Gene mutations in evolution algebras by means of strong isotopisms

Presentation · June 2016

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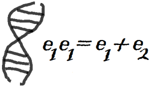
Gene Mutations in Evolution Algebras by means of Strong Isotopisms

Raúl Falcón

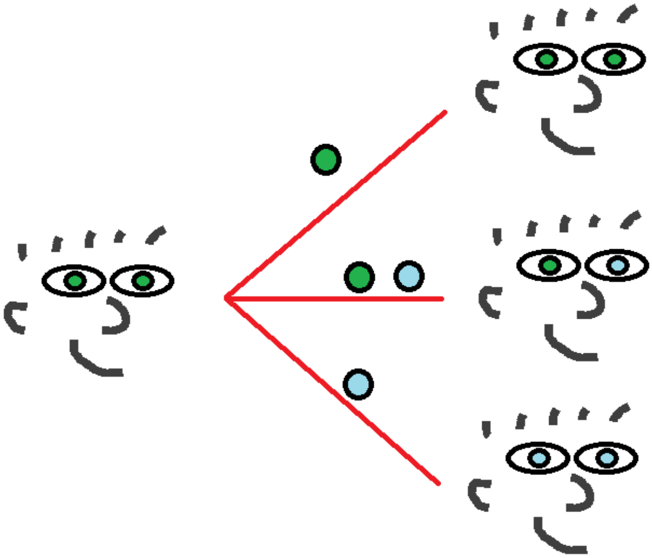


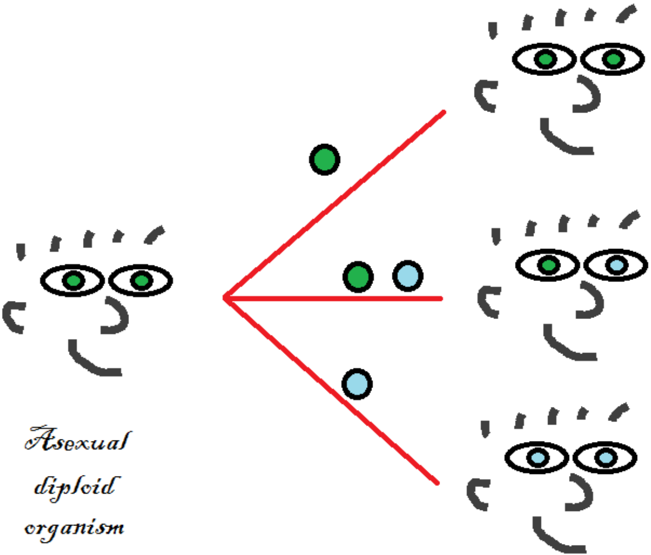
(Joint work with Óscar Falcón and Juan Núñez)

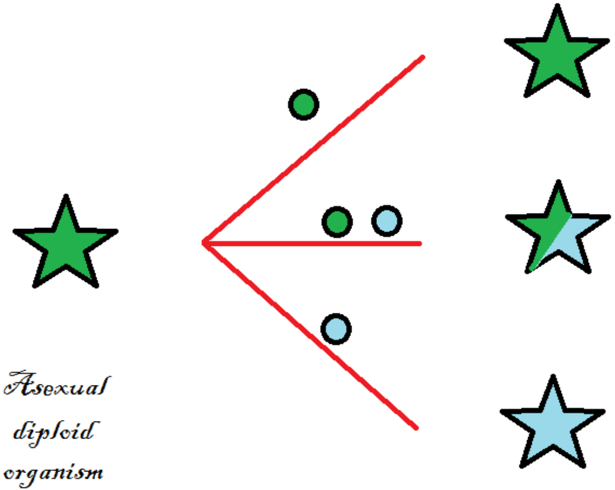
Department of Applied Mathematics I. University of Seville.

Pilsen. June 9, 2016.

