the system: n equations controlled with a **unique** distributed control.

Equivalently, the previous system can be written as

$$(21) \qquad \begin{cases} \partial_t y + \widehat{L}(t)y + C \cdot \nabla y + Ay = Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\widehat{L}(t)$ is the **matrix operator** given by $\widehat{L}(t) = \text{diag}\,(L_0^1(t), \cdots, L_0^n(t))$, $y = (y_i)_{1\le i\le n}$ is the state and $\nabla y = (\nabla y_i)_{1\le i\le n}$, and where

$$\begin{cases} y_0 = (y_0^i)_{1\le i\le n} \in L^2(\Omega;\mathbb{R}^n), \quad A(\cdot,\cdot) = (a_{ij}(\cdot,\cdot))_{1\le i,j\le n} \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^n)), \\ C(\cdot,\cdot) = (C_{ij}(\cdot,\cdot))_{1\le i,j\le n} \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{Nn})) \text{ and } B \equiv e_1 = (1,0,\dots,0)^* \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega;\mathbb{R}^n)$ and $v \in L^2(Q_T)$, System (21) admits a **unique weak solution**

$$y \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^n)) \cap C^0([0,T];L^2(\Omega;\mathbb{R}^n)).$$

5. Cascade system. Distributed controls

By **cascade system** we mean that matrices A and C have the following structure:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ 0 & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{nn} \end{pmatrix}$$

with $a_{ij} \in L^\infty(Q_T)$ and $C_{ij} \in L^\infty(Q_T;\mathbb{R}^N)$ and the coefficients $a_{i,i-1}$ satisfy

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \leq i \leq n,$$

with $\omega_0 \subseteq \omega$ a new open subset.

Remark

It is natural to assume that $a_{i,i-1} \not\equiv 0$ for any $i : 2 \leq i \leq n$. The previous assumption is **stronger** but will provide the controllability result. ■

5. Cascade system. Distributed controls

In this case, the corresponding **adjoint problem** has the form

$$\begin{cases} -\partial_t \varphi_i + L_0^i(t)\varphi_i - \displaystyle\sum_{j=1}^{i} [\nabla \cdot (C_{ji}\varphi_j) - a_{ji}\varphi_j] = -a_{i+1,i}\varphi_{i+1} & \text{in } Q_T, \\ \cdots \qquad (1 \le i \le n-1), & \\ -\partial_t \varphi_n + L_0^n(t)\varphi_n - \displaystyle\sum_{j=1}^{n} [\nabla \cdot (C_{jn}\varphi_j) - a_{jn}\varphi_j] = 0 & \text{in } Q_T, \\ \varphi_i = 0 \text{ on } \Sigma_T, \quad \varphi_i(\cdot, T) = \varphi_{i,T} \text{ in } \Omega, \quad 1 \le i \le n, & \end{cases}$$

where $\varphi_{i,T} \in L^2(\Omega)$ $(1 \le i \le n)$. Again, the null controllability of System (21) (with L^2-controls) at time T is equivalent to the existence of a constant $C > 0$ such that the so-called **observability inequality**

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega;\mathbb{R}^n)} \le C \iint_{\omega \times (0,T)} |\varphi_1(x,t)|^2$$

holds for every solution $\varphi = (\varphi_1, \ldots, \varphi_n)^*$ to the **adjoint problem**.

5. Cascade system. Distributed controls

Theorem

*Under the previous assumptions, let $M_0 = \max_{2\le i\le n} \|a_{i,i-1}\|_\infty$. Then, there exist a positive function $\alpha_0 \in C^2(\overline{\Omega})$ (only depending on Ω and ω_0), two positive constants C_0 and σ_0 (only depending on Ω, ω_0, c_0, M_0 and d) and $l \ge 0$ (only depending on n) such that, for every $\varphi_T \in L^2(Q_T;\mathbb{R}^n)$, the solution φ to the **adjoint problem** satisfies*

$$\sum_{i=1}^{n} \mathcal{I}(d+3(n-i),\varphi_i) \le C_0 s^{d+l} \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{d+l}|\varphi_1|^2,$$

$\forall s \ge s_0 = \sigma_0 \left[T + T^2 + T^2 \max_{i\le j} \left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} \right) \right]$. *In the previous inequality,* $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$ *and* $\mathcal{I}(d,z)$ *is given in Lemma 2.3 (with α instead of β).*

Combining the previous result and **energy inequalities** satisfied by the solutions of the **adjoint system** it is possible to prove an **observability inequality** for the **adjoint system** (as in the scalar case). Summarizing, we get

Corollary

Under assumptions of the previous result, there exists a positive constant C (only depending on Ω, ω, n, c_0 and M_0) such that for every $y_0 \in L^2(\Omega;\mathbb{R}^n)$ there is a control $v \in L^2(\Omega)$ which satisfies

$$\|v\|\|_{L^2(Q_T)}^2 \leq e^{C\,\mathcal{H}}\|y_0\|_{L^2(\Omega;\mathbb{R}^n)}^2,$$

and $y(\cdot,T)=0$ in Ω, with y the solution to (21) associated to y_0 and v. In the previous inequality, $\mathcal{H}$ is given by

$$\mathcal{H} \equiv 1+T+\frac{1}{T}+\max_{i\leq j}\left(\|a_{ij}\|_{\infty}^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_{\infty}^{\frac{2}{3(j-i)+1}} + T\left(\|a_{ij}\|_{\infty} + \|C_{ij}\|_{\infty}^2\right)\right).$$

Sketch of the proof of Theorem 5.1: Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma 2.3 and associated to Ω and $\mathcal{B} \equiv \omega_1$. We will do the proof in two steps:

Step 1. Let φ be the solution to **adjoint system** associated to φ_T. Each component satisfies

$$-\partial_t \varphi_i + L_0^i(t)\varphi_i = \boxed{\sum_{j=1}^{i} \left[\nabla \cdot (C_{ji}\varphi_j) - a_{ji}\varphi_j\right] - a_{i+1,i}\varphi_{i+1}}.$$

We begin applying inequality (12) with $\mathcal{B} = \omega_1$ to each function φ_i with $L_0 \equiv L_0^i$, $d = d + 3(n-i)$ and the corresponding right-hand side. Now if we take

$$s \geq s_0 = \sigma_0 \left(T + T^2 + T^2 \max_{i \leq j} \left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} \right) \right),$$

with $\sigma_0 = \sigma_0(\Omega, \omega_0, c_0, M_0) > 0$, we obtain the existence of a positive constants $C_1 = C_1(\Omega, \omega_0, c_0, M_0)$ such that if $s \geq s_0$, then

$$\sum_{i=1}^{n} \mathcal{I}(d+3(n-i),\varphi_i) \le C_1 \sum_{i=1}^{n} s^{s+3(n-i)} \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\gamma(t)^{s+3(n-i)}|\varphi_i|^2.$$

Step 2. Thanks to the assumption

$$a_{i,i-1} \ge c_0 > 0 \text{ or } -a_{i,i-1} \ge c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \le i \le n,$$

with $\omega_0 \subseteq \omega$ an open subset, and the cascade structure

$$a_{i,i-1}\varphi_i = \partial_t \varphi_{i-1} - L_0^{i-1}(t)\varphi_{i-1} + \sum_{j=1}^{i-1} \left[\nabla\cdot(C_{j,i-1}\varphi_j) - a_{j,i-1}\varphi_{i-1}\right] \text{ in } Q_T,$$

can eliminate the local terms for $2 \le i \le n$. In order to carry this process out, we will need the following result:

5. Cascade system. Distributed controls

Lemma

Under assumptions of Theorem 5.1 and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k \in \{2, ..., n\}$ and two open sets $\mathcal{O}_0$ and $\mathcal{O}_1$ such that $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$, there exist a constant C_k (only depending on Ω, $\mathcal{O}_0$, $\mathcal{O}_1$, c_0 and M_0) and $l_{kj} \in \mathbb{N}$, $1 \leq j \leq k-1$ (only depending on l, n, k and j), such that, if $s \geq s_0$, one has

$$s^l \iint_{\mathcal{O}_1 \times (0,T)} e^{-2s\alpha} \gamma(t)^l |\varphi_k|^2 \leq \varepsilon \left[\mathcal{I}(d + 3(n-k), \varphi_k) + \mathcal{I}(d + 3(n-k-1), \varphi_{k+1}) \right] + C_k \left(1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} s^{l_{kj}} \iint_{\mathcal{O}_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^{l_{kj}} |\varphi_j|^2 .$$

(In this inequality we have taken $\varphi_{k+1} \equiv 0$ when $k = n$).

