SOME INDEFINITE NONLINEAR EIGENVALUE PROBLEMS

A. SUÁREZ*

Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Fac. Matemáticas, C/ Tarfia s/n, C.P. 41012,
Univ. Sevilla, Spain
E-mail: suarez@us.es

Dedicated to Prof. Jean Mawhin for his first 60 years of Nonlinear Analysis

In this work we study the structure of the set of positive solutions of a nonlinear eigenvalue problem with a weight changing sign. Specifically, the reaction term arises from a population dynamic model. We use mainly bifurcation methods to obtain our results.

1. Introduction

The aim of this work is to study some nonlinear indefinite eigenvalue problems of the form

$$\begin{cases} -\Delta u = \lambda m(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a regular boundary $\partial\Omega$, $m \in C(\overline{\Omega})$ changes sign, f is a regular function and λ plays the role of real parameter. We focus our attention on the case $f(0) = 0$ and $\lambda > 0$; similar results can be obtained for negative values of λ.

Depending of the shape of f, Eq. (1) models different situations: population dynamics, population genetics, combustion theory,... see [10].

In the linear case, i.e., $f(u) = u$, (1) is the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda m(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

*Supported by the Spanish Ministry of Science and Technology under grants BFM2000-0797 and BFM2003-06446.

It is well known (see for instance [19] and [23]) that there exist two values of λ, $\lambda_-(m) < 0 < \lambda_+(m)$, called principal eigenvalues because they have associated positive eigenfunctions. In the present work, given $q \in L^\infty(\Omega)$ we denote by $\sigma_1^\Omega[-\Delta + q]$ (we delete the superscript Ω when no confusion arises) the principal eigenvalue of the problem

$$-\Delta u + q(x)u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

When in (1) the weight does not appear, i.e., $m \equiv 1$, the nonlinear problem

$$\begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega, \\ u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{3}$$

has been extensively studied. Classical references are [2] and [21], but many others can be given where, as well as existence results, uniqueness ones are shown: [4], [14], [26], [20], [22] and references therein.

Much less is known for problem (1). In [19], assuming for example that $f'(0) > 0$, the authors showed that there exists an unbounded continuum of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_+(m)/f'(0)$.

In [8] the authors assumed that $f : I \mapsto \mathbb{R}_+$, $I \subset \mathbb{R}$, and $f'' < 0$ and showed that every positive solution of (1) is stable. If, moreover, $I = [0, 1]$, $f(1) = 0$ and $f'(0) > 0$ they proved that there exists a positive solution if, and only if, $\lambda > \lambda_+(m)/f'(0)$, and in this case the solution is unique. Similar result was shown in [13], although the authors' motivation was to study the problem in the whole space. Very recently, in [9] the authors analyze the particular cases $f(u) = g_i(u)$, $i = 1, 2$ with

$$g_1(u) = u - u^2, \qquad\qquad g_2(u) = u + u^2. \tag{4}$$

Observe that the result of [8] can only be applied to g_1. In [9], without the assumption that f takes only values in $[0, 1]$, the main result of [8] was improved showing (by variational method) that, assuming some restriction in the space dimension, there exists positive solution if $\lambda \in (0, \lambda_+(m))$. For the case, $f = g_2$, they also proved the existence of positive solution for $\lambda \in (0, \lambda_+(m))$ and that there does not exist positive solution at $\lambda = \lambda_+(m)$. In [16] these results have been again completed. We prove for $f = g_1$ that there exist at least two positive solutions in $\lambda \in (\lambda_+(m), \infty)$, one of them linearly asymptotically stable and that for $f = g_2$ there exists positive solution if, and only if, $\lambda \in (0, \lambda_+(m))$.

In this work, we are going to analyze the following nonlinearities

$$f_1(u) = u - u^2 - K\frac{u}{1+u}, \qquad\qquad f_2(u) = u + u^2 - K\frac{u}{1+u}, \tag{5}$$

where $K \in \mathbb{R}$. Observe that the functions in (4) are included in (5). These last nonlinearities arise in population dynamics. Indeed, when $K = 0$, f_1 is the classical logistic reaction term and for $K \neq 0$ the predation one $Ku/(1+u)$ is called the Holling-Tanner term, see for example [7] for an ecological interpretation.