ESCO 2016 5th European Seminar on Computing
June 5 - 10, 2016
Pilsen, Czech Republic




Asexual
diploid
organism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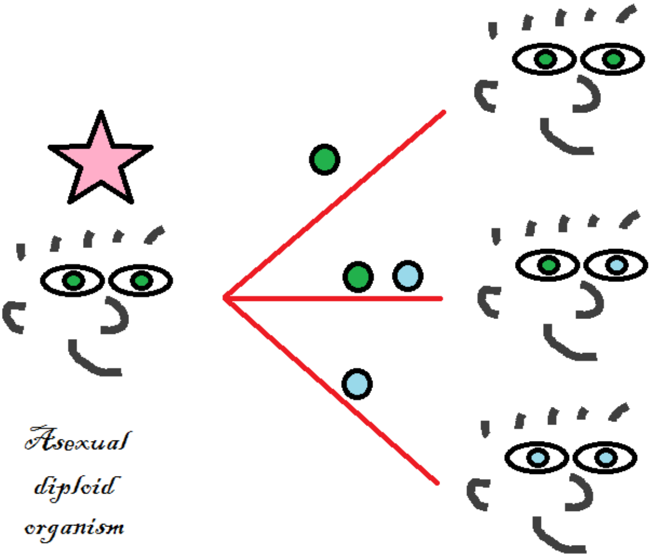
Asexual
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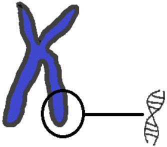
CONTENTS

I. Genetics and algebra

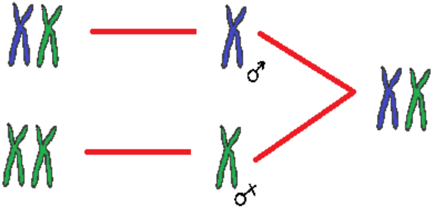
Genetics.

- **Gene**: Molecular unit of hereditary information.
- **Deoxyribonucleic acid (DNA)**: It contains the code to synthesize proteins and determines the attributes that characterize each organism.
- **Allele**: Distinct forms of genes related to an attribute (color of eyes: brown, green and blue alleles).
- **Chromosomes**: Long strands of DNA formed by ordered sequences of genes.
  - They carry the genetic code of any organism.
  - In the **process of reproduction**, the attributes of the offspring are inherited from the alleles contained in the chromosomes of the parents.



Genetics.

- **Diploid** organisms carry a double set of chromosomes (one of each parent).
- They reproduce by means of sex cells (**gametes**), each of them carrying a single set of chromosomes.
- The fusion of two gametes of opposite sex gives rise to a **zygote**, which contains a double set of chromosomes.



Sexual reproduction: Genetic algebras.

- **Nonassociative algebras** in genetics (*Etherington*, 1939).

Mendel's laws + Sexual reproduction + Inheritance

$\updownarrow$

Fusion of gametes into a zygote

$\|$

Algebraic multiplication with probabilistic structure constants related to the gametic output.

- **Genetic algebra**: Basis of alleles $\{e_1, \ldots, e_n\}$

$$\mathbf{e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k}$$

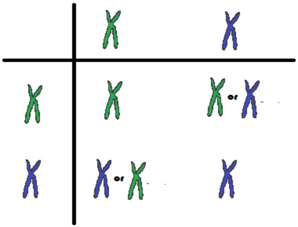
Hence,

$$\sum_{k=1}^{n} c_{ijk} = 1$$

Sexual reproduction: Genetic algebras.

- **Mendelian inheritance**: $e_i e_j = \frac{1}{2}(e_i + e_j)$.
- For a diploid organism:

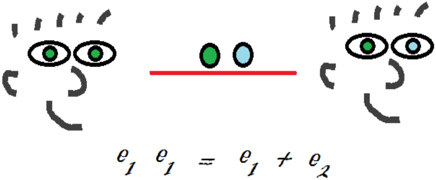
	e_1	e_2
e_1	e_1	$\frac{1}{2}(e_1 + e_2)$
e_2	$\frac{1}{2}(e_1 + e_2)$	e_2



Jianjun Paul Tian, 2004

An n-dimensional algebra over a field $\mathbb{K}$ is said to be an evolution algebra if it admits a natural basis $\{e_1, \ldots, e_n\}$ such that

(1) $e_i e_j = 0$ if $i \neq j$,
(2) $e_i e_i = \sum_{j \leq n} a_{ij} e_j$, for some $a_{i1}, \ldots, a_{in} \in \mathbb{K}$.



$T = (\sum_{j \leq n} a_{1j} e_j, \ldots, \sum_{j \leq n} a_{nj} e_j) \equiv$ Structure tuple of A_T.

Asexual reproduction: Evolution algebras.