The proof of Theorem 5.1 is a consequence of this Lemma 5.3. For the details, see [DE TERESA], Comm. PDE (2000), [G.-B., PÉREZ-GARCÍA], Asymp. Anal. (2006) and [G.-B., DE TERESA], Port. Math. (2010).

Remark

1. **Cascade systems** appear in the context of existence of **insensitizing controls** for a scalar parabolic equation: Equivalent to a null controllability result for a 2×2 parabolic system ($n = 2$) with one equation forward in time and the other one backward. The coupling coefficient a_{21} is $1_{\mathcal{O}}$ with $\mathcal{O} \subseteq \Omega$ an open set and $\boxed{\mathcal{O} \cap \omega \neq \emptyset}$.
2. The previous proof uses the assumption

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \leq i \leq n,$$

   in a crucial way. When $a_{i,i-1}$ are constant, this assumption is **necessary**. Is this condition **necessary** in the general case???
3. Is it possible to provide a **necessary** and **sufficient** (**Kalman condition**) condition for the null controllability of non-scalar systems? **YES** in some **constant coefficient systems**.

6. The Kalman condition for a class of parabolic systems. Distributed controls

6. The Kalman condition for a class of parabolic systems. Distributed controls

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

For $n, m \in \mathbb{N}$ we consider the following $n \times n$ parabolic system

$$(22) \qquad \begin{cases} \partial_t y + DL(t)y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where $L(t)$ is the operator given in (3), with coefficients satisfying (4) and (5), $y_0 \in L^2(\Omega; \mathbb{R}^n)$ is given and

$$\begin{cases} A \in C^{M-1}([0,T]; \mathcal{L}(\mathbb{R}^n)), \quad B \in C^M([0,T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)), \\ \quad D = \operatorname{diag}(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n), \quad (d_i > 0, \ \forall i), \end{cases}$$

with $M \in \mathbb{N}$ large enough. Again, $v \in L^2(Q_T; \mathbb{R}^m)$ is the control (m components).

4. The Kalman condition for a class of parabolic systems. Distributed controls

$$(22) \qquad \begin{cases} \partial_t y + DL(t)y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega. \end{cases}$$

Remark

This problem is well posed: For any $y_0 \in L^2(\Omega;\mathbb{R}^n)$ and $v \in L^2(Q_T;\mathbb{R}^m)$, problem (22) has a unique solution $y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

Remark

We want to control the whole system (n equations) with m controls. The most interesting case is $m < n$ or even $m = 1$.

Difficulties:

1. In general $m < n$.
2. D is not the identity matrix.
3. L, A and B depend on time. ■

The adjoint problem:

$$(23) \qquad \begin{cases} -\partial_t \varphi = (-DL(t) + A^*(t))\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

where $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$. Then, the exact controllability to the trajectories of system (22) is equivalent to the existence of $C > 0$ such that, for every $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$, the solution $\varphi \in C^0([0,T]; L^2(\Omega; \mathbb{R}^n))$ to the adjoint system (23) satisfies the **observability inequality**:

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega \times (0,T)} |B^*(t)\varphi(x,t)|^2.$$

6. The Kalman condition for a class of parabolic systems. Distributed controls

1. Identity diffusion matrix

We follow [AMMAR-KHODJA,BENABDALLAH,DUPAIX,G.-B.], Diff. Eq. Appl. (2009).

6. The Kalman condition for a class of parabolic systems. Distributed controls

1. Identity diffusion matrix

Assume that $D = Id$:

$$
(22) \qquad \begin{cases} \partial_t y + L(t)y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega. \end{cases}
$$

with $A \in C^{M-1}([0,T]; \mathcal{L}(\mathbb{R}^n))$ and $B \in C^M([0,T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$. This is the simplest case. It is possible to prove a null controllability result for this system which is very close to the finite-dimensional case (Theorem 3.1). Recall

$$
\begin{cases} B_0(t) = B(t), \\ B_i(t) = A(t)B_{i-1}(t) - \dfrac{d}{dt}B_{i-1}(t), \quad 1 \leq i \leq M. \end{cases}
$$

Theorem

Assume that $D = Id$ and $M \geq n$. Then, under the regularity assumptions on A and B, one has:

1. *If there exist $t_0 \in [0, T]$ and $p \in \{1, ..., M\}$ such that*

$$\text{rank } (B_0 \, , \, B_1 \, , \, \cdots \, , \, B_{p-1}) \, (t_0) = n,$$

*then System (22) is **null controllable** at time T.*

2. *System (22) is **totally null controllable** on $(0, T)$ if and only if there exists E, a dense subset of $(0, T)$, such that $\text{rank } [A \mid B](t) = n$ for every $t \in E$, (or, equivalently, $\text{rank } (B_0 \, , \, B_1 \, , \, \cdots \, , \, B_{p-1}) \, (t) = n$ for all $p \in \{n, ..., M\}$ and $t \in E$).*

(See Theorem 3.1).

Proof: The proof uses in an essential way the assumption $D = Id$. Using that $M \geq n$, it is possible to deduce the existence of an interval $(T_0, T_1) \subseteq (0, T)$ such that

$$\operatorname{rank} K_n(t) = n, \quad \forall t \in [T_0, T_1],$$

with $K_n(t) = (B_0 \,,\, B_1 \,,\, \cdots \,,\, B_{n-1})\,(t)$. This last condition allows to perform a change of variables on the interval $[T_0, T_1]$ and rewrite the system

$$(22) \qquad \begin{cases} \partial_t y + L(t)y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

as a **cascade system** on the interval (T_0, T_1). In particular, we can apply the results of the previous section. This implies the null controllability result on the interval (T_0, T_1) and then, at time T.

Let us see the proof in the simplest case $m = 1$ (**one control**) and $A(t) \equiv A \in \mathcal{L}(\mathbb{R}^n)$ and $B(t) \equiv B \in \mathbb{R}^n$, for all $t \in (0, T)$ (**autonomous case**).

...

$$(22) \qquad \begin{cases} \partial_t y + L(t)y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

Remark

(a) As in the finite-dimensional case, the existence of $t_0 \in (0, T)$ and $n \le p \le M$ s.t. $\operatorname{rank} K_p(t_0) = n$ ($K_p(t_0) := (B_0\,, B_1\,, \cdots\,, B_{p-1})\,(t_0)$) is not a **necessary** condition for the null controllability on $(0, T)$.

(b) When A and B are analytic functions on $(0, T)$ it is possible to prove that $\operatorname{rank} K_p(t_0) = n$ for $t_0 \in (0, T)$ and $n \le p \le M$ is a **necessary** and **sufficient** condition for the null controllability on $(0, T)$ (in particular in the **autonomous case**).