In order to state our main results we need some notations. Specifically, assume that

$$M_{\pm} := \{x \in \Omega : m^{\pm} > 0\}$$

are open and regular sets, where $m^{\pm}$ represent the positive and negative part of m respectively; and suppose that $m^{\pm}(x) \approx [\mathrm{dist}(x, \partial M_{\pm})]^{\gamma_{\pm}}$ for x close to $\partial M_{\pm}$ and some $\gamma_{\pm} \geq 0$. The following condition will provide us with *a priori* bounds of the solutions

$$2 < \min\left\{\frac{N+1+\gamma_{\pm}}{N-1}, \frac{N+2}{N-2}\right\}. \tag{6}$$

Finally, we define for $K \neq 1$ the values

$$\lambda_{+} := \frac{\lambda_{+}(m)}{1-K} \qquad \lambda_{-} := \frac{\lambda_{-}(m)}{1-K},$$

and $\Pi : \mathbb{R} \times C(\overline{\Omega}) \mapsto \mathbb{R}$ the projection map onto $\mathbb{R}$, i.e. $\Pi(\mu, u) = \mu$. The main results are:

Theorem 1.1. *Assume that* $K \neq 1$ *and* (6).

(1) *There exists an unbounded continuum* $\mathcal{C}$ *of positive solutions of* (1) *bifurcating from the trivial solution at* $\lambda = \lambda_{+}$ *if* $K < 1$ *and* $\lambda = \lambda_{-}$ *if* $K > 1$.

(2) *The bifurcation is supercritical for* $f = f_1$ *and for* $f = f_2$ *and* $K < -1$ *or* $K > 1$ *and subcritical for* $f = f_2$ *and* $K \in [-1, 1)$.

(3) *If* $f = f_1$ *and* $K < 1$ *(resp.* $f = f_2$ *and* $K > 1$*), then* $\Pi(\mathcal{C}) = (\lambda_{+}, \infty)$ *(resp.* (λ_{-}, ∞)*). Moreover, if* $(\lambda, u_\lambda) \in \mathcal{C}$*, then* u_λ *is linearly asymptotically and such that* $u_\lambda \leq \sqrt{1-K}$ *(resp.* $\sqrt{K}-1$*). Furthermore, there exists another positive solution* v_λ *for all* $\lambda > 0$.

(4) *If* $f = f_1$ *and* $K > 1$ *(resp.* $f = f_2$ *and* $K < -1$*) then* $\Pi(\mathcal{C}) = (0, \lambda_*]$ *for* $\lambda_* > \lambda_{-}$ *(resp.* λ_{+}*). Moreover, there exist* λ_0 *and* λ^* *with* $\lambda_0 < \lambda^*$ *such that for* $\lambda \geq \lambda^*$ *the problem* (1) *does not admit positive solutions and it possesses at least two positive solutions for* $\lambda \in (\lambda_{-}, \lambda_0)$ *(resp.* (λ_{+}, λ_0)*).*

(5) *If* $f = f_2$ *and* $K \in [-1, 1)$ *there exists positive solution for* $\lambda \in (0, \lambda_{+})$ *and* (1) *does not admit positive solutions for* $\lambda \geq \lambda^*$.

(6) In any case, if there exists a solution v_λ for $\lambda > 0$, then $\lim_{\lambda\to 0}\|v_\lambda\|_\infty = +\infty$.

Theorem 1.2. *Assume $K = 1$ and (6). Then there exists at least a solution u_λ for $\lambda > 0$ and $\lim_{\lambda\to 0}\|u_\lambda\|_\infty = +\infty$.*

Remark 1.1.

(1) The existence of $\mathcal{C}$ is true without assuming (6). In the cases (4) and (5) of Theorem 1.1, $\mathcal{C}$ could "go to infinity" in a value λ^0.
(2) In the particular case $f = f_2$ and $K = 0$, in [16] it was proved using a Picone inequality that (1) possesses a positive solution if, and only if, $\lambda \in (0, \lambda_+)$.

In Figs. 1 and 2 we have summarized these results (the case $f = f_2$ and $K = 1$ is similar to $f = f_1$ and $K = 1$).