Theorem (Casas et al., 2014)

Every two-dimensional non-abelian **complex** *evolution algebra A is isomorphic to exactly one of the next algebras*

- $\dim A^2 = 1$:
  - $E_1 := A_{(e_1,0)}$.
  - $E_2 := A_{(e_1,e_1)}$.
  - $E_3 := A_{(e_1+e_2,-e_1-e_2)}$.
  - $E_4 := A_{(e_2,0)}$.
- $\dim A^2 = 2$:
  - $E_{5_{a,b}} := A_{(e_1+ae_2,be_1+e_2)}$.

$$E_{5_{a,b}} \cong E_{5_{b,a}}.$$

  - $E_{6_a} := A_{(e_2,e_1+ae_2)}$.

$$E_{6_a} \cong E_{6_b} \Leftrightarrow \frac{a}{b} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}.$$

II. Isotopism classes of asexual diploid organisms

Abraham Adrian Albert

1905-1972

Two algebras A and A' are **isotopic** if

$$f(u)g(v) = h(uv), \text{ for all } u, v \in A,$$

where f, g and h are regular linear transformations from A to A'

- $(f, g, h) \equiv$ Isotopism ($\sim$)
- $f = g \Rightarrow$ Strong isotopism ($\simeq$)
- $f = g = h \Rightarrow$ Isomorphism ($\cong$)

Literature: Division, Lie, Jordan, absolute valued, structural algebras, . . .

What about genetic algebras?

Isotopisms of genetic algebras

- (Holgate, 1966; Campos, 1987).

Isotopisms of genetic algebras.
$\|$
Mutation of alleles in the inheritance process.

- **Mutation algebras**: Before the formation of a zygote, each allele e_i mutates to e_j with probability p_{ij}.

$$\mathbf{e_i} \circ \mathbf{e_j} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{p_{ik} e_k} \cdot \sum_{\mathbf{l=1}}^{\mathbf{n}} \mathbf{p_{jl} e_l} = \sum_{k,l=1}^{n} p_{ik} p_{jl} e_i \cdot e_k$$

What about evolution algebras?

Isotopisms of evolution algebras

Lemma

Two evolution algebras with equal structure tuples up to permutation of their components and basis vectors are strongly isotopic.

$$E_1 := A_{(e_1,0)} \simeq E_4 := A_{(e_2,0)}$$

Proposition

Any evolution algebra is strongly isotopic to an evolution algebra with a structure tuple of lower-triangular form.

$$E_3 := A_{(e_1+e_2,-e_1-e_2)} \simeq A_{(e_1,-e_1)}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

Lemma

Two evolution algebras with equal structure tuples up to permutation of their components and basis vectors are strongly isotopic.

$$E_1 := A_{(e_1,0)} \simeq E_4 := A_{(e_2,0)}$$

Proposition

Any evolution algebra is strongly isotopic to an evolution algebra with a structure tuple of lower-triangular form.

$$E_{5_{a,b}} := A_{(e_1+ae_2,be_1+e_2)} \simeq E_{5_{0,0}} := A_{(e_1,e_2)}$$

$$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Isotopisms of evolution algebras

Lemma

Two evolution algebras with equal structure tuples up to permutation of their components and basis vectors are strongly isotopic.

$$E_1 := A_{(e_1,0)} \simeq E_4 := A_{(e_2,0)}$$

Proposition

Any evolution algebra is strongly isotopic to an evolution algebra with a structure tuple of lower-triangular form.

$$E_{6_a} := A_{(e_2,e_1+ae_2)} \simeq E_{5_{0,0}} := A_{(e_1,e_2)}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Isotopisms of evolution algebras

- Annihilator: $\mathrm{Ann}(A) = \{u \in A \mid uv = vu = 0, \text{ for all } v \in A\}$
- Derived algebra: $A^2 = \{uv \mid u, v \in A\}$

Lemma

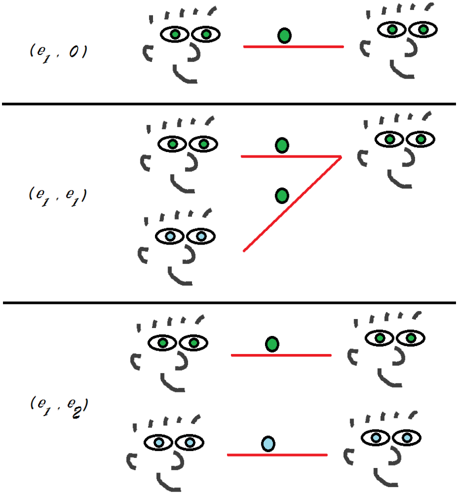
Dimensions of annihilators and derived algebras are preserved by isotopisms.