(c) It is possible to prove appropriate Carleman inequalities for the corresponding adjoint problem. ∎

Theorem (Autonomous case)

There exist a positive function $\alpha_0 \in C^2(\overline{\Omega})$ (only depending on Ω and ω), positive constants C and σ (only depending on Ω, ω, n, m, A and B) and a positive integer $\ell \geq 3$ (only depending on n and m) such that, if $\operatorname{rank}[A \mid B] = n$, *for every $\varphi_0 \in L^2(Q_T; \mathbb{R}^n)$, the solution φ to* (23) *satisfies*

$$\mathcal{I}_1(d, \varphi) \leq C\left(s^{d+\ell} \iint_{\omega\times(0,T)} e^{-2s\alpha}\gamma(t)^{d+\ell}|B^*\varphi|^2\right),$$

$\forall s \geq s_0 = \sigma\left(T + T^2\right)$. *In this inequality, $\alpha(x,t)$, $\gamma(t)$ and $\mathcal{I}_1(d,z)$ are as in Lemma 2.3 and Lemma 2.4.* ∎

For details, see [AMMAR-KHODJA, BENABDALLAH, DUPAIX, G.-B.], Diff. Eq. Appl. (2009).

6. The Kalman condition for a class of parabolic systems. Distributed controls

2. Diagonal diffusion matrix and autonomous systems

We follow [AMMAR-KHODJA, BENABDALLAH, DUPAIX, G.-B.], J. Evol. Eq. (2009).

We come back to System (22) in the autonomous case:

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ and $D = \text{diag}\,(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$. Now we assume that L_0 is the self-adjoint second order elliptic operator:

$$L_0 y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x)\frac{\partial y}{\partial x_j}\right)$$

with coefficients satisfying (4) and (5). Finally, $y_0 \in L^2(\Omega;\mathbb{R}^n)$ is given and $v \in L^2(Q_T;\mathbb{R}^m)$ is the control (*m* **distributed controls**).

Let us consider $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for L_0 with homogeneous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions.

Theorem (A Necessary Condition)

*If system (22) is **null controllable** at time T **then***

(24) $$\operatorname{rank}\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq 1.$$

where
$[-\lambda_k D + A \,|\, B] = [B\,,\,(-\lambda_k D + A)B\,,\,(-\lambda_k D + A)^2 B\,,\,\cdots\,,\,(-\lambda_k D + A)^{n-1} B]$.

Proof: Reasoning by contradiction: $\exists k \geq 1$ such that $\operatorname{rank}\,[-\lambda_k D + A \,|\, B] < n$. Then the o.d.s. $-Z' = (-\lambda_k D + A^*)Z \quad$ in $(0, T)$, is not B^*-observable at time T.

There exists $Z_0 \in \mathbb{R}^n$, $Z_0 \neq 0$, such that the solution Z to the previous system satisfies $B^*Z(\cdot) = 0$ on $(0, T)$. But $\varphi(x, t) = Z(t)\phi_k(x)$ is the solution to **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

associated to $\varphi_0(x) = Z_0 \phi_k \not\equiv 0$ and $\boxed{B^* \varphi(\cdot, \cdot) \equiv 0}$ in Q_T. Then, the **observability inequality**

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \le C \iint_{\omega \times (0,T)} |B^* \varphi(x, t)|^2,$$

fails and the system is not null controllable at time T. ■

Remark

If condition (24) is not satisfied, then system (22) is neither approximately controllable nor null controllable at time T (for any $T > 0$) even if $\omega \equiv \Omega$. ■

Question:

Is condition (24) rank $[-\lambda_k D + A \mid B] = n, \forall k \geq 1$, a **sufficient condition** for the null controllability of system (22)??? ■

Let us now introduce the unbounded matrix operator

Question:

Is condition (24) rank $[-\lambda_k D + A \,|\, B] = n$, $\forall k \geq 1$, a **sufficient condition** for the null controllability of system (22)??? ■

Let us now introduce the unbounded matrix operator

$$\mathcal{K} = [DL_0 + A \,|\, B] = [B\,,\ (-DL_0 + A)B\,,\ \cdots\ ,\ (-DL_0 + A)^{n-1}B],$$

$$\begin{cases} \mathcal{K} : D(\mathcal{K}) \subset L^2(\Omega;\mathbb{R}^{nm}) \to L^2(\Omega;\mathbb{R}^n), \ \text{with} \\ D(\mathcal{K}) := \{y \in L^2(\Omega;\mathbb{R}^{nm}) : \mathcal{K}y \in L^2(\Omega;\mathbb{R}^n)\}. \end{cases}$$

Then,

Proposition

$\ker \mathcal{K}^* = \{0\}$ ***if and only if*** *condition* (24), rank $[-\lambda_k D + A \,|\, B] = n$, $\forall k \geq 1$, *holds.* ■

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

Theorem (Kalman condition)

System (22) is ***exactly controllable to trajectories*** *at time T* ***if and only if***
System (22) is ***approximately controllable*** *at time T* ***if and only if***
$\boxed{\ker \mathcal{K}^* = \{0\}}$ $(\Longleftrightarrow \operatorname{rank}[-\lambda_k D + A \,|\, B] = n, \ \forall k \geq 1)$.

Remark

One can prove, either there exists $k_0 \geq 1$ such that

$$\boxed{\operatorname{rank}[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq k_0}$$

or

$$\boxed{\operatorname{rank}[-\lambda_k D + A \,|\, B] < n, \quad \forall k \geq 1}.$$

Controllability (outside a finite dimensional space) **if and only if** the algebraic Kalman condition $\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] = n}$ is satisfied for one frequency $k \geq 1$.

Remark

System (22) can be exactly controlled to the trajectories with one control force ($m = 1$ and $B \in \mathbb{R}^n$) even if $\boxed{A \equiv 0}$. Indeed, let us assume that $B = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$. Then,

$$[(-\lambda_k D + A) \,|\, B] = \begin{bmatrix} b_1 & (-\lambda_k d_1) b_1 & \cdots & (-\lambda_k d_1)^{n-1} b_1 \\ b_2 & (-\lambda_k d_2) b_2 & \cdots & (-\lambda_k d_2)^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & (-\lambda_k d_n) b_n & \cdots & (-\lambda_k d_n)^{n-1} b_n \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

and (24) holds **if and only if** $b_i \neq 0$ for every i and d_i are **distinct**. ■

Idea of the proof: We have proved the **necessary condition**. Therefore, let us prove that $\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] = n}$, for any k, is a **sufficient condition** for the null controllability at time T of the system.

Then, the objective is to prove the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |B^*\varphi(x,t)|^2,$$

for the solutions to the **adjoint problem**.

To this end we use two arguments:

- Prove a global Carleman estimate for a scalar parabolic equation of order n in time.
- Prove a **coercivity** property for the Kalman operator $\mathcal{K}$.

Let us fix $\varphi_0 \in D(L_0^i)$, $\forall i \geq 0$ and consider φ the corresponding solution to the **adjoint system** (23)

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi \quad \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega. \end{cases}$$

Let us take $\boxed{\Phi = \sum_{i=1}^{n} a_i \varphi_i}$, with $a_i \in \mathbb{R}$ $(1 \leq i \leq n)$. Then, Φ is a regular solution ($L_0^i \partial_t^j \Phi \in L^2(Q_T)$, $\forall i, j$) to the linear parabolic scalar equation of order n in time

$$\begin{cases} \det \left(I_d \partial_t - DL_0 + A^* \right) \Phi = 0 \quad \text{in } Q_T, \\ L_0^i \Phi = 0 \quad \text{on } \Sigma_T, \quad \forall i \geq 0. \end{cases}$$

The key point is to prove a Carleman inequality for the solutions to the previous problem. Fix $\omega_0 \subset\subset \omega$ a nonempty open subset. Recall Lemmas 2.3 and 2.4:

Lemma

There exist a $\alpha_0 \in C^2(\overline{\Omega})$ *(positive), and two constants* $C_0, \sigma_0 > 0$ *(only depending on* Ω, ω_0 *and d) s.t.*

$$\begin{cases} \mathcal{I}_1(d,\phi) \equiv \displaystyle\iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d-4} \left(|\phi_t|^2 + |L_0\phi|^2\right) \\ \quad + \displaystyle\iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d-2} |\nabla\phi|^2 + \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\phi|^2 \\ \leq C_0 \left(\displaystyle\iint_{\omega_0\times(0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\phi|^2 + \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d-3} |\phi_t \pm L_0\phi|^2 \right), \end{cases}$$

$\forall s \geq s_0 = \sigma_0(\Omega,\omega)(T+T^2)$, $\forall \phi \in L^2(0,T;H_0^1(\Omega))$ *s.t.* $\phi_t \pm L_0\phi \in L^2(Q_T)$.