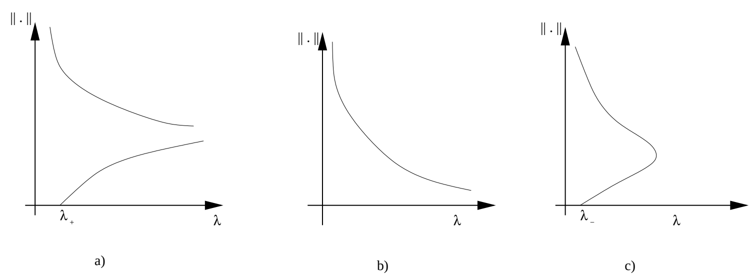

Figure 1. Bifurcation diagrams for $f = f_1$: a) $K < 1$; b) $K = 1$; c) $K > 1$.

The rest of the paper is organized as follows: Secs. 2 and 3 are devoted to prove Theorems 1.1 and 1.2, respectively.

2. Proof of Theorem 1.1

2.1. *Local bifurcation*

In this subsection we show the direction of bifurcation from the trivial solution for both cases f_1 and f_2. For that, we write the nonlinearity of the following manner

$$f(u) = u \mp u^2 - K\frac{u}{1+u} = u(1-K) + u^2\left(\frac{K}{1+u} \mp 1\right).$$

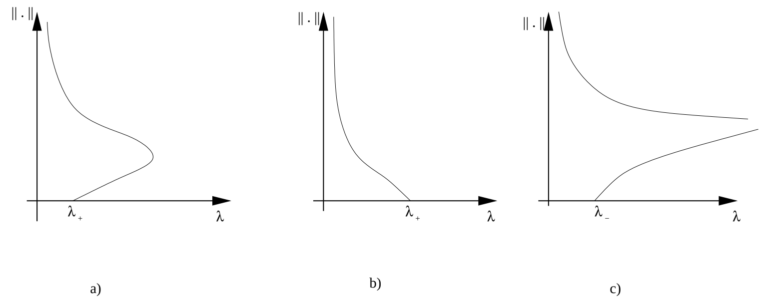

Figure 2. Bifurcation diagrams for $f = f_2$: a) $K < -1$; b) $K \in [-1, 1)$; c) $K > 1$.

It is clear that to study (1) is equivalent to find zeros of $\mathcal{L}(\lambda)u - N(\lambda, u) = 0$, where

$$\mathcal{L}(\lambda)u := u - \lambda(-\Delta)^{-1}m(x)(1-K)u,$$
$$N(\lambda, u) := \lambda(-\Delta)^{-1}m(x)u^2\left(\frac{K}{1+u} \mp 1\right).$$

We can prove that

$$N(\mathcal{L}(\lambda_+)) = \text{Span} < \varphi^+ > \quad \text{and} \quad \frac{d}{d\lambda}\mathcal{L}(\lambda_+)\varphi^+ \notin R(\mathcal{L}(\lambda_+)) \qquad (7)$$

where, given any linear continuous operator L, $N[L]$ and $R[L]$ stand for the null space and the range of L, respectively, and

$$-\Delta\varphi^+ = \lambda_+(m)m(x)\varphi^+ \quad \text{in } \Omega, \qquad \varphi^+ = 0 \quad \text{on } \partial\Omega. \qquad (8)$$

The first equality of (7) is trivial, for the second expression we need the following result.

Lemma 2.1. *For any $p \geq 2$ we have that*

$$\int_\Omega m(x)(\varphi^+)^p > 0.$$

Proof: Multiplying (8) by $(\varphi^+)^{p-1}$ we get

$$\lambda_+(m)\int_\Omega m(x)(\varphi^+)^p = \int_\Omega(-\Delta\varphi^+)(\varphi^+)^{p-1} = (p-1)\int_\Omega |\nabla\varphi^+|^2(\varphi^+)^{p-2} > 0.$$

◇

Now, we show (7). Assume that there exists u such that

$$\frac{d}{d\lambda}\mathcal{L}(\lambda_+)\varphi^+ = -(-\Delta)^{-1}m(x)(1-K)\varphi^+ = u - (-\Delta)^{-1}m(x)\lambda_+(1-K)u,$$

then

$$(-\Delta - \lambda_+(m)m(x))u = -(1-K)m(x)\varphi^+,$$

and so, multiplying by φ^+ we get a contradiction using Lemma 2.1.

