GENERALIZATION OF NONDIFFERENCIABLE CONVEX FUNCTIONS AND SOME CHARACTERIZATIONS.

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Abstract

In this paper we generalize the convex functions, defining the concept of preconvex function and we study some characterizations by intervals, some characterizations by polytopes, some characterizations by level sets, some properties of the extreme points and some relations whith the convex functions.

Also, we define the R-quasiconvex functions as a generalization of the quasiconvex functions, and we study some characterizations by level sets and by separation sets, and some relations with the quasiconvex functions.

1   INTRODUCTION.

The convex functions are continuous on the interior of the convex sets where they are defined, ROCKAFELLAR (1969). When we desire to generalize this concept, we can search functions whose local optimum is global , but then we lose the property of continuity. Therefore, we are going to generalize the convex functions in the class of continuous functions, getting some properties of these. This result follows the same line as that of FRENCHEL (1953) and ROCKAFELLAR (1969).

Following the MARTOS (1975), we define and we study the preconvex functions and we get some properties and some characterizations of the same, and furthermore, their behavior in the problems of optimization.

Also, we define the R-quasiconvex functions and the F-quasiconvex functions, which generalize the quasiconvex functions, and, moreover they are interesting in the study of some problems of mathemetical programming.

We finish this paper giving some interesting characterizations and some properties of these classes of functions.

Let $X \subseteq R^n$ be a convex set with at least two different points. We denote $C(X)$ the set of the functions

$$f : X \longmapsto R$$

continuous.

2 PRECONVEX FUNCTIONS.

DEFINITION 1.- A function $f \in C(X)$ is *preconvex* in the convex set X if for any $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$ and $x \in [x^1, x^2]$, implies $f(x) \leq f(x^2)$.

As immediate consequence from then definition we have the followin results:

THEOREM 1.- A function $f \in C(X)$ is preconvex if and only if any segment $[x^1, x^2]$ can be split into two subsegments so that $f(x)$ is decreasing in the first and increasing in the second, while one of these subsegments may be empty.

THEOREM 2.- A function $f \in C(X)$ is preconvex if and only if $\forall [x^1, x^2] \subset X$, the global maximum of $f(x)$ in $[x^1, x^2]$ is the point x^1 or x^2.

PROOF.- (By contradiction) Suppose that f is not preconvex .

Then there exist $x^1, x^2 \in X, x^0 \in [x^1, x^2]$ such that $f(x^0) > f(x^1) = f(x^2)$. This contradicts that the maximun is the point x^1 or x^2 .

Conversely , if there exist $x^1, x^2 \in X, x^0 \in [x^1, x^2]$ such that $f(x^0) > f(x^1)$ and $f(x^0) > f(x^2)$ then since f is continuous , there exist $x^3 \in [x^1, x^0), x^4 \in (x^0, x^2]$ such that $f(x^3) = f(x^4) < f(x^0)$. This contradicts that f is preconvex .

$\square$

THEOREM 3.- A function $f \in C(X)$ is preconvex if and only if for each nonempty polyhedron $X^\triangle$ the maximum is attained at a vertex of $X^\triangle$.

PROOF.- If the polyhedron , $X^\triangle$, has an unique point or it is a segment , the result can be inferred of THEOREM 2 .

If $X^\triangle = <x^1, x^2, x^3>$, we suppose that $f(x^1) \geq f(x^i), i = 1, 2, 3$. As f is preconvex, $f(x) \leq f(x^1)$ for any $x \in [x^1, x^2] \cup [x^1, x^3] \cup [x^2, x^3]$. If there exist $x^0 \in X$ such that $f(x^0) > f(x^1)$ let $z \in [x^2, x^3]$ such that $x^0 \in [x^1, z]$. Then f is not preconvex en $[x^1, z]$.

If $X^\triangle = <x^1, x^2, ..., x^n>$ the proof go similarly .

$\square$

THEOREM 4.-   A function $f \in C(X)$ is preconvex if and only if for any $[x^1, x^2] \subset X$, f can be defined for any $x \in [x^1, x^2]$ as

$$f(x) = max\{h_1(x), h_2(x)\}$$

with h_1, h_2 continuous monotonic functions in the convex set X.

