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# Combining linear and fast diffusion in a nonlinear elliptic equation

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**Abstract** In this paper we analyse an elliptic equation that combines linear and nonlinear fast diffusion with a logistic type reaction function. We prove existence and non-existence results of positive solutions using bifurcation theory and sub-supersolution method. Moreover, we apply variational methods to obtain a pair of ordered positive solutions.

**Keywords** Non-linear diffusion · Bifurcation · Sub-supersolution method · Variational Methods

**Mathematics Subject Classification (2000)** MSC 35B32 · 35J20 · 35J25 · 35J60

## 1 Introduction

In this paper we study the set of positive solutions of the following elliptic problem with nonlinear diffusion

$$\begin{cases} -\Delta(u + a(x)u^r) = \lambda u - bu^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega$ is a bounded and smooth domain of $\mathbb{R}^N, N \geq 1$, $\lambda \in \mathbb{R}$, $b \geq 0$, $0 < r < 1 < p$ and $a : \Omega \to [0, \infty)$ is a non-trivial regular function that can vanish on regions of $\Omega$. Thus, we will denote by

$$\Omega_{a+} := \{x \in \Omega;\ a(x) > 0\}$$

and

$$\Omega_{a0} := \Omega \setminus \overline{\Omega}_{a+}.$$

Once that $r < 1$, equation (1) provides us with the steady states of a porous medium equation where diffusion is linear in $\Omega_{a0}$ and fast in $\Omega_{a+}$. Thus, in the context of population dynamics, $\Omega$ represents an habitat, $u(x)$ the density of the population of a species at $x \in \Omega$ and $-\Delta(u+a(x)u^r)$ describes the diffusion of the species, that is, the spacial movement, which is fast in some region of $\Omega$ ($\Omega_{a+}$) and linear (or simple) in other ($\Omega_{a0}$). The function $\lambda u - bu^p$ is called logistic reaction term and, from biological point of view, $\lambda$ the intrinsic rate of natural increase of the species and $b$ denotes the maximum density supported locally by resources available, that is, the carrying capacity.

In particular, when $a \equiv 0$ in $\Omega$ (i.e., $\Omega_{a0} = \Omega$), (1) reduces to the classical linear eigenvalue problem for the Laplacian operator under Dirichlet boundary conditions in $\Omega$ if $b = 0$ and the classical logistic equation with linear diffusion if $b > 0$. Subsequently, for any potential $V \in L^\infty(\Omega)$, we shall denote by $\lambda_1[-\Delta + V; \Omega]$ the principal eigenvalue of $-\Delta + V$ in $\Omega$ under homogeneous Dirichlet boundary conditions. By simplicity, when $V \equiv 0$, we will denote

$$\lambda_1 = \lambda_1[-\Delta; \Omega].$$

Thus, in the case $a = b = 0$, according to the classical eigenvalue theory, (1) possesses a positive solution if, and only, if $\lambda = \lambda_1$. Actually, in such case, all positive solutions are the vector space generated by the principal eigenfunction. The study of case $b > 0$ began with works of [6]. In this paper, the authors proved that there exists a unique positive solution if, and only if, $\lambda > \lambda_1$ and this positive solution attracts all the positive solution of the associated parabolic problem (see also [5], [11]). Hence, since the case $a \equiv 0$ is well-know, in this paper we consider only the $\Omega_{a0} \neq \Omega$.

When $\Omega_{a0} \neq \emptyset$, another eigenvalue problem plays an important role on the existence of positive solutions of (1). Specifically, the problem

$$\begin{cases} -\Delta u = \lambda \mathcal{X}_{\Omega_{a0}} u \text{ in } \Omega, \\ u = 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

The existence of the principal eigenvalue of this problem is guaranteed by, for instance, [7] and [10]. Actually, denoting by $\lambda_{a0}$ the principal eigenvalue of (2), it is given by the following variational characterization

$$\lambda_{a0} = \min_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\varphi\|^2_{H_0^1}}{|\varphi|^2_{L^2(\Omega_{a0})}}. \tag{3}$$

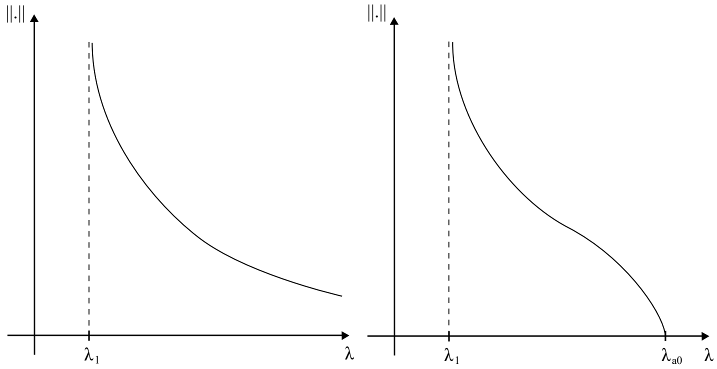


**Fig. 1** Bifurcation diagrams in the case $b = 0$ for $\Omega_{a0} = \emptyset$ and $\Omega_{a0} \neq \emptyset$, respectively.

This eigenvalue appears in problems that combine other types of nonlinear diffusion. For instance, [8] the authors analyzed the following problem

$$\begin{cases} -\Delta(u^{m(x)}) = \lambda u \text{ in } \Omega, \\ u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $m$ is a regular function with $m > 1$ in a smooth subdomain $\Omega_m$ of $\Omega$ with $\overline{\Omega}_m \subset \Omega$ and $m \equiv 1$ in $\Omega \setminus \Omega_m$, that is, there exists a zone of linear diffusion, $\Omega \setminus \overline{\Omega}_m$, and a zone of nonlinear diffusion, $\Omega_m$. The authors show that (4) possesses a positive solutions if, and only if, $\lambda \in (0, \lambda_m)$, where $\lambda_m$ is the principal eigenvalue of (2) with $\Omega \setminus \Omega_m$ instead of $\Omega_{a0}$. In fact, $\lambda = 0$ is a bifurcation point from the trivial solution and $\lambda_m$ is a bifurcation point from infinity.

To emphasize the dependence of the parameter $\lambda$, we will refer to (1) as $(1)_\lambda$. Thus, defining $\lambda_{a0} = \infty$ if $\Omega_{a0} = \emptyset$, our first main result is the following:

**Theorem 1** *If $b = 0$ in $\Omega$, then $(1)_\lambda$ possesses a positive solution if, and only if, $\lambda \in (\lambda_1, \lambda_{a0})$. Moreover, any family of positive solutions $u_\lambda$ of $(1)_\lambda$ satisfies*

$$\lim_{\lambda \to \lambda_1} \|u_\lambda\|_0 = \infty \tag{5}$$

*and*

$$\lim_{\lambda \to \lambda_{a0}} \|u_\lambda\|_0 = 0 \quad \text{if } \lambda_{a0} < \infty. \tag{6}$$

In Figure 1 we have represented the corresponding bifurcation diagram of positive solutions of $(1)_\lambda$ with $b = 0$. For the case $b > 0$ the bifurcation from infinity disappears, in fact, we have

**Theorem 2** *If $b > 0$, consider*

$$\Lambda_b = \{\lambda \in \mathbb{R};\ (1)_\lambda \text{ has a positive solution}\}.$$