Now, we can apply the Crandall-Rabinowitz Theorem [15] and conclude that there exists $\delta > 0$ such that in a neighborhood of $(\lambda_+, 0)$ the nontrivial solutions of (1) are of the form

$$u(s) = s\varphi^+ + s^2\varphi_2 + s^3\varphi_3 + o(s^3),$$
$$\lambda(s) = \lambda_+ + s\lambda_1 + s^2\lambda_2 + o(s^2).$$

Introducing these terms in (1), using (8) and a Taylor expression of the function $1/(1+u(s))$, we get

$$(-\Delta - \lambda_+(m)m(x))\varphi_2 = \lambda_+ m(x)(\varphi^+)^2(K \mp 1) + \lambda_1 m(x)(1-K)\varphi^+,$$

and so,

$$\lambda_1 = -\frac{\lambda_+(K \mp 1)}{1-K}\frac{\int_\Omega m(x)(\varphi^+)^3}{\int_\Omega m(x)(\varphi^+)^2}. \tag{9}$$

Observe that in the particular case $f = f_2$ and $K = -1$, $\lambda_1 = 0$, and so we have to calculate λ_2. It can be proved that

$$\lambda_2 = -\frac{\lambda_+}{2}\frac{\int_\Omega m(x)(\varphi^+)^4}{\int_\Omega m(x)(\varphi^+)^2}. \tag{10}$$

From (9) and (10), we conclude the paragraph (2) of Theorem 1.1. Analogously it can be treated the case λ_-.

2.2. *Non-existence results*

Lemma 2.2. *Assume $f = f_1$ and $K > 1$ or $f = f_2$ and $K < 1$. Then, there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$ (1) does not have positive solutions.*

Proof: Assume $f = f_1$ and $K > 1$. Firstly observe that

$$h(x) := x\left(\frac{K}{1+x} - 1\right) \leq (\sqrt{K} - 1)^2, \qquad \forall x \geq 0. \tag{11}$$

Let u be a positive solution of (1). Then, using the monotony of the principal eigenvalue with respect to the domain and (11) we get

$$\begin{aligned} 0 &= \sigma_1[-\Delta - \lambda m(x)(1-K) - \lambda m(x)u(\frac{K}{1+u} - 1)] < \\ &< \sigma_1^{M-}[-\Delta - \lambda m(x)((1-K) + (\sqrt{K}-1)^2)] = \\ &= \sigma_1^{M-}[-\Delta - \lambda m(x)2(1-\sqrt{K})], \end{aligned}$$

which is an absurdum for λ large.

Now, assume $f = f_2$ and $K < 1$. In this case,

$$x(\frac{K}{1+x}+1) \geq 0, \qquad \text{if } K \geq -1, \quad \forall x \geq 0,$$
$$x(\frac{K}{1+x}+1) \geq -(\sqrt{-K}-1)^2, \qquad \text{if } K < -1, \quad \forall x \geq 0.$$

So, if $-1 \leq K < 1$ we have

$$0 = \sigma_1[-\Delta - \lambda m(x)(1-K) - \lambda m(x)u(\frac{K}{1+u}+1)] < \sigma_1^{M+}[-\Delta - \lambda m(x)(1-K)];$$

on the other hand, for $K < -1$,

$$0 = \sigma_1[-\Delta - \lambda m(x)(1-K) - \lambda m(x)u(\frac{K}{1+u}+1)] < \sigma_1^{M+}[-\Delta - \lambda m(x)2\sqrt{-K}],$$

in both cases a contradiction for large λ. ◇

2.3. *Multiplicity results*

To obtain multiplicity results, we include (1) in the more general equation

$$\begin{cases} -\Delta u = \mu m(x)(1-K)u + \lambda m(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where g satisfies

$$(H_g) \qquad g(0) = g'(0) = 0, \quad g''(u) < 0, \quad \lim_{s\to+\infty} \frac{g(s)}{s^2} = \beta < 0.$$

Problem (12) has attracted a great deal of attention during last years (see for example [1], [3], [5], [6], [18] and [24]) when $m \equiv 1$ in the first term on the right-hand side of (12) and in [11], [12] and [13] with the right-hand side of the form $\mu h(x)u + g(x)u^p$ and restrictive conditions on h and g which are not satisfied in our case. In [16] was proved (see Fig. 3):