PROOF.- Let f be defined by the maximum of two monotonic functions , but f is not preconvex . Therefore , there exist $x^1, x^2 \in X$ with $f(x^1) = f(x^2)$ and $x^0 \in [x^1, x^2]$ such that

$$f(x^0) > f(x^1) = f(x^2)$$

or equivalently ,

$$max\{h_1(x^0), h_2(x^0)\} > max\{h_1(x^1), h_2(x^1)\} = max\{h_1(x^2), h_2(x^2)\}$$

Can be occur two cases :

   i) $f(x^1) = h_1(x^1)$ and $f(x^2) = h_1(x^2)$

   ii) $f(x^1) = h_1(x^1)$ and $f(x^2) = h_2(x^2)$

   In the case $i)$, since h is monotonic , should be $f(x^0) = h_2(x^0)$, that implies :

$$h_2(x^0) > h_1(x^1) \geq h_2(x^1)$$

$$h_2(x^0) > h_1(x^2) \geq h_2(x^2)$$

which contradicts that h_2 is a monotonic function . In the case $ii)$, since $f(x^1) = f(x^2)$, should be either :

$$f(x^0) = h_1(x^0) > h_1(x^1) \geq h_1(x^2)$$

or

$$f(x^0) = h_2(x^0) > h_2(x^2) \geq h_2(x^1)$$

In contradiction with the function h_1 are monotonic .

The converse is follows of THEOREM 1.

$\square$

THEOREM 5.-   A function $f \in C(X)$ is preconvex if and only if $\forall x^1, x^2 \in X$ the function $\varphi(\lambda) = f[(1 - \lambda)x^1 + \lambda x^2]$ is preconvex in $[0, 1]$.

The proof go similary of THEOREM 3.2.14 of MARTOS (1975).

Next we study the behavior of this functions in relation to their properties in the extreme points .

Let $M = \{x \in X \mid f(x) \leq f(y) \quad \forall y \in X\}$

THEOREM 6.-   If the function $f \in C(X)$ is preconvex, then M is convex.

PROOF.- Let $x^1, x^2 \in M$ be two points . Since f is preconvex , implies that for any $x \in [x^1, x^2]$ is $f(x) \leq f(x^1) = f(x^2)$. Thus $f(x) = f(x^1) = f(x^2)$ and $[x^1, x^2] \subset M$. Therefore M is convex .

$\square$

THEOREM 7.-   If $f \in C(X)$ is preconvex, then any strict local minimum point is a global minimum point.

PROOF.- Suppose that $x^0 \in X$ is a strict local minimum point of f and it is not a global minimum point.

Since x is a strict local minimum point there exist $\delta \in R, \delta > 0$ such that for any $x \in B(x^0, \delta) \cap X$ is :

$$f(x) > f(x^0)$$

On the other hand , since x^0 is not a global minimum point there exist a global minimum point , let x^1.

For any $x \in [x^1, x^0]$ we have that :

$$f(x) \geq f(x^1) \ and \ f(x^0) > f(x^1)$$

Thus there exist $x^2 \in [x^1, x^0] \backslash \{[x^1, x^0] \cap B(x^0, \delta)\}$ such that $f(x^2) = f(x^0)$ and therefore $f(x) \leq f(x^2) = f(x^0)$ for any $x \in [x^2, x^0]$, because of the function f is preconvex . This contradicts the hypothesis of x^0 is a strict minimum local point.

Therefore any strict local minimum point is a global minimum point for this class of functions.

$\square$

COROLLARY 1.-   Let $f \in C(X)$ be a function such that for any close segment any local minimum point is a global minimum point, then the function f is preconvex.

Next, we give some characterizations of this functions by means of the following sets .

Lower Set	$D(x^0)$	$=$	$\{x \in X \mid f(x) \leq f(x^0)\}$
Strict Lower Set	$D^<(x^0)$	$=$	$\{x \in X \mid f(x) < f(x^0)\}$
Level Set	$D(x^0)$	$=$	$\{x \in X \mid f(x) = f(x^0)\}$

THEOREM 8.-   A function $f \in C(X)$ is preconvex if and only if for any $x^0 \in X$ the set $D(x^0)$ is convex.

PROOF.- Suppose that $D(x)$ is convex $\forall x \in X$, and let $x^1, x^2 \in X$, such that $f(x^1) = f(x^2)$. Then , since $D(x^1)$ is convex , it is $f(x^0) \leq f(x^1)$ for any $x^0 \in [x^1, x^2]$. Thus if $D(x)$ is convex for any $x \in X$, the function is preconvex .