$\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$.

Theorem

Let $n, k_1, k_2 \in \mathbb{N}$ and $d \in \mathbb{R}$. There exist two constants C and σ (only depending on Ω, ω, n, D, A, k_1, k_2 and d), and $r_0 = r_0(n) \in \mathbb{N}$ such that

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(d - 4(i+j), L_0^i \partial_t^j \Phi) \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{3+r_0} |\Phi|^2, \quad ,$$

$\forall s \geq s = \sigma(\Omega, \omega)(T + T^2)$, Φ solution to the previous problem and

$$\mathcal{J}(\tau, z) := \mathcal{I}_1(\tau + 3(n-1), z) + \sum_{i=1}^{n} \mathcal{I}_1(\tau + 3(n-2), P_i z) + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \mathcal{I}_1(\tau + 3(n-p-1), P_{i_p} \cdots P_{i_1} z).$$

$(P_i \equiv \partial_t - d_i L_0)$

Sketch of the proof: We will give the main ideas in the case $\boxed{k_1 = k_2 = 0}$. If we use the notation $P_i \equiv \partial_t - d_i L_0$ $(1 \le i \le n)$, one has:

$$\det\left(I_d \partial_t - DL_0 + A^*\right) \equiv P_n \cdots P_1 + \sum_{p=2}^{n-1} \sum_{1\le i_1 < \cdots < i_p \le n} b_{i_1,\ldots,i_p} P_{i_1} \ldots P_{i_p}$$
$$+ \sum_{i=1}^{n} b_i P_i + b := P_n \cdots P_1 - F,$$

with $b_{i_1,\ldots,i_p}, b_i, b \in \mathbb{R}$ only depending on D and A.
We have a function Φ s.t. $L_0^i \partial_t^j \Phi \in L^2(Q_T)$, $\forall i, j$, and it is solution to

$$\begin{cases} \det\left(I_d \partial_t - DL_0 + A^*\right) \Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 \quad \text{on } \Sigma, & \forall i \ge 0. \end{cases}$$

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T.

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T. We rewrite the order-n equation as a system performing the change of variables:

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{i-1}\psi_{i-1} \equiv (\partial_t - d_{i-1})\psi_{i-1}, \quad 2 \leq i \leq n. \end{cases}$$

Then, $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)^*$ satisfies the **cascade system**

$$\begin{cases} (\partial_t - d_1 L_0)\,\psi_1 = \psi_2 & \text{in } Q_T, \\ (\partial_t - d_2 L_0)\,\psi_2 = \psi_3 & \text{in } Q_T, \\ \qquad\vdots & \\ (\partial_t - d_n L_0)\,\psi_n = F(\Phi) & \text{in } Q_T, \\ \psi_i = 0 \text{ on } \Sigma_T, & \forall i : 1 \leq i \leq n. \end{cases}$$

We can apply Theorem 5.1 (cascade systems) and obtain:

We can apply Theorem 5.1 and obtain (cascade systems) ($d \in \mathbb{R}$ is given):

$$\sum_{i=1}^{n} \mathcal{I}_1(d+3(n-i),\psi_i) \le C_0 \left(\iint_{\omega\times(0,T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\psi_1|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d} |F(\Phi)|^2 \right),$$

$\forall s \ge s_0 = \sigma_0 \left(T+T^2\right)$ with $r_0 = r_0(n)$ and

$$\mathcal{I}_1(d,z) \equiv \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d} \{ [s\gamma(t)]^{-4} (|\partial_t z|^2 + |L_0 z|^2) + [s\gamma(t)]^{-2} |\nabla z|^2 + |z|^2 \}.$$

Coming to the original variables, one has

$$\mathcal{I}_1(d+3(n-1),\Phi) + \sum_{i=2}^{n} \mathcal{I}_1(d+3(n-i), P_{i-1}\cdots P_1\Phi)$$

$$\le C_0 \left(\iint_{\omega\times(0,T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d} |F(\Phi)|^2 \right).$$

We can reproduce the previous argument for a general permutation Π of the set $\{1, 2, \ldots, n\}$, taking

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{\Pi(i-1)}\psi_{i-1} \equiv (\partial_t - d_{\Pi(i-1)})\psi_{\Pi(i-1)}, \quad 2 \le i \le n. \end{cases}$$

Thus,

$$\mathcal{I}_1(d + 3(n-1), \Phi) + \sum_{i=2}^{n} \mathcal{I}_1(d + 3(n-i), P_{\Pi(i-1)} \cdots P_{\Pi(1)}\Phi)$$
$$\le C_0 \left(\iint_{\omega \times (0,T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d} |F(\Phi)|^2 \right),$$

$\forall s \ge s_0 = \sigma_0 \left(T + T^2\right)$. Adding all these inequalities (for any permutation Π) with $d = 3$, we get

Adding all these inequalities (for any permutation Π) with $d = 3$, we get

$$\mathcal{J}(d,\Phi) \le C\left(\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2 + \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}|F(\Phi)|^2 \right),$$

$\forall s \ge s_0 = \sigma_0 \left(T + T^2\right)$ ($\mathcal{J}(\tau, z)$ given in the statement of Theorem 11 and

$$F(\Phi) = \sum_{p=2}^{n-1} \sum_{1\le i_1<\cdots<i_p\le n} b_{i_1,\dots,i_p} P_{i_1}\dots P_{i_p}\Phi + \sum_{i=1}^{n} b_i P_i \Phi + b\Phi).$$

From these expressions, it is possible to absorb the last term of the previous inequality and obtain

$$\mathcal{J}(d,\Phi) \le C \iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2,$$

for a new constant C, with $s \ge s = \sigma\left(T + T^2\right)$. This ends the proof in the case $k_1 = k_2 = 0$. ∎

Remark

Theorem 11 is, in fact, a Carleman inequality for the regular solutions Φ to the linear parabolic scalar equation of order n in time

$$\begin{cases} \det\left(I_d\partial_t - DL_0 + A^*\right)\Phi = 0 & \text{in } Q_T, \\ L_0^i\Phi = 0 \quad \text{on } \Sigma, & \forall i \geq 0. \end{cases}$$

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t\varphi + DL_0\varphi = A^*\varphi \quad \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^{n} a_i\varphi_i$ satisfies Theorem 11. In particular any component of $B^*\varphi$. ■

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t\varphi + DL_0\varphi = A^*\varphi \quad \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^n a_i\varphi_i$ satisfies Theorem 11. In particular any component of $B^*\varphi$. ∎

Recall $\mathcal{K} = [DL_0 + A \,|\, B] = [B\,,\, (-DL_0 + A)B\,,\, \cdots\,,\, (-DL_0 + A)^{n-1}B]$, then

$$\begin{aligned} \mathcal{K}^*\varphi(\cdot, t) &= [B^*\varphi\,,\, B^*(-DL_0 + A^*)\varphi\,,\, \cdots\,,\, B^*(-DL_0 + A^*)^{n-1}\varphi]^{tr}(\cdot, t) \\ &= [B^*\varphi\,,\, -\partial_t(B^*\varphi)\,,\, \cdots\,,\, (-1)^{n-1}\partial_t^{n-1}(B^*\varphi)]^{tr}(\cdot, t) \in \mathbb{R}^{nm}. \end{aligned}$$

We apply Theorem 11 with $k_1 = n - 1$ and $k_2 = k \geq 0$. Then, after some computations, we deduce ($d = 3$)

Then, after some computations, we deduce ($d = 3$)

$$\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\gamma(t)]^3 \, \|L_0^k \mathcal{K}^* \varphi\|^2_{L^2(\Omega)^{nm}} \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha} [s\gamma(t)]^{3+r_0} |B^*\varphi|^2$$

for every $s \geq \sigma\left(T + T^2\right)$. In this inequality, $M_0 = \max_{\overline{\Omega}} \alpha_0$ and $r_0 \geq 0$ is an integer only depending on n.