Proposition 2.1. *Assume that g satisfies (H_g), (6), $K \neq 1$ and fix $\lambda > 0$. Denote by*

$$\Lambda_+ := \lambda_+(m(x)(1-K)), \qquad \Lambda_- := \lambda_-(m(x)(1-K)).$$

Then, (12) *possesses a positive solution if* $\mu > \Lambda_-$. *Moreover, from the trivial solution* $u = 0$ *emanate two unbounded in* $\mathbb{R} \times C(\overline{\Omega})$ *continua of positive solutions* $\mathcal{C}_+ := \{(\mu, u_\mu)\}$ *and* $\mathcal{C}_- := \{(\mu, w_\mu)\}$ *at* $\mu = \Lambda_+$ *and* $\mu = \Lambda_-$*, respectively. Both continua bifurcate to the right and* $\Pi(\mathcal{C}_-) \supset (\Lambda_-, +\infty)$, $\Pi(\mathcal{C}_+) = (\Lambda_+, +\infty)$. *Finally, for* $\mu > \Lambda_+$, u_μ *is linearly asymptotically stable and* $u_\mu \neq w_\mu$.

Remark 2.1. Observe that for $K < 1$,

$$\Lambda_+ = \lambda_+ \quad \text{and} \quad \Lambda_- = \lambda_-,$$

and for $K > 1$,

$$\Lambda_+ = \lambda_- \quad \text{and} \quad \Lambda_- = \lambda_+.$$

Indeed, for example for $K > 1$, it follows that

$$\Lambda_+ = \lambda_+(m(x)(1-K)) = \frac{\lambda_+(-m(x))}{K-1} = \frac{-\lambda_-(m(x))}{K-1} = \frac{\lambda_-(m(x))}{1-K} = \lambda_-.$$

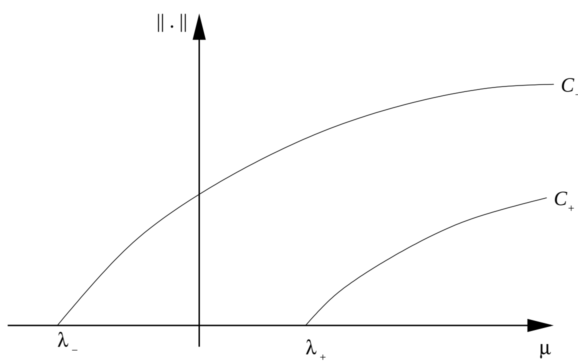

Figure 3. Bifurcation diagram for (12) and $K < 1$.

2.4. *Proof of Theorem 1.1:*

Before proving the result, we generalize a well-known result for $m \equiv 1$. The proof is coming from [8].

Lemma 2.3. *Assume that* f *is a regular function and* $f(0) = 0$. *Let* u_0 *be a positive solution of* (1) *such that* $f(u_0) > 0$*, it holds:*

(1) If $f''(u_0) < 0$, then u_0 is linearly asymptotically stable.
(2) If $f''(u_0) > 0$, then u_0 is unstable.

Proof: We have to calculate the sign of the eigenvalue $\sigma_1[-\Delta - \lambda m(x) f'(u_0)]$. Take $\psi := f(u_0) > 0$, then

$$(-\Delta - \lambda m(x) f'(u_0))\psi = -f''(u_0)|\nabla u_0|^2.$$

So, if f is concave (resp. convex) the function ψ is a supersolution (resp. subsolution) of $-\Delta - \lambda m(x) f'(u_0)$, and then (see [23]) $\sigma_1[-\Delta - \lambda m(x) f'(u_0)] > 0$ (resp. < 0). ◇

The following result is proved in Theorem 3.4 of [3] and provides us with *a priori* bounds for the positive solutions of (1).

Lemma 2.4. *Assume* (6)*. If (λ, u) is a positive solution of* (1) *and $\lambda \in J$, where J is a compact subset such that $J \subset (0, \infty)$, then there exists a positive constant C (independent from λ) such that*

$$\|u\|_\infty \leq C.$$

Finally, the following result is proved in [17].