Now , we suppose that the function f is preconvex . Let $x \in X$ and $x^1, x^2 \in D(x)$.

i) If $f(x^1) = f(x^2) < f(x)$, then $[x^1, x^2] \subset D(x)$ because of f is preconvex .

*ii)*If $f(x^1) < f(x^2) \leq f(x)$, then $f(x^0) \leq f(x^2) \leq f(x)$ for any $x^0 \in [x^1, x^2]$ as consequence of THEOREM 1 .

Therefore in each case is $[x^1, x^2] \subset D(x)$. Thus $D(x)$ is convex .

<div align="right">□</div>

COROLLARY 2.-   The function $f \in C(X)$ is preconconvex if and only if the set $D^<(x)$ is convex for any $x \in X$.

STRICTLY PRECONVEX AND PREMONOTONIC FUNCTIONS

DEFINITION 2.-   A function $f \in C(X)$ is *strictly preconvex* in the convex set X if for any $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$, is $f(x) < f(x^1)$ for any $x \in (x^1, x^2)$.

As in the anterior case , we have the following properties :

THEOREM 9.-   A function $f \in C(X)$ is strictly preconvex if and only if any $[x^1, x^2] \subset X$ can be split into two subsegment so that f is strictly decreasing in the first and strictly increasing in the second, while one of them may be empty.

THEOREM 10.- A function $f \in C(X)$ is strictly preconvex if and only if for any $[x^1, x^2] \subset X$, f can be defined for any $x \in [x^1, x^2]$ as

$$f(x) = max\{h_1(x), h_2(x)\}$$

with h_1 and h_2 continuous strictly monotonic functions in the set X.

THEOREM 11.- If $f \in C(X)$ is strictly preconvex then there exist an unique local minimum point of f in X.

THEOREM 12.- If $f \in C(X)$ is strictly preconvex, then the sets $D(x)$ and $D^<(x)$ are convex for any $x \in X$.

DEFINITION 3.- A function $f \in C(X)$ is *preconcave* in X if $[-f]$ is preconvex in X.

DEFINITION 4.- A function $f \in C(X)$ is *premonotonic* if it is preconvex and preconcave.

Of this definition may be deduce that the class of premonotonic functions is the same as the class of continuous monotonic functions and, therefore, they will have the same properties (the maximum and the minimum are in the extremes of the intervals, as that the lower and upper sets are convex), although in relation to these sets, we have some results more powerfull as the following.

THEOREM 13.- A function $f \in C(X)$ is premonotonic in the set X if and only if each level set $N(X)$ is convex.

Others results that can be obtain are :

THEOREM 14.- A function $f \in C(X)$ is premonotonic in the set X if and only if each level set $N(x)$ is the intersection of X with two closed halfspaces whose union contains X.

PROOF.- If f is premonotonic , by THEOREM 4.3.16 of MARTOS (1975) , we have that for any $x \in X$ is :

$$D(x) = X \cap S_1 \ and \ (D^<(x))^c = X \cap S_2$$

therefore $N(x) = D(x) \cap (D^<(x))^c = X \cap S_1 \cap S_2$.

Let $z \in X$, then we have that $f(z) \geq f(x)$ *or* $f(z) \leq f(x)$, which implies that $X \subset D(x) \cup (D^<(x))^c \subset S_1 \cup S_2$.

Conversely , if $N(x) = X \cap S_1 \cap S_2$ then $N(x)$ is convex . Therefore , by THEOREM 13 the function f is premonotonic .

$\square$

Similarly , we have :

THEOREM 15.-   A function $f \in C(X)$ is premonotonic if and only if any lower (upper) set is the intersection of X and a closed halfspace.

3   F-PRECONVEX FUNCTIONS.

DEFINITION 5.-   Let $X \subset R^n$ and $x, y \in X$, we define the set $\Re(x, y - x)$ as the intersection of the set X with the halfstraight line wich start to x and it cross y.

We define this set as *ray which start to x and it cross y.* The intersection of the other halfstraight line with X is $\Re(x, x - y)$.