Proposition

There exist three non-abelian strongly isotopism classes of two-dimensional complex evolution algebras:

$$A_{(e_1,0)} \qquad A_{(e_1,e_1)} \qquad \textit{and} \qquad A_{(e_1,e_2)}$$

Isotopisms of evolution algebras



What about non-complex evolution algebras?

- $\mathcal{E}_n(\mathbb{K}) \equiv n$-dimensional evolution algebras over $\mathbb{K}$.
- $\mathcal{E}_{n;m}(\mathbb{K}) = \{A \in \mathcal{E}_n(\mathbb{K}) \mid \dim \mathrm{Ann}(A) = n - m\}$

Lemma

Every evolution algebra in $\mathcal{E}_{n;m}(\mathbb{K})$ holds, up to isomorphism,

$$e_i e_i \neq 0 \Leftrightarrow i \leq m$$

Proposition

$$\begin{cases} A \in \mathcal{E}_{n;m}(\mathbb{K}) \\ A' \in \mathcal{E}_{n;m'}(\mathbb{K}) \end{cases} \Rightarrow A \not\sim A' \text{ if } m \neq m'.$$

Isotopisms of evolution algebras

Proposition

a) $\mathcal{E}_{n;0}(\mathbb{K}) = \{A_{(0,0)}\}$.

b) $\mathcal{E}_{1;1}(\mathbb{K}) \cong \{A_{(e_1,0)}\}$.

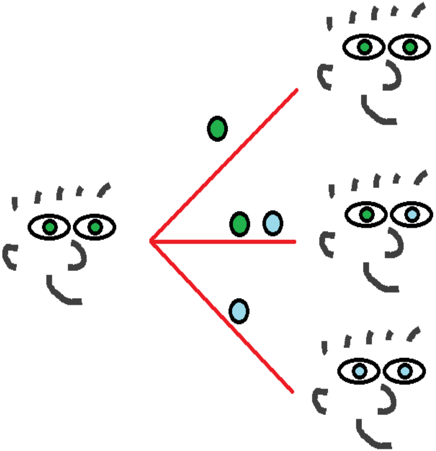
c) $\mathcal{E}_{n;1}(\mathbb{K}) \cong \{A_{(e_1,0)}, A_{(e_2,0)}\}$, *for all* $n > 1$.

d) $\mathcal{E}_{n;1}(\mathbb{K}) \sim \{A_{(e_1,0)}\}$.

e) $\mathcal{E}_{2;2}(\mathbb{K}) \cong \{A_{(e_1,*)}, A_{(e_1+e_2,*)}, A_{(e_2,*)}\}$.

f) $\mathcal{E}_{n;2}(\mathbb{K}) \sim \{A_{(e_1,e_1)}, A_{(e_1,e_2)}\}$.

$$\mathcal{E}_{2;2}(\mathbb{K}) \cong \{A_{(e_1,*)}, A_{(e_1+e_2,*)}, A_{(e_2,*)}\}$$

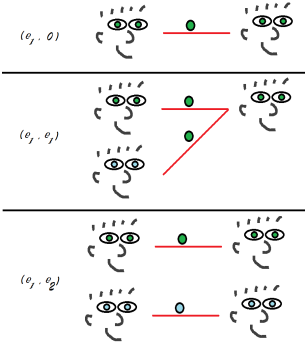


Isotopisms of evolution algebras

Proposition

There exist three non-abelian strongly isotopism classes of two-dimensional evolution algebras over any field:

$$A_{(e_1,0)} \qquad A_{(e_1,e_1)} \qquad \textit{and} \qquad A_{(e_1,e_2)}$$



III. Isomorphism classes of asexual diploid organisms

Isomorphisms of evolution algebras

Theorem

The isomorphism group between $A, A' \in \mathcal{E}_n(\mathbb{F}_q)$ is identified with the algebraic set defined by the ideal

$$I^{\mathrm{Isom}}_{A,A'} = \langle \sum_{k,l=1}^{n} \mathfrak{f}_{ik}\mathfrak{f}_{jl}{c'}^{m}_{kl} - \sum_{s=1}^{n} c^{s}_{ij}\mathfrak{f}_{sm} \mid i,j,m \leq n \rangle + \langle \det(F)^{q-1} - 1 \rangle,$$

where F denotes the regular matrix related to an isomorphism f.