Lemma 2.5. *Assume that $\Sigma \subset I \times C_0^2(\Omega)$, $I \subset \mathbb{R}$ an interval, is a connected set of positive solutions of* (1)*. Consider $\overline{u} : I \mapsto C_0^2(\Omega)$ a continuous map of supersolution for each $\lambda \in I$, but not a solution. If $u_0 < \overline{u}(\lambda_0)$ for some $(\lambda_0, u_0) \in \Sigma$, then $u < \overline{u}(\lambda)$ for all $(\lambda, u) \in \Sigma$.*

We are ready to prove the result. By subsec. 2.1 we know that there exists bifurcation from the trivial solution at $\lambda = \lambda_+$ or $\lambda = \lambda_-$ when $K < 1$ or $K > 1$, respectively. Moreover, we can apply Theorem 6.4.3 of [25], and conclude that from $\lambda = \lambda_+$ or $\lambda = \lambda_-$ bifurcates an unbounded continuum $\mathcal{C}$ of positive solutions of (1). We would like to remark that the a detailed proof that $\mathcal{C}$ is unbounded and it does not satisfy the other alternatives of the above mentioned result will be presented elsewhere.

Now assume $f = f_1$ and $K < 1$. It is clear that

$$\overline{u} := \sqrt{1 - K}$$

is a supersolution of (1). So, we can apply Lemma 2.5 (taking $\lambda_0 = \lambda_+$) and conclude that

$$\text{for all } (\lambda, u_\lambda) \in \mathcal{C}, \text{ we have that } u_\lambda < \sqrt{1 - K}. \tag{13}$$

Moreover, $f_1(u_\lambda) > 0$ and $f_1''(u_\lambda) < 0$, and so by Lemma 2.3 we get that u_λ is linearly asymptotically stable.

Now, we are going to apply Proposition 2.1. Recall that in this case $\Lambda_+ = \lambda_+$ and $\Lambda_- = \lambda_-$. Taking as

$$g(u) = u^2(\frac{K}{1+u} - 1),$$

we obtain a positive solution for $\mu = \lambda$ and $\lambda \in (0, \lambda_+]$ and at least two positive solutions for $\lambda > \lambda_+$.

Similarly, it can be considered the case $f = f_2$ and $K > 1$. Indeed, we only have to write $\mu m(x)(1-K)u + \lambda m(x)u^2(K/(1+u)+1)$ as

$$\mu(-m(x))(K-1)u + \lambda(-m(x))u^2(-K/(1+u) - 1).$$

Observe that $g(u) = u^2(-K/(1+u) - 1)$ satisfies (H_g) for $K > -1$, and so, Proposition 2.1 is true for

$$\Lambda_+ = \lambda_+(-m(x)(K-1)), \quad \text{and} \quad \Lambda_- = \lambda_-(-m(x)(K-1)).$$

And, since $K > 1$ it follows by Remark 2.1 that $\Lambda_+ = \lambda_-$.

The paragraphs (4) and (5) follow easily from the existence of $\mathcal{C}$ and Lemmas 2.2 and 2.4.

In order to prove paragraph (6), assume that there exist a sequence $(\lambda_n, u_n)_{n\in\mathbb{N}}$ of positive solution with $\lambda_n \to 0$ and $\|u_n\|_\infty \le C$ for some $C > 0$. Since there does not exist positive solution of (1) for $\lambda = 0$, we obtain that $\|u_n\|_\infty \to 0$. We claim that this is impossible. Indeed, we define

$$w_n = \frac{u_n}{\|u_n\|_\infty},$$

then w_n is uniformly bounded and, by passing to a suitable sequence again denoted by w_n, $w_n \to w^*$ as $n \to \infty$ for some $w^* \in C(\overline{\Omega})$ with $\|w^*\|_\infty = 1$. But,

$$-\Delta w_n = \lambda_n m(x) \frac{f(u_n)}{\|u_n\|_\infty},$$

and so $-\Delta w^* = 0$, which is an absurd. This concludes the proof. ◇

3. The particular case $K = 1$

In this case, the bifurcation from the trivial solution disappears. Consider

$$\begin{cases} -\Delta u = \mu u + \lambda m(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where

$$g(u) = u^2(\frac{1}{1+u} - 1) \qquad \text{or} \qquad g(u) = u^2(\frac{1}{1+u} + 1).$$

Proposition 3.1. *There exists a positive solution of* (14) *for* $\mu = 0$.
In particular, for all $\lambda > 0$ *there exists a positive solution of* (1).