DEFINITION 6.-   A function $f \in C(X)$ is *F-preconvex* in X if for any $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$, is

$$f(x) \leq f(x^2) \qquad \forall x \in \Re(x^2, x^1 - x^2)$$

The following results relate this type of functions with the premonotonic functions, wich allow the use of the results of this paper .

THEOREM 16.-   All function F-preconvex is premonotonic.

PROOF.- Let $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$ and we suppose that $\exists x^0 \in [x^1, x^2]$ such that $f(x^0) < f(x^1)$. Since f is continuous in the set X there exist $x^3 \in (x^1, x^0)$,$x^4 \in (x^0, x^2)$ with $f(x^0) \leq f(x^3) = f(x^4) < f(x^1)$, and since f is F-preconvex , $f(x^1) \leq f(x^3) = f(x^4)$, since $x^1 \in \Re(x^4, x^3 - x^4)$, which is absurd .

$\square$

The converse is not true, for instance the function $f : R^+ \longrightarrow R$:

$$
\begin{array}{llll}
f(x) & = & x^2 & 0 \le x \le 1 \\
f(x) & = & x1 & 1 \le x \le 2 \\
f(x) & = & x^3 - 7 & x \ge 2
\end{array}
$$

it is premonotonic in R and it is not F-preconvex .

In relation to the sets that this function define in each point , since it is premonotonic , its level sets, its lower (upper) sets and its strictly lower (strictly upper) sets are convex .

For this function we obtain the following characterization.

THEOREM 17.-   A function $f \in C(X)$ is F-preconvex in X if and only if for any $x^1, x^2 \in X$, the interval $[x^1, x^2]$ can be split into two subintervals, so that f is constant in the first and strictly decreasing in the second or strictly increasing in the first and constant in the second, while one of them may be empty.

PROOF.- Let f F-preconvex and $[x^1, x^2] \subset X$. Let $[z^1, z^2] \subset [x^1, x^2]$ the interval where f is constant . Then it is necessary that $f(x^1) \le f(z^1)$ and $f(x^2) \le f(z^2)$, but at least one inequality should be equality .

Therefore we have one of the following three cases :

$$
\begin{array}{lllllll}
i) & f(x^1) & = & f(z^1) & \text{and} & f(x^2) & = & f(z^2) \\
ii) & f(x^1) & = & f(z^1) & \text{and} & f(x^2) & < & f(z^2) \\
iii) & f(x^1) & < & f(z^1) & \text{and} & f(x^2) & = & f(z^2)
\end{array}
$$

In the first case the function is constant in all the interval $[x^1, x^2]$.

In the second case the function is constant in the subinterval $[x^1, z^2]$ and strictly decreasing in the subinterval $(z^2, x^2]$.

In the third case the function is strictly increasing in $[x^1, z^1]$ and constant in $(z^1, x^2]$.

Suppose now that there not exist $z^1, z^2 \in [x^1, x^2]$ such that $f(z^1) = f(z^2)$. Then f should be strictly increasing and strictly decreasing in all the segment . Conversely , if f is as the statement of theorem in each segment $[x^1, x^2]$ then f is F-preconvex .

$\square$

As consequence, we obtain the following result :

COROLLARY 3.-   If the function $f \in C(X)$ is F-preconvex, then for any segment $[x^1, x^2] \subset X$ the maximum and the minimum of f in such segment occur at x^1 or x^2.

For the optimum points of this class of functions we obtain the result:

DEFINITION 7.-   Let $Y \subseteq X \subseteq R^n$ with X a convex set. Y is a *extreme set of X* if for any $x^1, x^2 \in X$ such that

$$(x^1, x^2) \cap Y \neq \emptyset \Rightarrow [x^1, x^2] \subset Y$$

THEOREM 18.-   If $f \in C(X)$ is a F-preconvex function, the minimum set is a extreme subset of X.

PROOF.- Let M be the minimum set of the function f in X. Let $x^1, x^2 \in X$ such that $(x^1, x^2) \cap M \neq \emptyset \Rightarrow \exists x^0 \in (x^1, x^2) \cap M$ (1)

One of the following three cases hold :

i) f is constant in $[x^1, x^2] \Rightarrow [x^1, x^2] \subset M$ and M is a extreme set.

ii) $\exists y \in [x^1, x^2]$ such that f is strictly increasing in $[x^1, y]$ and it is constant in $[y, x^2]$ but, since f is strictly increasing in $[x^1, y]$ we have $f(x^1) < f(x)$ $, \forall x \in [x^1, x^2]$. This contradict (1).

iii) $\exists y \in [x^1, x^2]$ such that f is constant in $[x^1, y]$ and strictly decreasing in $[y, x^2] \Rightarrow$ $f(x^2) < f(x)$ $\forall x \in [x^1, x^2]$. This contradics (1).

$$\square$$

Of this result we deduce that if a F-preconvex function which has its minimum in the interior of the set X then it is constant.

