Lineability of holomorphic non-extendability

Luis Bernal González

Universidad de Sevilla

Workshop Functional Analysis Valencia 2015
on the occasion of the 60th birthday of José Bonet
Valencia, Spain, June 15–19th, 2015

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Weierstrass, 1872

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2\pi(8\pi + 3)^n x)$$

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Banach, 1931

The family of Weierstrass' monsters forms a residual subset of $C(\mathbb{R})$, endowed with the topology of uniform convergence in compacta.

- That is, the last family is, topologically speaking, large. But, is it large in some algebraic sense? More specifically, does it contains large vector spaces? Note that, clearly, {Weierstrass' monsters} is not a vector space. Surprisingly, we have:

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- Many concepts concerning lineability have been coined, but don't worry: I'll (try to) recall them each time they appear (if I remember doing it!).

Aron, Bayart, Gurariy, PérezGª, Quarta, Seoane, LBG, 2004-10

Assume that X is a TVS and that α is a cardinal number. A subset $A \subset X$ is called:

- lineable if $A \cup \{0\}$ contains an ∞-dimensional vector space [so the family of Weiestrass' monsters is lineable],
- α-lineable if $A \cup \{0\}$ contains a vector space M with $\dim(M) = \alpha$,
- dense-lineable whenever $A \cup \{0\}$ contains a dense vector subspace of X,
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- spaceable whenever $A \cup \{0\}$ contains a closed infinite dimensional vector subspace of X, and
- (densely, closely) α-algebrable if X is contained in some linear algebra and $A \cup \{0\}$ contains some (dense, closed, resp.) infinitely α-generated algebra.

Bartoszewicz and Glab, 2013

Assume that X is contained in some linear algebra and that $A \subset X$. Then A is said to be (densely, closely) strongly α-algebrable if $A \cup \{0\}$ contains a (dense, closed, resp.) α-generated free algebra M with $M \setminus \{0\} \subset A$ [if $\alpha = \aleph_0$, we simply say (densely, closely) strongly algebrable].

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- Just one remark: if X is contained in a commutative algebra then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$.
  Then strong algebrability $\Longrightarrow$ algebrability, but NOT$\Longleftarrow$.

- Now, we recall the concept of non-extendable holomorphic function. Our setting will be: $E =$ a complex separable Banach space, $G =$ a domain in E, and $H(G) =$ the space of all holomorphic functions $G \to \mathbb{C}$, endowed with the topology of convergence in compacta.
  In the special case $E = \mathbb{C}^N$ $(N \in \mathbb{N})$ the space $H(G)$ is a Fréchet space.

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Domains of existence

(a) We say that a function $f \in H(G)$ is holomorphically non-extendable across any boundary point [synonymous expressions are: f is analytically non-continuable beyond ∂G, f is holomorphic exactly on G, G is a domain of existence of f] whenever there do not exist two domains G_1 and G_2 in E and $\widetilde{f} \in H(G_1)$ such that

$$G_2 \subset G \cap G_1,\ G_1 \not\subset G \text{ and } \widetilde{f} = f \text{ on } G_2.$$

(b) We denote $H_e(G) := \{f \in H(G) : f \text{ is holomorphic exactly on } G\}$. A domain G is said to be a domain of existence if $H_e(G) \neq \varnothing$.

(c) We say that a function $f \in H(G)$ is holomorphic weakly exactly on G if there do not exist a domain $\widetilde{G}$ and $\widetilde{f} \in H(\widetilde{G})$ such that $\widetilde{G} \supsetneq G$ and $\widetilde{f}|_G = f$.
The set of such functions will be denoted by $H_{we}(G)$.

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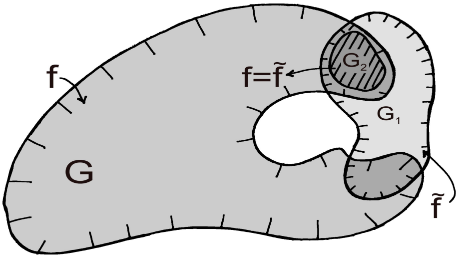
Non-extendable holomorphic functions, II

- The figure shows a function f that is not in $H_e(G)$:



- Always $H_e(G) \subset H_{we}(G)$. If, for instance, $f(z) = \log_p z$ and $G = \mathbb{C} \setminus (-\infty, 0]$, then $f \in H_{we}(G) \setminus H_e(G)$.
- If $G \subset \mathbb{C}$ is a Jordan domain then $H_{we}(G) = H_e(G)$.
- Let $G \subset \mathbb{C}$. Then: $f \in H_e(G) \iff \rho(f, a) = \text{dist}(a, \partial G)$ $\forall a \in G$, where $\rho(f, a) :=$ the radius of convergence of the Taylor series of f with center at a.

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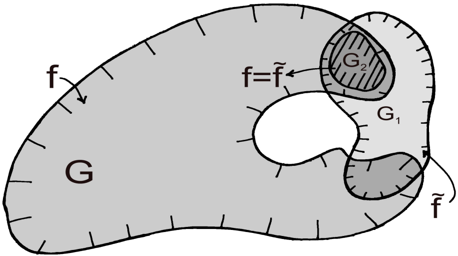

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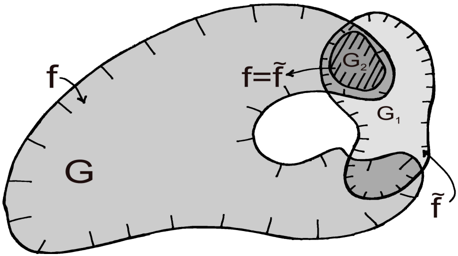

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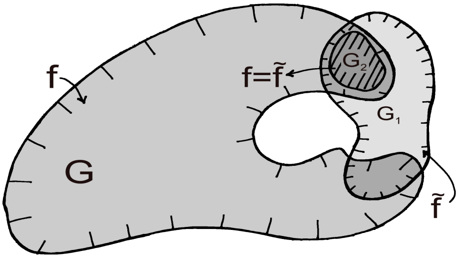

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Mittag-Leffler, 1884

Every domain G in $\mathbb{C}$ is a domain of existence.