*Then $\Lambda_b \neq \emptyset$ and denoting by $\lambda^*(b) = \inf \Lambda_b$, we have $\lambda_1 < \lambda^*(b) \leq \lambda_{a0}$. Moreover,*

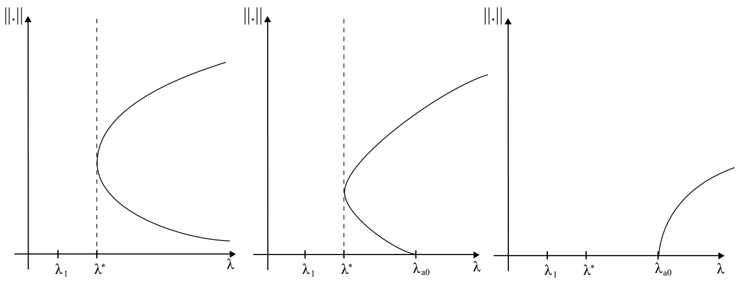


**Fig. 2** Possible bifurcation diagrams. From the left to the right, the case $\Omega_{a0} = \emptyset$. the case $\Omega_{a0} \neq \emptyset$ with subcritical bifurcation and the case $\Omega_{a0} \neq \emptyset$ with supercritical bifurcation.

*(a)* *If $\Omega_{a0} = \emptyset$, then* $(1)_\lambda$ *possesses a positive solution for all $\lambda \geq \lambda^*$.*
*(b)* *If $\Omega_{a0} \neq \emptyset$, then $\lambda_{a0}$ is a bifurcation point of (1) from the trivial solution and it is the only one for positive solutions. Furthermore, if the direction of the bifurcation is subcritical (resp. supercritical), then* $(1)_\lambda$ *possesses a positive solution for all $\lambda \geq \lambda^*$ (resp. $\lambda > \lambda^*$).*
*(c)* *In the case that $\lambda^* < \lambda_{a0}$, then for each $\lambda \in (\lambda^*, \lambda_{a0})$,* $(1)_\lambda$ *possesses two ordered positive solutions, that is, $w_\lambda$ and $v_\lambda$ positive solutions of* $(1)_\lambda$ *satisfying*
$$w_\lambda < v_\lambda.$$

Figure 2 shows some admissible situations within the setting of Theorem 2. We point out that in the case $b > 0$ we do not have bifurcation from infinity and if $\Omega_{a0} = \emptyset$ we also have not bifurcation from trivial solutions, and to conclude existence of positive solution we use the sub-supersolution method. For the case $\Omega_{a0} \neq \emptyset$, in Proposition 4 we give conditions on $p, r$, $a$ and $b$ that provide us the direction of the bifurcation. This result show us an effect of the interaction between the fast diffusion $u + a(x)u^r$ and the logistic non-linearity $\lambda u - bu^p$. Specifically, if $1/r < p$, then bifurcation from trivial solution is subcritical, while if $1/r > p$ it is supercritical. In the case $1/r = p$, $a$ and $b$ affect the direction of the bifurcation according to (20) and (21).

The next result gives us more information about the positive solutions with respect to the parameter $b$:

**Theorem 3** *Assume $b > 0$.*

*(a)* *For each $\lambda \geq \lambda^*(b)$, (1) possesses a maximal solution. That is, denoting it by $W_{\lambda(b)}$, then any positive solution, $w$, of (1) satisfies*
$$w \leq W_{\lambda(b)}.$$
*Moreover, if $\lambda^* \leq \mu < \lambda$, then $W_{\mu(b)} < W_{\lambda(b)}$.*
*(b)* *It holds*
$$\lambda^*(b) \to \lambda_1 \quad \text{as } b \to 0. \tag{7}$$
*(c)* *We have*
$$\lim_{b \to 0} \|W_{\lambda(b)}\|_0 = \infty \quad \forall \lambda(b) > \lambda^*(b). \tag{8}$$

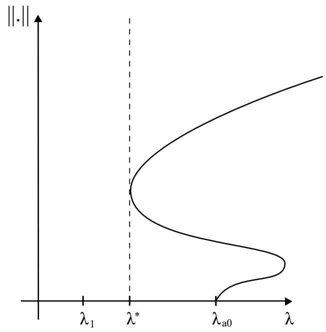


**Fig. 3** An admissible bifurcation diagram when $b > 0$ is small, $\Omega_{a0} \neq \emptyset$ and the bifurcation is supercritical.

As a consequence, an interesting bifurcation diagram is admissible in case that $b$ is small and the bifurcation is supercritical. The paragraph (b) of Theorem 3 gives us that, for $b > 0$ sufficiently small, $\lambda^*(b) < \lambda_{a0}$. Then, if the bifurcation from the trivial solution is supercritical, the continuum of positive solutions which emanates from $\lambda_{a0}$ goes to the right and, on the other hand, there exists positive solutions for $\lambda \in (\lambda^*(b), \lambda_{a0})$. Then, this leads us to a bifurcation diagram as in Figure 3.

The distribution of this paper is the following: in Section 2 we collect some useful previous results. Section 3 is dedicated to proof of Theorem 1. Theorems 2 and 3 are proved in Section 4, with the exception of the existence of a second positive solution, which will be considered in Section 5.

## 2 Previous results

We will present some basic results that will be used throughout this work. First, to deal with (1), we introduce the following change of variable

$$I(x,u) = w = u + a(x)u^r \Leftrightarrow u = q(x,w)$$

getting the following equivalent problem

$$\begin{cases} -\Delta w = \lambda q(x,w) - bq(x,w)^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Since we are interested in positive solutions of $(1)_\lambda$, we can define

$$q(x,s) = 0, \quad \forall x \in \Omega, s \leq 0.$$

Thus, by the Strong Maximal Principle, any non-trivial solution of $(1)_\lambda$ is in fact strictly positive. Hence $u > 0$ is a positive solution of $(1)_\lambda$ if, and only if, $w = u + a(x)u^r$ is a positive solution of (9). Therefore, we analyze the equivalent problem (9). Again, we will refer to (9) as $(9)_\lambda$.

Let us prove some useful properties of the function $q(x,s)$

**Lemma 1** *1. For each $x \in \Omega$, the map $s \mapsto q(x,s),\ s \geq 0$ is of class $\mathcal{C}^1$.*

*2. For all $x \in \Omega$, the map*

$$s \mapsto \frac{q(x,s)}{s} \quad s \geq 0,$$

*is non-decreasing and satisfies*

$$\mathcal{X}_{\Omega_{a0}}(x)s \leq q(x,s) \leq s \quad \forall x \in \Omega, \tag{10}$$

$$\lim_{s\to 0} \frac{q(x,s)}{s} = \mathcal{X}_{\Omega_{a0}}(x) = \begin{cases} 0 \text{ if } a(x) > 0, \\ 1 \text{ if } a(x) = 0. \end{cases} \tag{11}$$

*and*

$$\lim_{s\to\infty} \frac{q(x,s)}{s} = 1. \tag{12}$$

*3. For all $x \in \Omega$, the map*

$$s \mapsto \frac{q(x,s)^p}{s}$$

*is increasing and satisfies*

$$\lim_{s\to 0} \frac{q(x,s)^p}{s} = 0, \tag{13}$$

*and*

$$\lim_{s\to\infty} \frac{q(x,s)^p}{s} = +\infty \tag{14}$$

*Proof* 1. Since $q(x,\cdot)$ is the inverse function of $I(x,s) = s + a(x)s^r$, we get

$$q'(x,s) = \frac{1}{1 + ra(x)q(x,s)^{r-1}}.$$

Therefore $q'(x,s)$ is continuous in $(0,\infty)$. On the other hand,

$$\lim_{s\to 0^+} q'(x,s) = \lim_{s\to 0^+} \frac{1}{1 + a(x)rq(x,s)^{r-1}} = \mathcal{X}_{\Omega_{a0}}(x) = q'(x,0),$$

showing the continuity at 0.