Next, we study in which conditions a local vertex-optimum point is a global optimum point, if the function is defined in a polyhedron.

THEOREM 19.-   Let $f \in C(X^{\triangle})$ be a F-preconvex function in the polyhedron $X^{\triangle}$, then any local vertex-maximum point in $X^{\triangle}$ is a global maximum point of the function f in $X^{\triangle}$.

PROOF.- From THEOREMS 3 and 16, only we have to show that any local vertex-maximum is global vertex-maximum in the polyhedron $X^{\triangle}$.

If $X^{\triangle}$ has an unique point, it is a segment or it is engendered with three vertices, the result is immediate.

So that let $X^{\triangle} = < x^1, x^2, \ldots, x^n >$ be with $n \geq 4$ and let x^i be a local vertex-maximum and x^j global vertex-maximum, being $x^i \neq x^j$. Let us consider the convex

hull of the contiguous vertices to x^i, this has at least a common point with the segment $[x^i, x^j]$, let x^0 be. Therefore, we can split the segment $[x^i, x^j]$ in two subsegments such that the function is decreasing in $[x^i, x^0]$ and it is strictly increasing in $[x^0, x^j]$. This contradicts the THEOREM 16.

$\square$

When we restrict to segments, we have the following theorem.

THEOREM 20.-   A function $f \in C(X)$ is F-preconvex if and only if for any closed segment in X either the function is constant in the segment or only a vertex is minimum of the function f in the segment.

PROOF.- The necessary condition is a consequence of THEOREM 16.

For to prove the sufficiency, we have two points $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$.

By the hypothesis, $\forall x \in [x^1, x^2]$ should be :

$$f(x) = f(x^1) = f(x^2)$$

since any point in the segment is minimum. Otherwise, $\forall x^0 \in \Re(x^2, x^1 - x^2)$ we have :

$$f(x^0) \leq f(x^1) = f(x^2)$$

since, else we should have a segment $[x^0, x^2]$ with $x^1 \in [x^0, x^2]$ minimum, but not the entire segment. Hence, a function satisfying this condition is F-preconvex.

$\square$

THEOREM 21.-   If $f \in C(X)$ is a F-preconvex function on X, then every level set is the intersection of X with two closed subspaces whose union contains the set X.

The proof is a consequence of THEOREM 16.

THEOREM 22.-   If $f \in C(X)$ is a F-preconvex function in X, and $\exists x \in X$ such that its lower set and strict upper sets or its upper sets and strict lower sets are nonempty, then they are separable.

The theorem holds from the convexity if both sets and from the separation theorem.

Next, we study a property for this class of functions, giving a characterization by mean of their level sets ans their maximum sets.

THEOREM 23.- A function $f \in C(X)$ is F-preconvex on X if and only if one of the following two alternatives hold :

  i) f is constant on X.

  ii) Every level set nonempty is the intersection of X with a plane or with a closed halfspace if the above level set is the set of global maximum.

PROOF.- Let f be F-preconvex. If f is constant in X, the theorem hold since any points is maximum and their level sets are the intersection of X with the entire space. Else, let $x^* \in X$ be a maximum of the function in X. By THEOREM 16, f is premonotonic, and by THEOREM 15, the upper set :

$$(D^<(x^*))^c = \{x \in X \mid f(x) \geq f(x^*)\} = \{x \in X \mid f(x) = f(x^*)\}$$

is the intersection of X with a closed halfspace and they coincide with the level sets.

Now, let $x^* \in X$ a nonmaximum point of the function in X. By THEOREM 14, the level set $N(x^*)$ is the intersection of X with two closed halfspaces. If $N(x^*)$ is not reduced to an hyperplane, there is a neighborhood of x^*, contained in $N(x^*)$ and containing some points with superior values and some points with inferior values, contradicting the definition of $N(x^*)$.