Require: A set S of n-dimensional algebras over a finite field.
Ensure: C, the set of isomorphism classes of S.

$C = \emptyset$.
while $S \neq \emptyset$ **do**
  Take $A \in S$.
  $S := S \setminus \{A\}$.
  $C := C \cup \{A\}$.
  for $A' \in S$ **do**
    if $|\mathcal{V}(I^{\mathrm{Isom}}_{A,A'})| > 0$ **then**
      $S := S \setminus \{A'\}$.
    end if
  end for
end while
return C.

Isomorphisms of evolution algebras

q	Structure tuples			
2	$(0,0)$	$(e_2,0)$	$(e_1,0)$	(e_1,e_1)
	(e_1+e_2,e_1+e_2)	(e_2,e_1)	(e_2,e_1+e_2)	(e_1,e_1+e_2)
	(e_1,e_2)			
3	$(0,0)$	$(e_2,0)$	$(e_1,0)$	$(e_1,2e_1)$
	$(e_1+e_2,2e_1+2e_2)$	(e_1,e_1)	(e_2,e_1)	(e_2,e_1+e_2)
	(e_2,e_1+2e_2)	$(e_1+e_2,2e_1+e_2)$	(e_1,e_1+e_2)	$(e_1,2e_1+e_2)$
	(e_1,e_2)			
5	$(0,0)$	$(e_2,0)$	$(e_1,0)$	(e_1,e_1)
	$(e_1+e_2,4e_1+4e_2)$	$(e_1,2e_1)$	(e_2,e_1)	(e_2,e_1+e_2)
	(e_2,e_1+2e_2)	(e_2,e_1+3e_2)	(e_2,e_1+4e_2)	(e_1+e_2,e_1+2e_2)
	(e_1+e_2,e_1+3e_2)	(e_1+e_2,e_1+4e_2)	$(e_1+e_2,2e_1+e_2)$	$(e_1+e_2,3e_1+e_2)$
	$(e_1+e_2,2e_1+3e_2)$	$(e_1+e_2,3e_1+2e_2)$	(e_1,e_1+e_2)	$(e_1,2e_1+e_2)$
	$(e_1,3e_1+e_2)$	$(e_1,4e_1+e_2)$	(e_1,e_2)	
7	$(0,0)$	$(e_2,2e_1+e_2)$	(e_1+e_2,e_1+2e_2)	$(e_1+e_2,3e_1+5e_2)$
	$(e_1,0)$	$(e_2,2e_1+3e_2)$	(e_1+e_2,e_1+3e_2)	$(e_1+e_2,3e_1+6e_2)$
	$(e_2,0)$	$(e_2,3e_1+e_2)$	(e_1+e_2,e_1+4e_2)	$(e_1+e_2,4e_1+3e_2)$
	(e_1,e_1)	$(e_2,3e_1+3e_2)$	(e_1+e_2,e_1+5e_2)	$(e_1+e_2,4e_1+5e_2)$
	$(e_1,2e_1)$	(e_1,e_1+e_2)	(e_1+e_2,e_1+6e_2)	$(e_1+e_2,4e_1+6e_2)$
	$(e_1,3e_1)$	(e_1,e_1+2e_2)	$(e_1+e_2,2e_1+e_2)$	$(e_1+e_2,6e_1+3e_2)$
	(e_1,e_2)	(e_1,e_1+3e_2)	$(e_1+e_2,2e_1+3e_2)$	$(e_1+e_2,6e_1+5e_2)$
	(e_2,e_1)	$(e_1,3e_1+e_2)$	$(e_1+e_2,2e_1+4e_2)$	$(e_1+e_2,6e_1+6e_2)$
	(e_2,e_1+e_2)	$(e_1,3e_1+2e_2)$	$(e_1+e_2,2e_1+5e_2)$	
	(e_2,e_1+3e_2)	$(e_1,3e_1+3e_2)$	$(e_1+e_2,2e_1+6e_2)$	

Theorem

The sets $\mathcal{E}_2(\mathbb{F}_q)$, with $q \in \{2,3,5,7\}$, are respectively distributed into 9, 13, 23 and 38 isomorphism classes.

Complexity time: $q^{O(n^2)} + O(n^6 n!)$.

q	Run time	Usage memory
2	0 seconds	0 MB
3	3 seconds	0 MB
5	38 seconds	80 MB
7	278 seconds	1360 MB

Intel Core i7-2600, with a 3.4 GHz processor and 16 GB of RAM

Is it possible a general classification?