2. Observe that

$$I(x, q(x,s)) = s = q(x,s) + a(x)q(x,s)^r,$$

and therefore

$$\frac{q(x,s)}{s} = \frac{1}{1 + a(x)q(x,s)^{r-1}}, \tag{15}$$

where we deduce (10). Moreover, since $s \mapsto q(x,s)$ is increasing and $r < 1$, (15) provides that $q(x,s)/s$ is non-decreasing.

To calculate the limits (11)–(12), observe that if $a(x) = 0$ we have $q(x,s)/s = 1$ and it is immediate. If $a(x) > 0$, using

$$\lim_{s\to 0} q(x,s) = 0 \quad \text{and} \quad \lim_{s\to\infty} q(x,s) = \infty,$$

(15) gives

$$\lim_{s\to 0} \frac{q(x,s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{q(x,s)}{s} = 1.$$

3. Analogously, observe that

$$\frac{q(x,s)^p}{s} = \frac{1}{q(x,s)^{1-p} + a(x)q(x,s)^{r-p}}. \tag{16}$$

By the monotonicity of $s \mapsto q(x,s)$ and since $r < 1 < p$, it follows that $q(x,s)/s$ is increasing in $s$, for all $x \in \Omega$. Moreover, letting $s \to 0$ and $s \to \infty$ in (16), yields to (13)–(14).

The following function will play a crucial role in our exposition

$$\mu(\lambda) := \lambda_1[-\Delta - \lambda \mathcal{X}_{\Omega_{a0}}; \Omega], \quad \lambda \in \mathbb{R}. \tag{17}$$

It is well defined because $-\lambda\mathcal{X}_{\Omega_{a0}} \in L^\infty(\Omega)$ for all $\lambda \in \mathbb{R}$ and the next result provides some properties of this function and that will be useful throughout the work.

**Proposition 1** *The function $\mu$ defined in (17) is decreasing and possesses a unique zero, say $\lambda_{a0}$. Moreover, $\mu(\lambda) > 0$ if, and only if, $\lambda < \lambda_{a0}$. Furthermore, it satisfies*

$$\lambda_1 < \lambda_{a0}, \tag{18}$$

*and $\lambda_{a0}$ is the principal eigenvalue of (2).*

*Proof* Observe that, by the monotonicity of $\lambda_1[-\Delta - \lambda\mathcal{X}_{\Omega_{a0}}; \Omega]$ with respect of the potential, we get

$$\lambda_1 - \lambda < \mu(\lambda) < \lambda_1[-\Delta; \Omega_{a0}] - \lambda,$$

consequently, $\mu(\lambda) \to -\infty$ as $\lambda \to +\infty$ and

$$\lambda_1 - \lambda_{a0} < \mu(\lambda_{a0}) = 0.$$

Moreover, by [9], $\mu'(\lambda) < 0$ (see [10] for further details). Therefore, since $\mu$ is a continuous function and $\mu(0) = \lambda_1[-\Delta; \Omega] > 0$, there exists a unique $\lambda_{a0} \in \mathbb{R}$, such that $\mu(\lambda_{a0}) = 0$. Furthermore, since $\mu$ is decreasing, it follows that $\mu(\lambda) > 0$ if, and only if, $\lambda < \lambda_{a0}$.

Finally, note that

$$\mu(\lambda_{a0}) = \lambda_1[-\Delta - \lambda_{a0}\mathcal{X}_{\Omega_{a0}}; \Omega] = 0$$

is equivalent to say that $\lambda_{a0}$ is the principal eigenvalue of (2).

*Proof* Observe that, by the monotonicity of $\lambda_1[-\Delta - \lambda\mathcal{X}_{\Omega_{a0}};\Omega]$ with respect of the potential, we get

$$\lambda_1 - \lambda < \mu(\lambda) < \lambda_1[-\Delta;\Omega_{a0}] - \lambda,$$

consequently, $\mu(\lambda) \to -\infty$ as $\lambda \to +\infty$ and

$$\lambda_1 < \mu(0).$$

Moreover, by [9], $\mu'(\lambda) < 0$ (see [10] for further details). Therefore, since $\mu$ is a continuous function and $\mu(0) = \lambda_1 > 0$, there exists a unique $\lambda_{a0} \in \mathbb{R}$, such that $\mu(\lambda_{a0}) = 0$. Furthermore,

$$\lambda_1 - \lambda_{a0} < \mu(\lambda_{a0}) = 0$$

and, since $\mu$ is decreasing, it follows that $\mu(\lambda) > 0$ if, and only if, $\lambda < \lambda_{a0}$ and
Finally, note that

$$\mu(\lambda_{a0}) = \lambda_1[-\Delta - \lambda_{a0}\mathcal{X}_{\Omega_{a0}};\Omega] = 0$$

is equivalent to say that $\lambda_{a0}$ is the principal eigenvalue of (2).

To end this section, we will study an auxiliary problem that will provide us the existence of a maximal solution to $(9)_\lambda$ and a priori bound for positive solutions of $(9)_\lambda$. Specifically, consider the problem

$$\begin{cases} -\Delta w = \lambda w - bq(x,w)^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

**Proposition 2** *(19) possesses a positive solution if, and only if $\lambda > \lambda_1$. Moreover, it is unique if it exists and we will denote it by $\theta_\lambda$ and*

$$\theta_\mu \le \theta_\lambda \quad \text{if } \lambda_1 < \mu \le \lambda.$$

*Proof* If $w > 0$ is a solution of (19), then

$$\lambda = \lambda_1[-\Delta + bq(x,w)^p/w;\Omega] > \lambda_1[-\Delta;\Omega] = \lambda_1.$$

Consequently, $\lambda > \lambda_1$ is a necessary condition for the existence of positive solutions. Now, suppose $\lambda > \lambda_1$. To prove the existence of positive solution, observe that $(\varepsilon\varphi_1, K)$ is a pair of sub-supersolution of (19) for constants $\varepsilon > 0$ small and $K > 0$ large.

The uniqueness follows by Theorem 1 of [5], once that

$$s \mapsto \lambda - b\frac{q(x,s)^p}{s}$$

is decreasing for all $x \in \Omega$. Finally, the monotonicity with respect to $\lambda$ follows from the comparison principle.