Proof: It easy to prove that this problem is in the setting of some works, see for example [3] and references therein, and then there exists an unbounded continuum $\mathcal{S}$ of positive solutions of (14) bifurcating from $\mu = \sigma_1[-\Delta]$ and it satisfies that $\Pi(\mathcal{S}) \supset (-\infty, \sigma_1[-\Delta])$ (see Theorem 7.1 in [3]). This concludes the proof. ⋄

References

1. S. Alama, G. Tarantello, *On the semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. Partial Differential Equations, **1** (1993), 439-475.
2. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., **18** (1976), 620-709.
3. H. Amann, J. López-Gómez, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems*, J. Differential Equations, **146** (1998), 336-374.
4. A. Ambrosetti, P. Hess, *Positive solutions of asymptotically linear elliptic eigenvalue problems*, J. Math. Anal. Appl., **73** (1980), 411-422.
5. H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, *Superlinear indefinite problems and nonlinear Liouville theorems*, Topol. Methods Nonlinear Anal., **4** (1994), 59-78.
6. H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA Nonlinear Differential Equations Appl., **2** (1995), 553-572.
7. J. Blat, K. J. Brown, *Global bifurcation of positive solutions in some system of elliptic equations*, SIAM J. Math. Anal., **17** (1986) 1339-1353.
8. K. J. Brown, P. Hess, *Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem*, Differential Integral Equations, **3** (1990), 201-207.
9. K. J. Brown, B. Ko, *The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problem*, Nonlinear Anal., **39** (2000), 587-597.
10. N. Y. Các, J. A. Gatica, Y. Li, *Positive solutions to semilinear problems with coefficient that changes sign*, Nonlinear Anal., **37** (1999), 501-510.
11. S. Cano-Casanova, *Compact components of positive solutions for superlinear indefinite elliptic problems of mixed type*, to appear in Topol. Meth. Nonl. Anal.
12. S. Cano-Casanova, J. López-Gómez, M. Molina-Meyer, *Isolas*, submitted.
13. S. Cingolani, J. L. Gámez, *Positive solutions of a semilinear elliptic equation on* $\mathbb{R}^N$ *with indefinite nonlinearity*, Adv. Differential Equations, **1** (1996), 773-791.
14. P. Clément, G. Sweers, *Existence and multiplicity results for a semilinear*

elliptic eigenvalue problem, Annal. Scuola Norm. Sup. Pisa, Cl Sci., **4** (1987), 97-121.
15. M. G. Crandall, P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal., **8** (1971), 321-340.
16. M. Delgado, A. Suárez, *On the existence and multiplicity of positive solutions for some indefinite nonlinear eigenvalue problem*, to appear in Proc. Am. Math. Soc.
17. J. L. Gámez, *Sub- and super-solutions in bifurcation problems*, Nonlinear Anal., **28** (1997), 625-632.
18. R. Gómez-Reñasco, J. López-Gómez, *The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations*, J. Differential Equations, **167** (2000), 36-72.
19. P. Hess, T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differential Equations, **5** (1980), 999-1030.
20. P. Korman, J. Shi, *New exact multiplicity results with an application to a population model*, Proc. Royal Soc. Edin., **131A** (2001), 1167-1182.
21. P. L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev., **24** (1982), 441-467.
22. Z. Liu, *Positive solutions of a class of nonlinear elliptic eigenvalue problems*, Math. Z., **242** (2002), 663-686.
23. J. López-Gómez, *The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems*, J. Differential Equations, **127** (1996), 263-294.
24. J. López-Gómez, *On the existence of positive solutions for some indefinite superlinear elliptic problems*, Comm. Partial Differential Equations, **22** (1997), 1787-1804.
25. J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, Research Notes in Mathematics 426, CRC Press, Boca Raton, 2001.
26. T. Ouyang, J. Shi, *Exact multiplicity of possitive solutions for a class of semilinear, Parts I and II*, J. Differential Equations, **146** (1998), 121-156 and **158** (1999), 94-151.