In order to obtain the reciprocal, it is obvious that if f is constant then it is F-preconvex. Therefore, let us suppose *ii*) hold. Let $x^1, x^2 \in X$ such that $f(x^1) = f(x^2)$. If $\exists x \in \Re(x^2, x^1 - x^2)$ with $f(x) > f(x^2)$ then x^2 is not global maximum of f. But X is convex $\Rightarrow N(x^1) = N(x^2)$ is not the intersection of X with a plane, contradicting the assumption.

<div align="right">□</div>

4   R-QUASICONVEX FUNCTIONS.

DEFINITION 8.- A function f is *R-quasiconvex* in X if $\forall x^1, x^2 \in X$, such that $f(x^1) \leq f(x^2) \Rightarrow f(x) \leq f(x^2)$, $\forall x \in \Re(x^2, x^1 - x^2)$.

This definition is a generalization of the quasiconvexity because

$$[x^1, x^2] \subseteq \Re(x^2, x^1 - x^2).$$

Therefore, these functions are quasiconvex in the normal sense. Moreover, we will show that they are strictly quasiconcave.

The R-quasiconvex functions verifies the KARAMARDIAN's anomaly. Thus, we give the following definition :

DEFINITION 9.-   A function f is *F-quasiconvex* in X if $\forall x^1, x^2 \in X$ such that $f(x^1) \leq f(x^2) \Rightarrow f$ is R-quasiconvex in $\Re(x^2, x^1 - x^2)$ and $f(x) \leq f(x^1)$, $\forall x \in \Re(x^1, x^1 - x^2)$.

We note that if a F-quasiconvex function decrease then cannot increase. Thus, we elude the KARAMARDIAN's anomaly.

THEOREM 24.-   The function f is R-quasiconvex if and only if $\forall x^1, x^2 \in X$, f has one of the following forms in the segment $[x^1, x^2]$:

i) f is constant except an unique point where its value is lower.

ii) The segment $[x^1, x^2]$ can be split into two segments, at least one closed, such that in the first is constant and strictly decreasing in the second, or strictly increasing in the first and constant in the second, while one of them may be empty.

PROOF.- Let f be R-quasiconvex and we suppose $\exists y^1, y^2 \in [x^1, x^2]$ such that $f(y^1) = f(y^2)$. Thus, $f(y) \leq f(y^1) = f(y^2), \forall y \in [x^1, x^2]$. Let $y^1 \in [x^1, y^2)$. If $\exists y^0 \in [x^1, y^1)$ with $f(y^0) < f(y^1)$ then $\forall y \in (y^1, x^2] \Rightarrow f(y^0) < f(y)$, therefore, f is strictly decreasing in any subsegment and constant in the rest (ii) or the point y is unique (i).

If $\exists y^0 \in (y^2, x^2]$ such that $f(y^0) < f(y^2)$, we obtain a similar result.

If $\exists y^0 \in [y^1, y^2]$ such that $f(y^0) < f(y^1) = f(y^2)$, then is unique (i).

If not one of the cases above-mentioned hold, then f is constant in the entire segment (ii).

Now, we suppose that there are not $y^1, y^2 \in [x^1, x^2]$ with $f(y^1) = f(y^2)$, then f should be strictly increasing or strictly decreasing along the segment (ii).

Reciprocally, if f satisfies (i) or (ii) in any segment $[x^1, x^2]$, then f is R-quasiconvex.

$\square$

THEOREM 25.-   The function f is F-quasiconvex in X if and only if any segment $[x^1, x^2] \subset X$ can be split into two segments, at least one closed, such that in the first is constant and strictly decresing in the second, or strictly increasing in the first and constant in the second, while one of them may be empty.

THEOREM 26.-   The function f is R-quasiconvex in X if and only if it is quasiconvex and strictly quasiconcave in X.

PROOF.- Let f be R-quasiconvex in X, then $[x^1, x^2] \subset \Re(x^2, x^1 - x^2) \Rightarrow f(x) \leq f(x^2)$, $\forall x \in [x^1, x^2] \Rightarrow f$ is quasiconvex.