**Corollary 1** *For any $\lambda \ge \mu > \lambda_1$, any positive solution $w_\mu$ of $(9)_\mu$ satisfies*

$$w_\mu \le \theta_\mu \le \theta_\lambda.$$

*Proof* Just observe that $w_\mu$ is a subsolution of (19) and $K$ sufficiently large is a supersolution. Hence, by the uniqueness of solution of (19), necessarily

$$w_\mu \le \theta_\mu \le \theta_\lambda.$$

## 3 Case $b = 0$.

This section is dedicated to study the case $b = 0$. To this, we use bifurcation techniques. Thus, we consider the map $\Phi_\lambda : \mathcal{C}_0(\overline{\Omega}) \longrightarrow \mathcal{C}_0(\overline{\Omega})$ defined by

$$\Phi_\lambda(w) = I - (-\Delta)^{-1}(\lambda q(x, w)),$$

here $(-\Delta)^{-1}$ is the inverse of Laplace operator under homogeneous Dirichlet boundary condition. Observe that $w \in \mathcal{C}_0(\overline{\Omega})$ is a positive solution of (9) if, and only if, $\Phi_\lambda(w) = 0$. Denoting by $\Sigma$ the closure of the set

$$\{(\lambda, w) \in \mathbb{R} \times \mathcal{C}_0(\overline{\Omega}) \text{ such that } \Phi_\lambda(w) = 0,\ w \neq 0\},$$

we get

**Proposition 3** *Suppose $b = 0$ in $\Omega$,*

1. *If there exists a positive solution of* $(9)_\lambda$*, then* $\lambda \in (\lambda_1, \lambda_{a0})$.
2. $\lambda_1$ *is the unique bifurcation point from the infinity of positive solutions of* $(9)_\lambda$*. Moreover, there exists a unbounded component* $\Sigma_\infty \subset \Sigma$ *such that*

$$\overline{\Sigma}_\infty = \left\{(\lambda, w) \text{ with } w \neq 0;\ \left(\lambda, \frac{w}{\|w\|_0^2}\right) \in \Sigma_\infty\right\} \cup \{(\lambda_1, 0)\}$$

*is connected and unbounded.*

*Proof* 1. If $w > 0$ is a solution of $(9)_\lambda$, we have

$$\begin{cases} \left[-\Delta - \lambda \dfrac{q(x, w)}{w}\right] w = 0, \text{ in } \Omega, \\ w = 0, \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Using (10), we obtain

$$0 = \lambda_1 \left[-\Delta - \lambda \frac{q(x, w)}{w}; \Omega\right] > \lambda_1[-\Delta - \lambda; \Omega] = \lambda_1 - \lambda.$$

In the case $\Omega_{a0} \neq \emptyset$, using again (10), we derive that

$$0 = \lambda_1 \left[-\Delta - \lambda \frac{q(x, w)}{w}; \Omega\right] < \lambda_1[-\Delta - \lambda \mathcal{X}_{\Omega_{a0}}; \Omega] = \mu(\lambda).$$

By the properties of function $\mu$, it follows that $\lambda < \lambda_{a0}$.

2. In view of (12) and since $f(\lambda, x, s) := \lambda q(x, s)$ satisfies $f(0, x, s) \equiv 0$ for all $x \in \Omega$ and $s \geq 0$, we can apply the Theorem 3.4 of [3] and get the results.

Proof of Theorem 1:

By Proposition 3 2., $\lambda_1$ is a bifurcation point of $(9)_\lambda$ from infinity and it is the only one for positive solutions. In order to prove the existence of solution for $\lambda \in (\lambda_1, \lambda_{a0})$, we will consider two cases: $\Omega_{a0} = \emptyset$ and $\Omega_{a0} \neq \emptyset$.

Case $\Omega_{a0} = \emptyset$: To conclude the results, it is sufficient to check the following:

*Claim:* for all compact set $\Lambda \subset [\lambda_1, \infty)$ there exists $\varepsilon > 0$ such that $(9)_\lambda$ has no positive solution with $(\lambda, w) \in \Lambda \times B_\varepsilon(0)$.

Indeed, because the global nature of $\Sigma_\infty$ implies that it is unbounded with respect to $\lambda$ and, since $(9)_\lambda$ has no positive solution for $\lambda < \lambda_1$ (Proposition 3), the result follows.

Let us prove the claim. Arguing by contradiction, there exists $(\lambda_n, w_n)$ a sequence of solutions of $(9)_{\lambda_n}$ such that $\lambda_n \in \Lambda$ for all $n \in \mathbb{N}$ and $\|w_n\|_0 \to 0$. Since $\Lambda$ is compact, up to subsequence if necessary, we have

$$(\lambda_n, w_n) \to (\lambda^*, 0) \quad \text{in } \mathbb{R} \times \mathcal{C}_0(\overline{\Omega})$$

From (11) and previous limit we get that for all $\delta > 0$, there exists $n_\delta \in \mathbb{N}$ such that

$$\frac{q(x, w_n)}{w_n} \leq \delta \quad \forall n > n_\delta.$$

Thus, since $(\lambda_n, w_n)$ is a solution of $(9)_{\lambda_n}$, we obtain

$$0 = \lambda_1 \left[ -\Delta - \lambda_n \frac{q(x, w_n)}{w_n}; \Omega \right] > \lambda_1[-\Delta - \lambda_n \delta; \Omega] = \lambda_1 - \lambda_n \delta \quad \forall n > n_\delta,$$

that is,

$$\lambda_n \delta > \lambda_1.$$

Letting $n \to \infty$ and thanks to $\lambda_n \to \lambda^* < \infty$, the above inequality provides $\lambda_1 \leq \lambda^* \delta$, for all $\delta > 0$, which is a contradiction.

Case $\Omega_{a0} \neq \emptyset$

In view of (11), we can apply Theorem 4.4 of [3] and obtain that $\lambda_{a0}$ is a bifurcation point from the trivial solution of positive solutions, and it is the only one in $\mathbb{R}_0^+$. Furthermore, there exists an unbounded component $\Sigma_0 \subset \Sigma$ meeting $\lambda_{a0}$. Once that these bifurcation points are unique, we get

$$\Sigma_\infty = \Sigma_0.$$

As a consequence, by global nature of these continuum, we obtain that there exist positive solutions for all $\lambda \in (\lambda_1, \lambda_{a0})$.

## 4 Case $b > 0$

In this section we will prove Theorems 2 and 3, except the existence of a second solution that will be treated in the next section.

First, denoting by $\varphi_{a0}$ the principal positive eigenfunction associated to $\lambda_{a0}$ with $\|\varphi_{a0}\|_0 = 1$, we have the following result of existence and non-existence of positive solutions.

**Proposition 4** *1. If $(9)_\lambda$ possesses a positive solution, then $\lambda > \lambda_1$.*

*2. If $\Omega_{a0} \neq \emptyset$, then $\lambda_{a0}$ is a bifurcation point of (9) from the trivial solution and it is the only one for positive solutions. Furthermore, the bifurcation is*

*(a) Subcritical if* $1/r < p$.
*(b) Subcritical if* $1/r = p$ *and*

$$\int_{\Omega_{a+}} \frac{\varphi_{a0}^{p+1}}{a(x)^p} > b \int_{\Omega_{a0}} \varphi_{a0}^{p+1}. \tag{20}$$

*(c) Supercritical if* $1/r = p$, $a(x)^{-p} \in L^1(\Omega_{a+})$ *and*

$$\int_{\Omega_{a+}} \frac{\varphi_{a0}^{p+1}}{a(x)^p} < b \int_{\Omega_{a0}} \varphi_{a0}^{p+1}. \tag{21}$$

*(d) Supercritical if* $1/r > p$.

*3. There exists* $\overline{\lambda} > \lambda_1$ *such that* $(9)_{\overline{\lambda}}$ *has a positive solution*

*Proof* The proof of first paragraph is similar to first one of Proposition 3. Thus, we will prove only 2 and 3.