Remark

The previous inequality is a partial observability estimate. It is valid even if the Kalman condition does not hold, i.e., even if $\ker \mathcal{K}^* \neq \{0\}$. ■

The **coercivity** property of $\mathcal{K}^*$:

Theorem

Assume that $\ker \mathcal{K}^* = \{0\}$ *and consider* $k = (n-1)(2n-1)$. *Then there exists* $C > 0$ *such that if* $z \in L^2(\Omega)^n$ *satisfies* $\mathcal{K}^* z \in D(L_0^k)^{nm}$, *one has*

$$\|z\|^2_{L^2(\Omega)^n} \le C \|L_0^k \mathcal{K}^* z\|^2_{L^2(\Omega)^{nm}} .$$

So, from the previous inequality we get

$$\boxed{\int_0^T e^{\frac{-2sM_0}{t(T-t)}} \left[s\gamma(t)\right]^3 \|\varphi\|^2_{L^2(\Omega)^{nm}} \le C \iint_{\omega\times(0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{3+r_0} |B^*\varphi|^2}$$

and the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \le C \iint_{\omega\times(0,T)} |B^*\varphi(x,t)|^2.$$

Summarizing

1. We have established a **Kalman condition**

$$\ker \mathcal{K}^* = \{0\}$$

which characterizes the controllability properties of system (22).

2. The **Kalman condition** for system (22) $\ker \mathcal{K}^* = \{0\}$ generalizes the **algebraic Kalman condition** $\ker[A \mid B]^* = \{0\}$ for o.d.s.

3. This **Kalman condition** is also equivalent to the **approximate controllability** of system (22) at time T. Again, **approximate** and **null controllability** are equivalent concepts for system (22).

References

1. F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. G.-B., *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, Differ. Equ. Appl. 1 (2009), no. 3, 139–151.

$$\boxed{D = I_d, \ A = A(t) \text{ and } B = B(t).}$$

2. F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. G.-B., *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. 9 (2009), no. 2, 267–291.

$$\boxed{D \text{ diagonal matrix, } \ A \text{ and } B \text{ constant matrices.}}$$

Open problem

Null controllability properties of

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

when $A(t)$ and $B(t)$ depend on t (for instance, $A \in C^\infty([0,T]; \mathcal{L}(\mathbb{R}^n))$ and $B \in C^\infty([0,T]; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$) and $D = \operatorname{diag}(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$.

7. The Kalman condition for a class of parabolic systems. Boundary controls

[AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA], J. Math. Pures Appl. (2011).

7. The Kalman condition for a class of parabolic systems. Boundary controls

Let us consider the **boundary controllability problem**:

$$(25) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ are two given matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ is the initial datum. In system (25), $v \in L^2(0,T;\mathbb{C}^m)$ is the control function (to be determined).

Simpler problem: One-dimensional case and $D = Id$.

This problem has been studied in the case $n = 2$:

- E. Fernández-Cara, M. G.-B., L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

7. The Kalman condition for a class of parabolic systems. Boundary controls

We consider again $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for $-\partial_{xx}$ in $(0,\pi)$ with homogenuous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions:

$$\boxed{\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}}\sin kx, \quad k \geq 1, \quad x \in (0,\pi).}$$

Theorem ($n=2, m=1$)

Let $A \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A^*. Then (25) is ***exactly controllable to the trajectories*** *at any time $T > 0$* ***if and only if*** $\boxed{\operatorname{rank}[A \mid B] = 2}$ *and*

$$\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j.}$$

■

Remark ($n = 2$, $m = 1$)

For the previous **boundary controllability problem**, one has

1. A complete characterization of the **exact controllability to trajectories** at time T: **Kalman condition**.
2. **Boundary controllability** and **distributed controllability** are not equivalent
3. **Approximate controllability** $\Longleftrightarrow$ **null controllability**.

What happens if $n > 2$??

As in the “simple example" seen in Subsection 2, we will work in the following finite-dimensional space:

$$X_k = \{\varphi_0 = \sum_{\ell=1}^{k} a_\ell \phi_\ell : a_\ell \in \mathbb{C}^n\} \subset H_0^1(0, \pi; \mathbb{C}^n).$$

7. The Kalman condition for a class of parabolic systems. Boundary controls

Adjoint Problem:

$$(26) \qquad \begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi), \end{cases}$$

with $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. Then, system (25) is exactly controllable to trajectories at time $T \iff$ for a constant $C > 0$ one has (**observability inequality**)

$$\|\varphi(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{C}^n)} \le C \int_0^T |B^*\varphi_x(0,t)|^2 \, dt.$$

Taking initial data in X_k, we deduce that an appropriate o.d. system in $\mathbb{C}^{nk}$ also satisfies an **observability inequality**. Let us analyze this finite-dimensional system.

7. The Kalman condition for a class of parabolic systems. Boundary controls

Notation

For $k \geq 1$, we introduce $L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n)$ and the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}),$$

and let us write the Kalman matrix associated with the pair $(\mathcal{L}_k, B_k)$:

$$\mathcal{K}_k = [\mathcal{L}_k \,|\, B_k] = [B_k \,,\, \mathcal{L}_k B_k \,,\, \mathcal{L}_k^2 B_k \,,\, \cdots \,,\, \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

With this notation, the o.d. system associated to the **adjoint system** (26) for $\varphi_0 \in X_k$ is $\boxed{-Z' = \mathcal{L}_k^* Z \text{ on } (0,T), Z(T) = Z_0 \in \mathbb{C}^{nk}}$, and the solutions must be B_k^*-observable, i.e., rank $\mathcal{K}_k = nk$: **necessary condition**. One has:

7. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ *and* $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$. *Then, system* (25) *is* ***exactly controllable to trajectories*** *at time* T *if and only if*

(27) $$\text{rank}\, \mathcal{K}_k = nk, \quad \forall k \geq 1.$$

Remark

1. This result gives a complete characterization of the **exact controllability to trajectories** at time T: **Kalman condition**.
2. If for $k \geq 1$ one has rank $\mathcal{K}_k = nk$, then rank $[A \mid B] = n$ and system

$$\begin{cases} \partial_t y - \Delta y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

is **exactly controllable to trajectories** at time T. But rank $[A \mid B] = n$ does not imply condition (27). So **boundary controllability** and **distributed controllability** are not equivalent.