Now , let the segment $[x^1, x^2]$ such that $f(x^1) < f(x^2)$ and we suppose that $\exists x^0 \in [x^1, x^2]$ with $f(x^0) \leq f(x^1)$. Since f is R-quasiconvex , we have

$f(x) \leq f(x^1), \forall x \in \Re(x^1, x^0 - x^1)$. But $x^2 \in \Re(x^1, x^0 - x^1) \Rightarrow f(x^2) \leq f(x^1)$. This is a contradiction.

The proof of the converse implication has two part :

i) Let $x^1, x^2 \in X$ and we suppose $f(x^1) < f(x^2)$ (*)

Let $x^0 \in \Re(x^2, x^1 - x^2)$ such that $f(x^0) > f(x^2)$.

i1) If $x^0 \in [x^1, x^2]$ then the function is not quasiconvex. This contradicts our assumption.

i2) Let x^0 such that $x^1 \in [x^0, x^2]$. Since f is strictly quasiconvex, we have :

$$f(x) > min\{f(x^0), f(x^2)\} = f(x^2), \forall x \in [x^0, x^2]$$

$$\Rightarrow f(x^1) > f(x^2)$$

in contradiction with (*).

ii) Now, we suppose $f(x^1) = f(x^2)$. We have two cases :

ii1) f is constant in the segment $\Rightarrow f$ is R-quasiconvex.

ii2) $\exists x^0 \in [x^1, x^2]$ with $f(x^0) < f(x^2)$. If this point is unique, we have the KARA-MARDIAN's anomaly and the function is R-quasiconvex. If the point is not unique, we change x^1 by x^0 and we follow a similar reasoning to i1).

ii3) $\exists x^0 \in [x^1, x^2]$ with $f(x^0) > f(x^2)$. Then, the function is not quasiconvex, in contradiction with the hypothesis.

$\square$

THEOREM 27.-   The function f is F-quasiconvex in X if and only if it is quasi-monotonic and strictly quasiconcave.

PROOF.- By THEOREM 16 and the definition of F-quasiconvexity, we have to prove only :

a) If f is F-quasiconvex $\Rightarrow$ is quasiconvex.

b) If f is quasimonotonic and strictly quasiconcave $\Rightarrow f$ is F-quasiconvex.

In order to prove **a)** we have to show

$$\forall x^1, x^2 \in X \ and \ \forall x \in (x^1, x^2) \Rightarrow f(x) \geq min\{f(x^1), f(x^2)\}$$

Let $x^0 \in (x^1, x^2)$ with

$$f(x^0) < min\{f(x^1), f(x^2)\}$$

and we consider the ray $\Re(x^2, x^0 - x^2)$ then

$$f(x^0) < f(x^2) \Rightarrow f(x) \leq f(x^0) \Rightarrow f(x^1) \leq f(x^0)$$

That is a contradiction, Thus f is quasiconvex.

Now, we prove **b)** . Since f is quasimonotonic and strictly quasiconcave, f is R-quasiconvex. Therefore, we have to prove only the last part in the definition of F-quasiconvexity.

Let $x^0 \in \Re(x^1, x^1 - x^2)$ such that $f(x^0) > f(x^1)$ with $f(x^1) \leq f(x^2)$ and let us consider the segment $[x^0, x^2]$. We have :

i) f is quasimonotonic, then, if $f(x^0) = f(x^2) \Rightarrow$

$$min\{f(x^0), f(x^2)\} \leq f(x) \leq max\{f(x^0), f(x^2)\}   \forall x \in [x^0, x^2]$$

If $x = x^1$ we obtain a contradiction with $f(x^0) > f(x^1)$.

ii) If $f(x^0) < f(x^2)$, as f is strictly quasiconcave, we have :

$$\forall x \in [x^0, x^2], f(x) > min\{f(x^0), f(x^2)\} = f(x^0)$$

in particular, $f(x^1) > f(x^0)$. This is a contradiction.

<div align="right">□</div>

From THEOREM 26, we have that if f is a function R-quasiconvex in X, its lower and strictly lower sets are convex. And, by THEOREM 27, if f is a function F-quasiconvex, its level sets and the lower and strictly lower sets are convex.

For the minimum set of these functions, we have the following results.

THEOREM 28.-   If f is a F-quasiconvex function then it take its minimum in an extreme subset of X.

THEOREM 29.-   If f is a F-quasiconvex function in a polyhedron, then any local vertex-maximum point in the polyhedron is a global maximum point of f in the polyhedron.

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