We prove first the second paragraph. If $\Omega_{a0} \neq \emptyset$, by (11), we can apply the Theorem 4.4 of [3] to obtain that $\lambda_{a0}$ is the only bifurcation point from the trivial solution. To conclude the direction of bifurcation we will apply the paragraphs (i) and (ii) of Theorem 4.4 of [3] and argue as follows. Denote

$$g(\lambda, x, s) := \frac{\lambda q(x,s) - bq(x,s)^p - \lambda \mathcal{X}_{\Omega_{a0}}(x)s}{s^{1-\sigma}},$$

where $\sigma < 0$ to be chosen later.

(a) If $1/r < p$, we choose $\sigma = 1 - 1/r$. Thus, in $\Omega_{a+}$ we have

$$\begin{aligned} g(\lambda, x, s) &= \lambda \frac{(q(x,s)^r)^{1/r}}{(q(x,s) + a(x)q(x,s)^r)^{1/r}} - b\frac{(q(x,s)^{pr})^{1/r}}{(q(x,s) + a(x)q(x,s)^r)^{1/r}} \\ &= \lambda \frac{1}{(q(x,s)^{1-r} + a(x))^{1/r}} - b\frac{1}{(q(x,s)^{1-pr} + a(x)q(x,s)^{(1-p)r})^{1/r}} \end{aligned}$$

and, therefore,

$$\liminf_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda,x,s) = \frac{\lambda_{a0}}{a(x)^{1/r}} \quad \text{in } \Omega_{a+}.$$

On the other hand, in $\Omega_{a0}$ we have

$$g(\lambda,x,s) = \frac{\lambda s - bs^p - \lambda s}{s^{1/r}} = -bs^{p-1/r},$$

and, since $1/r < p$, we obtain that

$$\liminf_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda,x,s) = 0 \quad \text{in } \Omega_{a0}.$$

Consequently,

$$\underline{\mu}(x) \equiv \liminf_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda,x,s) \geq 0$$

and

$$\int_{\Omega} \underline{\mu}(x)\varphi_{a0}^{1/r+1} > 0.$$

Then, by Theorem 4.4 (i) of [3], the bifurcation of positive solutions at $\lambda = \lambda_{a0}$ is subcritical.

(b) If $1/r = p$, we choose $\sigma = 1 - p$. Thus, in $\Omega_{a+}$, we have

$$g(\lambda, x, s) = \lambda \frac{1}{(q(x,s)^{1-1/p} + a(x))^p} - b\left(\frac{q(x,s)}{s}\right)^p.$$

Implying that

$$\underline{\mu}(x) \equiv \liminf_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda, x, s) = \frac{\lambda_{a0}}{a(x)^p} \quad \text{in } \Omega_{a+}.$$

On the other hand, in $\Omega_{a0}$ we have

$$g(\lambda, x, s) = \frac{\lambda s - bs^p - \lambda s}{s^p} = -b.$$

Consequently,

$$\underline{\mu}(x) \equiv \liminf_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda, x, s) = \begin{cases} \dfrac{\lambda_{a0}}{a(x)^p} & \text{if } x \in \Omega_{a+}, \\ -b & \text{if } x \in \Omega_{a0}. \end{cases}$$

Therefore, $\underline{\mu}(x) \geq -b$ and (20) is equivalent to

$$\int_\Omega \underline{\mu}(x)\varphi_{a0}^{p+1} > 0.$$

Thus, by Theorem 4.4 (i) of [3], the bifurcation of positive solutions at $\lambda = \lambda_{a0}$ is subcritical.

(c) Analogously to the previous case, for $\sigma = 1 - p$ we have

$$\overline{\mu}(x) \equiv \limsup_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda, x, s) = \begin{cases} \dfrac{\lambda_{a0}}{a(x)^p} & \text{if } x \in \Omega_{a+}, \\ -b & \text{if } x \in \Omega_{a0}. \end{cases}$$

Once that $a(x)^{-p} \in L^1(\Omega_{a+})$, we get $\overline{\mu} \in L^1(\Omega)$ and since (21) is equivalent to

$$\int_\Omega \overline{\mu}(x)\varphi_{a0}^{p+1} < 0.$$

Theorem 4.4 (ii) of [3] implies that the bifurcation of positive solutions at $\lambda = \lambda_{a0}$ is supercritical.

(d) If $1/r > p$, we choose $\sigma = 1 - p$. Thus, in $\Omega_{a+}$, we have

$$g(\lambda, x, s) = \lambda \frac{1}{(q(x,s)^{1-1/p} + a(x)q(x,s)^{r-1/p})^p} - b\left(\frac{q(x,s)}{s}\right)^p$$

and, since $1/r > p$,

$$\limsup_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda, x, s) = 0 \quad \text{in } \Omega_{a+}.$$

On the other hand, in $\Omega_{a0}$ we have

$$g(\lambda, x, s) = \frac{\lambda s - bs^p - \lambda s}{s^p} = -b.$$

Consequently,

$$\overline{\mu}(x) \equiv \limsup_{(\lambda,s)\to(\lambda_{a0},0^+)} g(\lambda, x, s) = -\mathcal{X}_{\Omega_{a0}} b \in L^1(\Omega)$$

and

$$\int_{\Omega} \overline{\mu}(x)\varphi_{a0}^{p+1} < 0.$$

Then, by Theorem 4.4 (ii) of [3], the bifurcation of positive solutions at $\lambda = \lambda_{a0}$ is supercritical.

To prove the third paragraph, note that the case $\Omega_{a0} \neq \emptyset$ is a immediate consequence of the second paragraph.

If $\Omega_{a0} = \emptyset$, then we can not apply the bifurcation theorem, thus we will use the method of sub-supersolution to prove the existence of positive solution for $\lambda > \lambda_1$ large.

To build the subsolution, denoting by $\varphi_1 > 0$, the eigenvalue associated to $\lambda_1$ with $\|\varphi_1\|_0 = 1$, it satisfies

$$\begin{aligned}\Delta(\varphi_1^m) &= m(m-1)\varphi_1^{m-2}|\nabla\varphi_1|^2 + m\varphi_1^{m-1}\Delta\varphi_1.\\ &= m(m-1)\varphi_1^{m-2}|\nabla\varphi_1|^2 - m\lambda_1\varphi_1^m.\end{aligned}$$

Therefore, $\underline{w} = \varphi_1^m$ is a subsolution of $(9)_\lambda$ provided that

$$-\Delta(\varphi_1^m) \le \lambda q(x, \varphi_1^m) - bq(x, \varphi_1^m)^p \quad \forall x \in \Omega,$$

once that $q(x, \varphi_1^m) > 0$ for all $x \in \Omega$, this inequality is equivalent to

$$\frac{m\varphi_1^m}{q(x,\varphi_1^m)}\left((1-m)\frac{|\nabla\varphi_1|^2}{\varphi_1^2} + \lambda_1\right) + bq(x,\varphi_1^m)^{p-1} \le \lambda \quad \forall x \in \Omega. \tag{22}$$

Note that the term $bq(x, \varphi_1^m)$ is bounded. Let us show that the remaining terms are also bounded. Indeed, observe that

$$(1-m)\frac{|\nabla\varphi_1|^2}{\varphi_1^2} + \lambda_1 \le 0 \tag{23}$$

provided that

$$\left(\frac{\lambda_1}{m-1}\right)^{1/2} \le \frac{|\nabla\varphi_1|}{\varphi_1}.$$

Since $\varphi_1 = 0$ and $\partial\varphi_1/\partial\eta < 0$ in $\partial\Omega$, where $\eta = \eta(x)$ denote the outward normal derivative of $\varphi_1$ in the point $x \in \partial\Omega$, we can obtain $\delta > 0$ such that

$$\begin{aligned}\Omega_\delta := \{x \in \Omega; d(x, \partial\Omega) \le \delta\} \subset\\ \{x \in \Omega; (\lambda_1/(m-1))^{1/2} \le |\nabla\varphi_1(x)|/\varphi_1(x)\}.\end{aligned} \tag{24}$$

As a consequence, (23) occurs for all $x \in \Omega_\delta$.