Proof: Fix a sequence $A = \{a_1, a_2, \dots\} \subset G$ of distinct points satisfying: $A' \cap G = \varnothing$ and, for every open set U st $U \cap \partial G \neq \varnothing$ and every connected component V of $U \cap G$, we have $A \cap V \neq \varnothing$. For each $n \in \mathbb{N}$, choose $b_n \in \partial G$ with $|b_n - a_n| \leq |z - a_n| \ \forall z \notin G$. Then

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- NO LONGER TRUE for higher dimensions: $(\mathbb{D})^2 \setminus \overline{((1/2)\mathbb{D})^2}$ is not a domain of existence in $\mathbb{C}^2$.
- For domains in $\mathbb{C}^N$ ($N \in \mathbb{N}$), the Cartan–Thullen theorem characterizes domains of existence: G is a D.E. $\Longleftrightarrow$ G is holomorphically convex.

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- In the case $E = \mathbb{C}$, the phenomenon of non-extendability turns out to be topologically generic:

Kierst and Szpilrajn, 1933

For any domain $G \subset \mathbb{C}$, the set $H_e(G)$ is residual in $H(G)$.

Kahane, 2000

Let $G \subset \mathbb{C}$ be a domain and X be a Baire topological vector space with $X \subset H(G)$ such that all evaluation functionals

$$f \in X \mapsto f^{(k)}(a) \in \mathbb{C} \quad (a \in G,\ k \geq 0)$$

are continuous and, for every $a \in G$ and every $r > \mathrm{dist}\,(a, \partial G)$, $\exists f \in X$ st $\rho(f, a) < r$. Then $H_e(G) \cap X$ is residual in X.

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For any domain $G \subset \mathbb{C}$, the set $H_e(G)$ is residual in $H(G)$.

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Let $G \subset \mathbb{C}$ be a domain and X be a Baire topological vector space with $X \subset H(G)$ such that all evaluation functionals

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- Now: how large is $H_e(G)$ in the algebraic sense?

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If $G \subset \mathbb{C}$ then $H_e(G) \cup \{0\}$ contains a dense vector subspace of $H(G)$, as well as a closed infinitely generated algebra. In particular, $H_e(G)$ is dense-lineable, spaceable and algebrable.

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- The AGM-proof of the 2nd part is super-easy!: simply take

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The set $X \cap H_e(\mathbb{D})$ is dense-lineable and spaceable in X for $X = A^\infty(\mathbb{D})$, H^p or B^p $(0 < p < \infty)$, under their respective natural topologies.

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- Turning to the general case of a domain $G \subset \mathbb{C}$, it is possible to impose growth conditions to the non-extendability: Given $\varphi : G \to (0, +\infty)$, the set $\{f \in H(G) : \limsup_{z \to \xi} \frac{|f(z)|}{\varphi(z)} = +\infty \ \forall$ prime end ξ of $G\}$ is maximal dense-lineable and spaceable [LBG, 2006].
- Valdivia (2008) showed that the dense subspace contained in $H_e(G) \cup \{0\}$ $(G \subset \mathbb{C}^N)$ can be chosen to be nearly-Baire for any domain of existence $G \subset \mathbb{C}^N$.
  Recall that a subset A of a locally convex space E is said to be sum-absorbing whenever $\exists \lambda > 0$ s.t. $\lambda(A + A) \subset A$, and E is called nearly-Baire if, given a sequence (A_j) of sum-absorbing balanced closed subsets with $E = \bigcup_{j=1}^{\infty} A_j$, there is j_0 such that A_{j_0} is a neighborhood of 0.

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- A domain $G \subset \mathbb{C}$ is finite-length if $\exists M \in (0, +\infty)$ s.t. $\forall a, b \in G$ $\exists$ a curve $\Gamma \subset G$ joining a to b for which length $(\Gamma) \leq M$.

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(a) Let $G \subset \mathbb{C}$ be a finite-length regular domain such that $\mathbb{C} \setminus \overline{G}$ is connected. Then $H_e(G) \cap A^\infty(G)$ is dense-lineable in $A^\infty(G)$.
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In particular, if G is a Jordan domain then $H_e(G) \cap A^\infty(G)$ is dense-lineable in $A^\infty(G)$.

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Let $G \subset \mathbb{C}$ be a regular domain. Then $A^\infty(G) \cap H_{we}(G)$ is spaceable and maximal dense-lineable in $A^\infty(G)$.
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Suppose that G is a domain of existence of a separable complex Banach space E. Then $H_e(G)$ is closely strongly algebrable and densely strongly algebrable in $H(G)$.

- OPEN PROBLEM: Of course, the last result implies the spaceability and the dense-lineability of $H_e(G)$, but not in the maximal degree [i.e. one derives neither $\mathfrak{c}$-spaceability nor dense $\mathfrak{c}$-lineability nor $\mathfrak{c}$-algebrability], because the algebras discovered by Alves are $\aleph_0$-generated, and Baire's theorem is not at our disposal, because $H(G)$ is not barreled (if $\dim(E) = \infty$), hence not-Baire!!

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THANK YOU !