7. The Kalman condition for a class of parabolic systems. Boundary controls

Remark

Condition (27) is also a **necessary** and **sufficient condition** for the boundary approximate controllability of system (25). Then

$$\textbf{Approximate controllability} \Longleftrightarrow \textbf{null controllability}.$$

Remark (n controls)

If $\boxed{\operatorname{rank} B = n}$ (and thus $m \geq n$), then the pair (A, B) fulfills condition (27) and the system is **exactly controllable to trajectories** at time T. ∎

7. The Kalman condition for a class of parabolic systems. Boundary controls

Remark (One control, $m = 1$)

When $m = 1$, the **Kalman condition** (27) is equivalent to $\boxed{\text{rank}\,[A \mid B] = n}$ and $\boxed{\lambda_k - \lambda_l \neq \mu_i - \mu_j}$ for any $k, l \in \mathbb{N}$ and $1 \leq i, j \leq p$ with $(k, i) \neq (l, j)$, where $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*. We generalize the results of [FERNÁNDEZ-CARA, G.-B., DE TERESA], J. Funct. Anal. (2010). ■

One control, $m = 1$

We have imposed two conditions:

1. $\text{rank}\,[A \mid B] = n$: System (25) is not decoupled.
2. $\lambda_k - \lambda_l \neq \mu_i - \mu_j$: The adjoint system can be written ($\mathcal{R}_0 = I_d \partial_{xx} + A^*$)

$$(26) \qquad \begin{cases} -\varphi_t = \mathcal{R}_0 \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases}$$

and the eigenvalues of $\mathcal{R}_0$ are simple. ■

7. The Kalman condition for a class of parabolic systems. Boundary controls

Necessary implication. We reason as before: if rank $\mathcal{K}_k < nk$, for some $k \geq 1$, then the o.d.s.

$$-Z' = \mathcal{L}_k^* Z \quad \text{on } (0,T), \quad Z(T) = Z_0 \in \mathbb{C}^{nk}$$

is not B_k^*-observable on $(0,T)$, i.e., there exists $Z_0 \neq 0$ s.t. $B_k^* Z(t) = 0$ for every $t \in (0,T)$. From Z_0 it is possible to construct $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ with $\varphi_0 \not\equiv 0$ such that the corresponding solution to the adjoint problem (27) satisfies

$$B^* \varphi_x(0,t) = 0 \quad \forall t \in (0,T).$$

As a consequence: The **unique continuation property** and the previous **observability inequality** for the adjoint problem fail:

Neither approximate nor null controllability at any T for system (25).

7. The Kalman condition for a class of parabolic systems. Boundary controls

Sufficient implication. For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two "big" steps:

(I) We reformulate the null controllability problem for system (25) as a **vector moment problem**.

(II) Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

7. The Kalman condition for a class of parabolic systems. Boundary controls

Before describing the first point, let us analyze the **Kalman condition** (27) rank $\mathcal{K}_k = nk$, $\forall k \geq 1$:

Proposition

Let us denote by $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ *the set of distinct eigenvalues of* A^*. *Then,*

1. *There exists an integer* $k_0 = k_0(A) \in \mathbb{N}$, *only depending on* A, *such that,*

$$\boxed{\lambda_k - \lambda_l \neq \mu_i - \mu_j}, \quad \forall k > k_0,\ l \geq 1,\ k \neq l,\ \text{and } 1 \leq i, j \leq p.$$

2. *The following conditions are equivalent:*
   (a) rank $\mathcal{K}_k = nk$ *for every* $k \geq 1$.
   (b) rank $\mathcal{K}_k = nk$ *for every* $k : 1 \leq k \leq k_0$.
   (c) rank $\mathcal{K}_{k_0} = nk_0$.

■

7. The Kalman condition for a class of parabolic systems. Boundary controls

(I) The **vector moment problem**: As in the scalar case, $v \in L^2(0,T;\mathbb{C}^m)$ is a **null control** for system

$$(25) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

(i.e., the solution y to (25) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$) $\iff$ v satisfies

$$\boxed{-\langle y_0, \varphi(\cdot,0)\rangle = \int_0^T (v(t)\,,\, B^*\varphi_x(0,t))_{\mathbb{C}^m}\,dt, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n),}$$

where φ is the solution to the adjoint problem

$$(26) \qquad \begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

7. The Kalman condition for a class of parabolic systems. Boundary controls

(I) The **vector moment problem**:

Thus, the idea is to take firstly $\varphi_0 \in X_{k_0}$,
($X_{k_0} = \{\varphi_0 : \varphi_0 = \sum_{i=1}^{k_0} a_i \phi_i$ with $a_i \in \mathbb{C}^n\}$) and then $\varphi_0 = a\phi_k$, with $k > k_0$ and $a \in \mathbb{C}^n$. Therefore, we want $v \in L^2(0,T;\mathbb{C}^m)$ s.t.

$$\begin{cases} \displaystyle\int_0^T \left(v(T-t)\,,\, B_{k_0}^* e^{\mathcal{L}_{k_0}^* t}\Phi_0\right)_{\mathbb{C}^m} dt = \boxed{F(Y_0,\Phi_0)}, & \forall \Phi_0 \in \mathbb{C}^{nk_0}, \\ \displaystyle\int_0^T \left(v(T-t)\,,\, B^* e^{(-\lambda_k I_d + A^*)t} a\right)_{\mathbb{C}^m} dt = \boxed{f_k(y_0,a)}, & \forall a \in \mathbb{C}^n,\ \forall k > k_0, \end{cases}$$

In some sense, v has to solve an infinite number of null controllability problems for appropriate o.d. systems:

$$\begin{cases} \boxed{Y' = \mathcal{L}_{k_0} Y + B_{k_0} v \text{ on } (0,T), \quad Y(0) = Y_0}; \\ \boxed{Z' = (-\lambda_k I_d + A)Z + Bv \text{ on } (0,T), \quad Z(0) = y_{0k} := (y_0,\phi_k)}, \quad \forall k > k_0. \end{cases}$$

7. The Kalman condition for a class of parabolic systems. Boundary controls

(I) The **vector moment problem**:

Remark

Using the assumptions rank $\mathcal{K}_{k_0} = [\mathcal{L}_{k_0} \mid B_{k_0}] = nk_0$ and rank $[-\lambda_k I_d + A \mid B] = \text{rank}\,[A \mid B] = n$, it is possible to reformulate the boundary null controllability problem as a **vector moment problem**. ■

Remark

Technically, this reformulation of the null controllability problem is complex, but the difficulties come from the fact of having ordinary differential systems. We would have the same difficulties if we wanted to solve the null controllability problem for the o.d. system:

$$Y' = \mathcal{A}Y + \mathcal{B}v \text{ on } (0,T); \quad Y(0) = Y_0 \in \mathbb{C}^N,$$

using the moment method. In the previous system, $\mathcal{A} \in \mathcal{L}(\mathbb{C}^N)$ and $\mathcal{B} \in \mathcal{L}(\mathbb{C}^M; \mathbb{C}^N)$ are given and $v \in L^2(0,T;\mathbb{C}^M)$ is the control. ■

7. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.
From the previous step, we have obtained the **complex matrix exponentials**

$$e^{\mathcal{L}_{k_0}^* t} \quad \text{and} \quad \{e^{(-\lambda_k I_d + A^*)t}\}_{k>k_0}.$$

Let us denote $\{\gamma_\ell\}_{1\le\ell\le\widetilde{p}} \subset \mathbb{C}$ the set of distinct eigenvalues of $\mathcal{L}_{k_0}^*$ and recall that $\{\mu_i\}_{1\le i\le p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*. Then, the set $\Lambda = \{\gamma_\ell\}_{1\le\ell\le\widetilde{p}} \cup \{-\lambda_k + \mu_i\}_{k>k_0, 1\le i\le p}$ is the set of eigenvalues of the operator $\partial_{xx} Id + A^*$. Thus, our next purpose is:

Objective

As in the scalar case, construction of a **biorthogonal family** in $L^2(0,T;\mathbb{C})$ to

$$\left\{ t^j e^{\gamma_\ell t}, t^j e^{(-\lambda_k+\mu_i)t} : 1\le\ell\le\widetilde{p},\ 1\le i\le p,\ 0\le j\le\eta-1,\ k>k_0 \right\},$$

which satisfies appropriate bounds (see 5). In the previous expression, η is the maximal dimension of the Jordan blocks associated to γ_ℓ and μ_i.

7. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.
Let us fix $\eta \geq 1$, an integer, $T \in (0, \infty]$ and $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ a sequence s.t.

$$\Lambda_k \neq \Lambda_j, \quad \forall k, j \geq \text{ with } k \neq j.$$

Let us recall that the family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ is **biorthogonal** to $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ if one has

$$\int_0^T t^j e^{-\Lambda_k t} \varphi^*_{l,i}(t)\, dt = \delta_{kl}\delta_{ij}, \quad \forall (k,j),(l,i) : k,l \geq 1,\ 0 \leq i,j \leq \eta - 1.$$

In addition, we want the family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ to satisfy the property:

For any $\varepsilon > 0$, there is $C(\varepsilon, T) > 0$ s.t. $\boxed{\|\varphi_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}}$,
$\forall k \geq 1$ and $0 \leq j \leq \eta - 1$.

7. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Theorem

Let us fix $\boxed{T \in (0, \infty]}$ *and assume that for two positive constants* δ *and* ρ *one has*

$$\begin{cases} \Re \Lambda_k \geq \delta |\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1, \\ \displaystyle\sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty. \end{cases}$$

Then, $\exists \{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta - 1}$ ***biorthogonal*** *to* $\left\{t^j e^{-\Lambda_k t}\right\}_{k \geq 1, 0 \leq j \leq \eta - 1}$ *such that, for every* $\varepsilon > 0$*, there exists* $C(\varepsilon, T) > 0$ *satisfying*

$$\|\varphi_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}, \quad \forall (k,j) : k \geq 1, \ 0 \leq j \leq \eta - 1.$$

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Proof:

The proof of this result is very technical. It can be found in [AMMAR-KHODJA,BENABDALLAH,G.-B.,DE TERESA], *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (2011).
In fact, the result is proved, first for $\boxed{T=\infty}$ and then the general case is deduced.

Let us analyze the key points of the proof when $T=\infty$:

1. **EXISTENCE**
2. **BOUNDS**

7. The Kalman condition for a class of parabolic systems. Boundary controls

(II.1) **Biorthogonal families**: **EXISTENCE**.

Lemma

Assume that $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$, *with* $\Lambda_k \neq \Lambda_j$ $\forall k, j \geq$ *with* $k \neq j$, *and*

$$\sum_{k\geq 1} \frac{\Re\Lambda_k}{(1+\Re\Lambda_k)^2 + (\Im\Lambda_k)^2} < \infty.$$

Then, there exists a biorthogonal family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,\infty;\mathbb{C})$ *to* $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ *such that*

$$\|\varphi_{k,j}\|_{L^2} \leq C\left[1+\left(\frac{1}{\Re\Lambda_k}\right)^{(2\eta-j)(\eta-j-1)+1}\right](\Re\Lambda_k)^{\eta(\eta-j)}|1+\Lambda_k|^{2\eta(\eta-j)}\mathcal{P}_k^{\eta(\eta-j)},$$

with $C = C(\eta) > 0$, *a constant, and* $\mathcal{P}_k := \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left|\frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell}\right|$.

(II.1) **Biorthogonal families**: **EXISTENCE**.

Remark

Observe that the assumptions

$$\Re\Lambda_k \geq \delta|\Lambda_k| \quad \text{and} \quad \sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

in Theorem 16 guarantee the hypothesis in the previous lemma. Therefore, these two assumptions imply the existence of the **biorthogonal family** $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ to $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ in $L^2(0,\infty;\mathbb{C})$. In addition, the norm $\|\varphi_{k,j}\|_{L^2}$ is bound with respect to the Blaschke product

$$\mathcal{P}_k = \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left| \frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell} \right|.$$

7. The Kalman condition for a class of parabolic systems. Boundary controls

(II.2) Biorthogonal families: BOUNDS.

Proposition

Let $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ *be a sequence satisfying*

$$\Re\Lambda_k \geq \delta|\Lambda_k|, \quad \boxed{|\Lambda_k - \Lambda_l| \geq \rho|k-l|}, \ \forall k,l \geq 1, \ \textit{and} \ \sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

for $\delta, \rho > 0$. *Then, for every* $\varepsilon > 0$ *there exists a constant* $C(\varepsilon) > 0$ *such that*

$$\mathcal{P}_k := \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left| \frac{1 + \Lambda_k/\Lambda_\ell^*}{1 - \Lambda_k/\Lambda_\ell} \right| \leq C(\varepsilon) e^{\varepsilon \Re\Lambda_k}, \quad \forall k \geq 1. \qquad \blacksquare$$

For a proof of this result: [Fattorini,Russell] Quart. Appl. Math. (1974/75) (real case) or [Fernández-Cara,G.-B.,de Teresa], J. Funct. Anal. (2010) (general case).

7. The Kalman condition for a class of parabolic systems. Boundary controls

Summarizing

For the problem

$$(25) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

($A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$) we know:

"System (25) is approximate controllable at time $T \iff$ System (25) is null controllable at time $T \iff$ the ***Kalman condition*** rank $\mathcal{K}_k = nk, \quad \forall k \geq 1$".

ESSENTIAL ASSUMPTION: Diffusion matrix $D = I_d$

What happens if $D \neq I_d$???

7. The Kalman condition for a class of parabolic systems. Boundary controls

Let us now revisit the boundary controllability problem:

$$(18) \qquad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=1} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $d_1, d_2 > 0$, $d_1 \neq d_2$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We know:

1. **Approximate controllability**: System (18) is approximately controllable at time $T \iff \boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}} \iff$ the eigenvalues of $\mathcal{R} = D\partial_{xx} + A^*$ are **simple**.
2. **Null controllability**: There are d_1, d_2 s.t. $\sqrt{d_1/d_2} \notin \mathbb{Q}$ and System (18) is not null controllable at any time $T > 0$.

7. The Kalman condition for a class of parabolic systems. Boundary controls

Let us analyze a little more the null controllability problem for System (18) when $\nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$. It is possible to apply the **moment method** to (18) (now, much simpler) and reduce the null controllability problem to the **existence** (with **bounds**) of a biorthogonal family to appropriate exponentials.

Spectrum of $-\mathcal{R}^*$

The operator $-\mathcal{R}^* = -D\partial_{xx} - A^*$ has a sequence of positive real eigenvalues

$$\left\{ \frac{\ell^2}{d_1} \right\}_{\ell \geq 1} \cup \left\{ \frac{i^2}{d_2} \right\}_{i \geq 1} = \{\Lambda_k\}_{k \geq 1},$$

and these eigenvalues are simple $\iff$ $\nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$.

Question: Is it possible to construct a biorthogonal family to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ (in $L^2(0,\infty)$) which satisfies appropriate bounds??

7. The Kalman condition for a class of parabolic systems. Boundary controls

1. We can apply Lemma 17 ($\eta \equiv 1$): there exists $\{\varphi_k\}_{k\geq 1} \subset L^2(0,\infty)$ biorthogonal to $\left\{e^{-\Lambda_k t}\right\}_{k\geq 1}$ such that, for $C > 0$, one has

$$\|\varphi_k\|_{L^2} \leq C|1+\Lambda_k|^3 \prod_{\ell\geq 1, \ell\neq k} \left|\frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell}\right| := C|1+\Lambda_k|^3 \mathcal{P}_k.$$

2. The separability condition $\boxed{|\Lambda_k - \Lambda_l| \geq \rho|k-l|}$ ($\rho > 0$ a constant) does not hold. We cannot apply Proposition 18 and, in general, the following property fails:

   "For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ s.t. $\mathcal{P}_k \leq C(\varepsilon)e^{\varepsilon\Lambda_k}$, $\forall k \geq 1$".

   This property is crucial for proving the null controllability result for System (18) with boundary controls in $L^2(0,T)$ for **arbitrary final times** $T > 0$ (see the scalar case in 5).

Conclusion: This approach does not work when T is small.