On the other hand, since

$$M = \min_{x \in \Omega \setminus \Omega_\delta} \varphi_1^m(x) > 0$$

and the map $s \mapsto s/q(x,s)$ is non-increasing, it follows

$$\frac{\varphi_1^m}{q(x,\varphi_1^m)} \leq \frac{M}{q(x,M)} \quad \forall x \in \Omega \setminus \Omega_\delta. \tag{25}$$

Thus, thanks to (23) and (25), we get (22) for $\lambda$ large enough therefore $\underline{w} = \varphi_1^m$ is a subsolution of $(9)_\lambda$.

Now, let $K > 0$ a positive constant. Then $\overline{w} = K$ is a supersolution of $(9)_\lambda$, provided that

$$0 = -\Delta K \geq \lambda q(x,K) - bq(x,K)^p,$$

which is equivalent to

$$q(x,K)^{p-1} \geq \frac{\lambda}{b}. \tag{26}$$

Hence, choosing $K$ satisfying (26) and $K > \varphi_1^m$, $\overline{w} = K$ is a supersolution of $(9)_\lambda$. Consequently, there exists a positive soution $w$ of $(9)_\lambda$ for $\lambda$ large, satisfying

$$\varphi_1^m \leq w \leq K.$$

Proof of Theorem 2 (b) and (c): Once that $b > 0$ is fixed in this theorem, here we will denote $\lambda^*(b)$ simply by $\lambda^*$.

Thanks to Proposition 4 we already have that $\Lambda_b \neq \emptyset$ and $\lambda_1 \leq \lambda^* < \infty$. With the notation $\lambda_{a0} = \infty$ if $\Omega_{a0} = \emptyset$, we can deal with paragraphs (b) and (c) simultaneously to show existence of positive solution for $\lambda > \lambda^*$.

Thus, if $\lambda > \lambda^*$, by definition of $\lambda^*$, we can get that there exists $\overline{\lambda}$ with

$$\lambda^* < \overline{\lambda} < \lambda$$

such that $(9)_{\overline{\lambda}}$ possesses a positive solution, $w_{\overline{\lambda}}$. Since $\overline{\lambda} < \lambda$, $w_{\overline{\lambda}}$ is a subsolution of $(9)_\lambda$.

On the other hand, a constant $K > 0$ large enough satisfying (26) and $K > w_{\overline{\lambda}}$ is a supersolution. Consequently, $(9)_\lambda$ possesses a positive solutions, for all $\lambda > \lambda^*$.

If $\Omega_{a0} \neq \emptyset$ and the bifurcation direction at $\lambda_{a0}$ is subcritical or $\Omega_{a0} = \emptyset$, we need to show existence of positive solution for $\lambda = \lambda^*$. Indeed, in both cases we have

$$\lambda^* < \lambda_{a0}. \tag{27}$$

Thus, let $\sigma_n$ be a minimizer sequence such that $\sigma_n \downarrow \lambda^*$ and $w_n$ a respective positive solution. Then $w_n$ is bounded in $\mathcal{C}(\overline{\Omega})$. Since $\sigma_1 > \lambda_1$ and $\sigma_n \leq \sigma_1$, Corollary 1 gives

$$w_n \leq \theta_{\sigma_1} \quad \forall n \in \mathbb{N},$$

where $\theta_{\sigma_1}$ denote the unique solution of (19) with $\lambda = \sigma_1$. Thus, $\|w_n\|_0 \le \|\theta_{\sigma_1}\|_0$.

In addition, once that $(\sigma_n, w_n)$ is a solution of $(9)_{\sigma_n}$, we have

$$\int_\Omega \nabla w_n \cdot \nabla \phi = \int_\Omega (\sigma_n q(x, w_n) - bq(x, w_n)^p)\phi \quad \forall \phi \in H_0^1(\Omega) \tag{28}$$

Taking $\phi = w_n$ as a test function and using (10) we derive that

$$\begin{aligned}\|w_n\|^2_{H_0^1} &= \int_\Omega (\sigma_n q(x, w_n) - bq(x, w_n)^p) w_n \\ &\le \sigma_1 \int_\Omega q(x, w_n) w_n \le \sigma_1 \int_\Omega w_n^2 \le \sigma_1 \|\theta_{\sigma_1}\|_0^2 |\Omega|.\end{aligned}$$

As a consequence, $w_n$ is bounded in $H_0^1(\Omega)$. Thus, up to a subsequence if necessary,

$$w_n \rightharpoonup w^* \text{ in } H_0^1(\Omega) \quad \text{and} \quad w_n \to w^* \text{ in } L^m(\Omega) \quad m < 2^*.$$

Passing to the limit $n \to \infty$ in (28), it yields

$$\int_\Omega \nabla w^* \cdot \nabla \phi = \int_\Omega (\lambda^* q(x, w^*) - bq(x, w^*)^p)\phi \quad \forall \phi \in H_0^1(\Omega).$$

Hence $w^*$ is a weak solution of $(9)_{\lambda^*}$ and by the elliptic regularity, we obtain that $w^*$ is a classical non-negative solution. We claim that $w^* \neq 0$. Indeed, otherwise by elliptic regularity and the Morrey theorem, we have

$$\|w_n\|_{\mathcal{C}^1(\overline{\Omega})} \le C,$$

for some positive constant $C$. Thus, by the compact embeddeding of $\mathcal{C}^1(\overline{\Omega})$ into $\mathcal{C}(\overline{\Omega})$, up to a subsequence if necessary, we deduce that

$$\|w_n\|_0 \to 0 \quad .$$

In view of (11), for all $\delta > 0$, there exists $n_\delta \in \mathbb{N}$ such that

$$\frac{q(x, w_n)}{w_n} - \mathcal{X}_{\Omega_{a0}}(x) \le \delta \quad \forall n > n_\delta,\ x \in \Omega.$$

Consequently,

$$0 = \lambda_1 \left[ -\Delta - \sigma_n \frac{q(x, w_n)}{w_n} + b\frac{q(x, w_n)^p}{w_n}; \Omega \right] > \lambda_1[-\Delta - \sigma_n(\delta + \mathcal{X}_{\Omega_{a0}}; \Omega)]$$

Taking $\delta \to 0$ imply $n \to \infty$ and we deduce that

$$0 \ge \lambda_1[-\Delta - \lambda^* \mathcal{X}_{\Omega_{a0}}; \Omega] = \mu(\lambda^*).$$

By the properties of $\mu$ (see Proposition 1), the above inequality provides us that $\lambda^* \ge \lambda_{a0}$, which is a contradiction with (27).