Isomorphisms of evolution algebras

- $\begin{cases} A = A_{(ae_1+be_2, ce_1+de_2)} \\ A' = A_{(\alpha e'_1+\beta e'_2, \gamma e'_1+\delta e'_2)} \end{cases} \in \mathcal{E}_{2,2}(\mathbb{K})$

- $f \equiv$ Isomorphism such that

$$f(e_i) = f_{i1}e'_1 + f_{i2}e'_2, \text{ for all } i \in \{1,2\}$$

From the computation of the reduced Gröbner basis:

$$\begin{cases} (ad-bc)f_{11}f_{21} = 0, \\ (ad-bc)f_{12}f_{22} = 0. \end{cases}$$

One-dimensional derived algebra ($ad = bc$)

- $A' = A_{(e_1', \gamma e_1')}$ $\qquad \gamma \neq 0$

$$f_{11} f_{21} = f_{12} f_{22} = 0$$

  - $f_{11} = f_{22} = 0 \Rightarrow f_{21} = 0$!!!
  - $f_{21} = f_{12} = 0 \Rightarrow c = \gamma f_{22}^2$

$$\mathbf{A_{(e_1, ce_1)} \cong A_{(e_1, cm^2 e_1)}}$$

- $A' = A_{(e_2', \delta e_2')}$ $\qquad \delta \neq 0$

$$\begin{cases} f_{11} = f_{22} = 0, \\ f_{12} = 1/\delta, \\ f_{21}^2 = c/\delta. \end{cases}$$

$$\mathbf{A_{(e_2, ce_2)} \cong A_{(e_1, ce_1)}}$$

One-dimensional derived algebra ($ad = bc$)

- $A' = A_{(e'_1+e'_2,\gamma(e'_1+e'_2))}$ $\qquad \gamma \neq 0$

$$\begin{cases} \gamma \neq -1, \\ f_{11} = f_{12} = 1/(\gamma+1), \\ f_{21} = -\gamma f_{22}, \\ c = \gamma(\gamma+1)^2 f_{22}^2. \end{cases}$$

$$\mathbf{A}_{(\mathbf{e_1}+\mathbf{e_2},\mathbf{c}(\mathbf{e_1}+\mathbf{e_2}))} \cong \mathbf{A}_{(\mathbf{e_1},\mathbf{c}(\mathbf{c}+\mathbf{1})^2\mathbf{e_1})}$$

Two-dimensional derived algebra ($ad \neq bc$)

Proposition

a) $A_{(e_1,de_2)} \cong A_{(e_1,e_2)}$ $\quad d \neq 0$

b) $A_{(e_1+e_2,de_2)} \cong A_{(e_1,de_1+e_2)}$ $\quad d \neq 0$

c) $A_{(e_2,c^2m^3e_1)} \cong A_{(e_2,ce_1)}$ $(c, m \neq 0)$

d) $A_{(e_1+e_2,ce_1)} \cong A_{(e_2,\frac{1}{c}(e_1+e_2))}$ $\quad c \neq 0$

e) $A_{(e_1+e_2,ce_1+de_2)} \cong A_{(e_1+e_2,\frac{c}{d^2}(\frac{c}{d}e_1+e_2))}$ $\quad c, d \neq 0,\ c \neq d$

f) $A_{(e_1,ce_1+de_2)} \cong A_{(e_1,\gamma e_1+\delta e_2)}$ $\quad d, \delta \neq 0,\ c\delta^2 = d^2\gamma$

g) $A_{(e_2,ce_1+de_2)} \cong A_{(e_2,\frac{c}{m^3}e_1+\frac{d}{m^2}e_2)}$ $\quad c, d, m \neq 0$

Isomorphisms of evolution algebras

Theorem

Isomorphism classes in $\mathcal{E}_{2,2}(\mathbb{K})$:

- $A_{(e_1,ce_1)}$ $c \neq 0.$
- $A_{(e_1+e_2,-e_1-e_2)}.$
- $A_{(e_1,ce_1+de_2)}$ $d \neq 0$
- $A_{(e_2,ce_1+de_2)}$ $c \neq 0$
- $A_{(e_1+e_2,ce_1+de_2)}$ $c, d \neq 0$

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Many thanks!!

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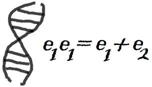

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