7. The Kalman condition for a class of parabolic systems. Boundary controls

In a forthcoming paper

- AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA, *Condensation index and necessary and sufficient conditions for the null controllability of abstract systems. Application to the boundary null controllability of coupled parabolic systems,* in preparation,

the authors prove for System (18) the following result:

Theorem

1. *Given* $\nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$*, there exists* $T_0 = T_0(\nu) \in [0, \infty]$ *such that System* (18) *is null controllable at time* T *with controls* $v \in L^2(0,T)$ $\iff T > T_0$.
2. *Given* $T_0 \in [0, \infty]$*, there exists a positive* $\nu \notin \mathbb{Q}$ *such that System* (18) *(for* $d_1 = \nu^2$ *and* $d_2 = 1$*) is null controllable at time* T *with controls* $v \in L^2(0,T) \iff T > T_0$. ■

7. The Kalman condition for a class of parabolic systems. Boundary controls

We had two important differences between the controllability problem for **scalar** and **non-scalar** parabolic problems:

1. First difference, see slide 7, Section 4.2.
2. Second difference, see slide 10, Section 4.2.

Third difference with scalar problems

In general, we can get a null controllability result at time $T > 0$ for a **non-scalar parabolic** problem if T is **large enough**. ■

8. Further results

8. Further results: I. First and second order coupling terms

All previous results concern non-scalar parabolic equations with zero order coupling terms (a matrix A). For null controllability results for some 2×2 parabolic systems with first and second order coupling terms see [GUERRERO] SIAM J. Control Optim (2007). For system

$$(28) \qquad \begin{cases} \partial_t y - \Delta y + cy + D \cdot \nabla y = \partial_{x_1}(w\theta_1) + v1_\omega & \text{in } Q_T, \\ \partial_t w - \Delta w + hw + K \cdot \nabla w = \Delta(y\theta_2) & \text{in } Q_T, \\ y = w = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0, \quad w(\cdot,0) = w_0 & \text{in } \Omega, \end{cases}$$

with $c, h \in \mathbb{R}$ and $D, K \in \mathbb{R}^N$ and $\theta_1, \theta_2 \in C^2(\overline{\Omega})$, one has:

Theorem

Assume that there exists a nonempty open subset $\omega_2 \subset \omega$ and $C > 0$ such that $\boxed{|\theta_2| \geq C > 0}$ *in ω_2. Then System (28) is null controllable at any time $T > 0$.*

Remark

Again the control open set ω have to meet the support of the function $|\theta_2|$.

In [Benabdallah, Cristofol, Gaitan, de Teresa] CRAS (2010), the following 3×3 control problem has been studied:

$$(29) \qquad \begin{cases} \partial_t y = (\mathcal{L} + A)y + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{L} = \text{diag}\,(L_1, L_2, L_2)$ with $(L_i)_{i=1,2}$ operators as in (3) satisfying (4) and (5), $A = (a_{ij})_{1\le i,j\le 3} \in C^4(\overline{Q}_T; \mathcal{L}(\mathbb{R}^3))$, $B = (1, 0, 0)^* \in \mathbb{R}^3$, $v \in L^2(Q_T)$ is the control, and $y_0 \in L^2(\Omega; \mathbb{R}^3)$ is the initial condition.

Theorem

Suppose that a_{21} *and* a_{31} *are time independent and* $j \in \{2,3\}$ *such that* $|a_{j1}(\cdot)| \geq C > 0$ *in* ω, $C > 0$. *For* $j \in \{2,3\}$, *we set* $k_j = \frac{6}{j}$ *and*

$$B_{k_j} \in C^3(\overline{Q}_T; \mathbb{R}^N); \quad B^i_{k_j} := \sum_{\ell=1}^{N} \alpha^{(2)}_{i\ell} \left(\partial_l a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \partial_l a_{j1} \right), \quad 1 \leq i \leq N.$$

Assume that $\partial\omega \cap \partial\Omega = \gamma$, *with* $|\gamma| \neq 0$, *and* $B_{k_j} \cdot \nu \neq 0$ *on* γ, *where* ν *is the outward unit normal vector. Then, System* (29) *is null controllable at time* T.

Remark

A different **sufficient** condition has been obtained by K. Mauffrey. ■

Consider the following system

$$(30)\qquad \begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 + v 1_\omega & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot,0) = y_{0,1}, \quad y_2(\cdot,0) = y_{0,2} & \text{in } \Omega, \end{cases}$$

with p a smooth function and $\delta > 0$. In [ALABAU-BOUSSOUIRA, LÉAUTAUD], CRAS (2011). One has:

Theorem

Let $p \geq 0$ on Ω. Assume that $\exists p_0 > 0$ and $\omega_p \subset \Omega$ satisfying the Geometric Control Condition (GCC) with $\boxed{p \geq p_0 \text{ in } \omega_p}$. Assume that ω also satisfies GCC. Then there exists $\delta_0 > 0$ such that for all $0 < \sqrt{\delta}\|p\|_{L^\infty(\Omega)} \leq \delta_0$ System (30) is null controllable at any positive time T.

With the same kind of arguments, in the same paper, a new boundary control result is proved. For simplicity, consider

$$(31) \qquad \begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = b v, \quad y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot,0) = y_{0,1}, \quad y_2(\cdot,0) = y^{0,2} & \text{in } \Omega, \end{cases}$$

where b is a function on $\partial\Omega$, $p \in L^\infty(\Omega)$ and $\delta > 0$. One has

Theorem

Let p satisfy assumptions of Theorem 8.1. Suppose that there $\exists \Gamma_b \subset \partial\Omega$ *satisfying GCC and* $b \geq b_0 > 0$ *on* Γ_b*. Then there exists* $\delta_0 > 0$ *such that for all* $0 < \sqrt{\delta}\|p\|_{L^\infty(\Omega)} \leq \delta_0$ *System* (31) *is null controllable at any time* $T > 0$.

Even if the geometrical assumptions are too strong, these two theorems give the first examples on controllability of cascade system with coupling terms vanishing on the control domain. Moreover it also gives the first result on boundary control of two coupled parabolic equations for $N > 1$.

9. Comments and open problems

Most of the controllability results for parabolic systems are open.

A.- Let us consider the distributed controllability problem

$$(22) \qquad \begin{cases} \partial_t y - D\Delta y = Ay + I_d v 1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega. \end{cases}$$

with $A \in \mathcal{L}(\mathbb{R}^n)$ (as before), $\boxed{B = I_d}$ and with $D \in \mathcal{L}(\mathbb{R}^n)$ a non-symmetric matrix such that the Jordan canonical form J is real and positive definite, i.e., $J \in \mathcal{L}(\mathbb{R}^n)$ and

$$\xi J \xi^* > 0, \quad \forall \xi \in \mathbb{R}^n, \xi \neq 0.$$

Some partial results by
E. Fernández-Cara, M. G.-B., L. de Teresa, in preparation.

9. Comments and open problems

B.- The null controllability property of non-scalar parabolic problems with coupling matrices depending on x is open. A first problem:

Consider the distributed null controllability problem

$$\begin{cases} y_t - L_0 y = q(x) A_0 y + B v 1_\omega & \text{in } Q_T = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ y(\cdot, 0) = y_0, & \text{in } \Omega, \end{cases}$$

with $A_0 \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, $y_0 \in L^2(\Omega; \mathbb{R}^n)$, $v \in L^2(Q_T; \mathbb{R}^m)$ and q is a given scalar function.

Simple case:

A_0 is a "cascade matrix" and $q(x) = 1_{\mathcal{O}}$ with $\mathcal{O} \subset \Omega$ a new open set s.t.

$$\boxed{\mathcal{O} \cap \omega = \emptyset.}$$

9. Comments and open problems

C.- Kalman condition: Only in the cases presented here.

Other situations ?

D.- Boundary controllability for $N > 1$.

Reference:

[AMMAR-KHODJA,BENABDALLAH,G.-B.,DE TERESA], *Recent results on the controllability of linear coupled parabolic problems: a survey*, Mathematical Control and Related Fields **1** (2011), no. 3, 267–306.

Very important: See the references therein.

Thanks for your attention !

¡ Gracias por vuestra atención !