To complete the proof, it remains to show that $\lambda_1 < \lambda^* \leq \lambda_{a0}$. Indeed, If $\Omega_{a0} = \emptyset$ then $\lambda_{a0} = \infty$ and $\lambda^* \leq \lambda_{a0}$ is immediate. If $\Omega_{a0} \neq \emptyset$ then $\lambda_{a0}$ is a bifurcation point from the trivial solution and, by definition of $\lambda^*$, it follows that $\lambda^* \leq \lambda_{a0}$. In order to prove $\lambda_1 < \lambda$, if $\lambda^* < \lambda_{a0}$, then we have already know, that $(9)_\lambda$ possesses a positive solution for $\lambda = \lambda^*$ and since $\lambda > \lambda_1$ is a necessary condition for the existence, it follows that $\lambda^* > \lambda_1$. If $\lambda^* = \lambda_{a0}$, since we are considering only the case $a \neq 0$ in $\Omega$, this implies that $\lambda_1 < \lambda_{a0} = \lambda^*$.

Proof of Theorem 3 (a): Recall that, by Corollary 1, every solution $w > 0$ of $(9)_\lambda$ satisfies

$$w \leq \|\theta_\lambda\|_0.$$

Thus, let us consider the function

$$f(x, s) := \lambda q(x, s) - bq(x, s)^p + Ks.$$

Since

$$f_s(x, s) = \lambda q_s(x, s) - bpq(x, s)^{p-1} q_s(x, s) + K \quad \forall s > 0,$$

and $q_s(x, s)$ is bounded for $0 < s < \|\theta_\lambda\|_0$, we can choose $K > 0$ large enough such that this function is increasing on $[0, \|\theta_\lambda\|_0]$. Thus, the monotonic interaction

$$-\Delta w_{n+1} + K w_{n+1} = \lambda q(x, w_n) - bq(x, w_n)^p + K w_n, \quad w_0 = \theta_\lambda$$

provides a maximal solution in $[0, \theta_\lambda]$. Once that every positive solution $w > 0$ satisfies $w < \theta_\lambda$, we get the result.

Now, given $\lambda^*(b) \leq \mu < \lambda$, then $W_\mu$ is a subsolution of $(9)_\lambda$. Since $K > 0$ large enough is a super solution of $(9)_\lambda$, we derive that $(9)_\lambda$ possesses a positive solution $w$ with

$$W_\mu < w \leq K.$$

The strict inequality occurs because $W_\mu$ is not a solution of $(9)_\lambda$. Once that $W_\lambda$ is a maximal solution of $(9)_\lambda$, we deduce

$$W_\mu < w \leq W_\lambda.$$

This completes the proof.

In order to prove (7), we need the following result

**Lemma 2** *If* $b_1 < b_2$, *then* $\inf \Lambda_{b_1} \leq \inf \Lambda_{b_2}$.

*Proof* Just note that $\Lambda_{b_2} \subset \Lambda_{b_1}$. Indeed, if $\lambda \in \Lambda_{b_2}$, then $w_{\lambda(b_2)}$ is a subsolution of $(9)_\lambda$ with $b = b_1$. Choosing $K$ large enough satisfying (26) and $K \geq w_{\lambda(b_2)}$, it follows that there exists a positive solution of $(9)_\lambda$ with $b = b_1$. Moreover,

$$w_{\lambda(b_2)} \leq w_{\lambda(b_1)}.$$

Proof of Theorem 3 (b): Fix $\lambda > \lambda_1$, we can choose $\lambda = \lambda_1 + \varepsilon_0$, with $\varepsilon_0 > 0$. Let be $C > 0$ a constant, then $\underline{w} = C\varphi_1^m$ is a subsolution of $(9)_\lambda$ if

$$Cm(1-m)|\nabla\varphi_1|^2 \frac{\varphi_1^{m-2}}{q(x, C\varphi_1^m)} + \lambda_1 \left( m \frac{C\varphi_1^m}{q(x, C\varphi_1^m)} - 1 \right) + bq(x, C\varphi_1^m)^{p-1} \le \varepsilon_0, \qquad (29)$$

for all $x \in \Omega$. Let us obtain conditions for that (29) is fulfilled in $\Omega_\delta$ as well as in $\Omega \setminus \Omega_\delta$, where $\Omega_\delta$ is given as in (24).

Firstly, fix $m = m(\lambda) > 1$ such that

$$\lambda_1(m-1) < \frac{\epsilon_0}{2} \qquad (30)$$

For this $m$, we pick $\delta = \delta(m)$ as in Proposition 4. Observe that $\delta$ does not depend on $C$.

Now, recall that the map $s \mapsto q(x,s)/s$ is increasing and $\lim_{s\to\infty} q(x,s)/s = 1$ (see Lemma 1), therefore

$$\frac{s}{q(x,s)} \downarrow 1 \quad \text{as } s \to \infty$$

Since

$$\min_{\Omega\setminus\Omega_\delta} \varphi_1^m > 0$$

from (30) and the above limit, we can get $C > 0$ large such that

$$\lambda_1 \left( m \frac{C\varphi_1^m}{q(x, C\varphi_1^m)} - 1 \right) \le \frac{\varepsilon_0}{2} \quad \forall x \in \Omega \setminus \Omega_\delta.$$

As a consequence, for $b > 0$ satisfying

$$bq(x, C\varphi_1^m)^{p-1} \le \frac{\varepsilon_0}{2} \quad \forall x \in \Omega, \qquad (31)$$

we derive that (29) occurs for all $x \in \Omega \setminus \Omega_\delta$.

On the other hand, if $x \in \Omega_\delta$ we have

$$m(1-m)|\nabla\varphi_1|^2\varphi_1^{m-2} + m\lambda_1\varphi_1^m \le 0$$

implying

$$Cm(1-m)|\nabla\varphi_1|^2 \frac{\varphi_1^{m-2}}{q(x, C\varphi_1^m)} + m\lambda_1 \frac{C\varphi_1^m}{q(x, C\varphi_1^m)} \le 0.$$

In view of (31), it follows that (29) also meets in $\Omega_\delta$ and therefore $\underline{w} = C\varphi_1^m$ is a subsolution of $(9)_\lambda$. Taking $K$ satisfying (26) and $K \ge C\varphi_1^m$ it is a supersolution of $(9)_\lambda$. Hence,

$$C\varphi_1^m \le w_{[\lambda,b]} \le K. \qquad (32)$$

As a consequence, given $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that

$$\lambda_1 < \lambda^*(b_\varepsilon) \le \lambda_1 + \varepsilon.$$

by Proposition 2, the above inequality is verified for all $0 < b \le b_\varepsilon$, showing (7).

**Proposition 5** *Let $(w_{\lambda^*(b)})_{b>0}$ be a family of positive solutions, then*

$$\lim_{b\to 0} \|w_{\lambda^*(b)}\|_0 = \infty. \tag{33}$$

*Proof* Arguing by contradiction, suppose that $\|w_{\lambda^*(b)}\|_0 \le M$, for each $b < b_0$. Hence

$$\begin{aligned} 0 = \lambda_1 \left[ -\Delta - \lambda^*(b) \frac{q(x, w_{\lambda^*(b)})}{w_{\lambda^*(b)}} + b \frac{q(x, w_{\lambda^*(b)})^p}{w_{\lambda^*(b)}}; \Omega \right] \\ \ge \lambda_1 \left[ -\lambda^*(b) \frac{q(x, M)}{M}; \Omega \right]. \end{aligned}$$

Letting to $b \to 0$, yields

$$0 \ge \lambda_1 \left[ -\Delta - \lambda_1 \frac{q(x, M)}{M}; \Omega \right].$$

Since $\Omega_{a0} \ne \Omega$, then $q(x, M)/M < 1$ and it imply

$$0 > \lambda_1[-\Delta - \lambda_1; \Omega] = 0,$$

which is a contradiction.

As a consequence of this result, we get
Proof of Theorem 3 (c): By Theorem 3 (a), for all $b > 0$ we have

$$w_{\lambda^*(b)} \le W_{\lambda^*(b)} \le W_{\lambda(b)}.$$

Thus, by the Proposition 5, we obtain the result.

## 5 Multiplicity of positive solutions

This section is dedicated to obtain a second positive solution of $(9)_\lambda$ and for this propose, we use variational methods. The arguments presented here are inspired by [1] and [2].

For each $\lambda > \lambda_1$, let $M > 0$ be such that $\|\theta_\lambda\|_0 < M$ where $\theta_\lambda$ is stands for the unique solutions of (19), see Proposition 2. Fix $\varepsilon > 0$, we define

$$\overline{q}(x, s) = \begin{cases} q(x, s) & \text{if } s \le M \\ \phi(x, s) & \text{if } M \le s \le M + \varepsilon \\ q(x, M + \varepsilon) & \text{if } M + \varepsilon < s \end{cases}$$

where $\phi(x, s)$ is a regular function such that the map $s \in (0, \infty) \mapsto \overline{q}(x, s)$ is of class $\mathcal{C}^1$. Defining the functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\lambda(w) = \frac{1}{2}\|w\|^2_{H_0^1} - \lambda \int_\Omega Q(x, w)dx + b \int_\Omega Q_p(x, w)dx,$$

where

$$Q(x,w) := \int_0^w \overline{q}(x,s)ds \quad \text{and} \quad Q_p(x,w) := \int_0^w \overline{q}(x,s)^p ds.$$

Thus, $I_\lambda$ is well-defined and of class $\mathcal{C}^2$, for all $\lambda > \lambda_1$. Moreover, since every positive solution of $(9)_\lambda$ is bounded from above by $M$ (according to Corollary 1), then critical points of $I_\lambda$ are weak positive solutions of $(9)_\lambda$ and by elliptic regularity, are classical solution of $(9)_\lambda$

Let us collect some properties of this functional.

**Proposition 6** *The functional $I_\lambda$ is coercive and bounded from below.*

*Proof* For each $w \in H_0^1(\Omega)$ we have

$$\begin{aligned} I_\lambda(w) &= \frac{1}{2}\|w\|^2_{H_0^1} - \lambda \int_\Omega Q(x,w)dx + b\int_\Omega Q_p(x,w)dx \\ &= \frac{1}{2}\|w\|^2_{H_0^1} - \int_\Omega \int_0^w \left(\lambda \overline{q}(x,w) - b\overline{q}(x,w)^p\right) dsdx \end{aligned}$$

since the map

$$s \mapsto \lambda s - bs^p, \; s \geq 0$$

is bounded above, we can obtain a constant $C > 0$ such that

$$\lambda \overline{q}(x,s) - b\overline{q}(x,s)^p \leq C, \quad s \geq 0.$$

In this way, we get

$$I_\lambda(w) \geq \frac{1}{2}\|w\|^2_{H_0^1} - C\int_\Omega w dx \geq \frac{1}{2}\|w\|^2_{H_0^1} - C|w|_1.$$

By the continuous embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ it follows

$$I_\lambda(w) \geq \frac{1}{2}\|w\|^2_{H_0^1} - C_1\|w\|_{H_0^1}.$$

Showing that $I_\lambda$ is coercive and bounded below.

**Proposition 7** *If $w_n$ is a sequence in $H_0^1(\Omega)$ with $I_\lambda(w_n)$ bounded, then, up a subsequence if necessary,*

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega)$$

*and*

$$I_\lambda(w) \leq \liminf_{n\to\infty} I_\lambda(w_n).$$

*In particular, $I_\lambda$ attains its infimum on $H_0^1(\Omega)$.*

coercive

*Proof* Thanks to the coercivity of $I_\lambda$, the sequence $w_n$ is bounded in $H_0^1(\Omega)$. Thus, up to a subsequence if necessary,

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega)$$

and

$$w_n \to w \text{ in } L^s(\Omega),\ \ s \in [1, 2^*).$$

Consequently,

$$I_\lambda(w) - I_\lambda(w_n) = \frac{1}{2}(\|w\|^2_{H_0^1} - \|w_n\|^2_{H_0^1}) + \int_\Omega [(\lambda Q(x, w_n) - bQ_p(x, w_n)) - (\lambda Q(x, w) - bQ_p(x, w))]dx.$$

Writing $F(x, s) = \lambda Q(x, s) - bQ_p(x, s),\ \ s \geq 0$, we have

$$I_\lambda(w) - I_\lambda(w_n) = \frac{1}{2}(\|w\|^2_{H_0^1} - \|w_n\|^2_{H_0^1}) + \int_\Omega [F(x, w_n) - F(x, w)]dx. \tag{34}$$

By the properties of $\overline{q}$,

$$F_s(x, s) = \lambda\overline{q}(x, s) - b\overline{q}(x, s)^p$$

is bounded in $\Omega \times [0, \infty)$. Thus, (34) implies

$$\begin{aligned} &I_\lambda(w) - I_\lambda(w_n) = \frac{1}{2}(\|w\|^2_{H_0^1} - \|w_n\|^2_{H_0^1}) + \\ &\int_\Omega \left[\int_0^1 (\lambda\overline{q}(x, tw_n + (1-t)w) - b\overline{q}(x, tw_n + (1-t)w)^p dt (w_n - w)\right] dx \\ &\leq \frac{1}{2}(\|w\|^2_{H_0^1} - \|w_n\|^2_{H_0^1}) + C\int_\Omega |w_n - w| dx \end{aligned}$$

Since $w_n \to w$ in $L^1(\Omega)$ and $w_n \rightharpoonup w$ in $H_0^1(\Omega)$, it follows

$$I_\lambda(w) - \liminf_{n\to\infty} I_\lambda(w_n) \leq 0.$$

Finally, since $I_\lambda$ is coercive and bounded below (Proposition 6), we obtain $I_\lambda$ attains its infimum on $H_0^1(\Omega)$.

In order to apply Theorem II.11.8 of [12], let us prove that $I_\lambda$ has two solutions that are local minimum of $I_\lambda$ in $H_0^1(\Omega)$.

**Proposition 8** *For all $\lambda > \lambda^*$, $(9)_\lambda$ possesses a solution $w$ that is a local minimum for $I_\lambda$ in $H_0^1(\Omega)$.*

*Proof* By Theorem 3 (a), the maximal solution of $(9)_{\lambda^*}$, $W_{\lambda^*}$, is a strict subsolution of $(9)_\lambda$ for all $\lambda > \lambda^*$. Thus, we obtain a solution $v_\lambda$ for $(9)_\lambda$ via minimization

$$I_\lambda(v_\lambda) = \inf\{I_\lambda(w);\ w \in H_0^1(\Omega),\ w(x) \geq W_{\lambda^*}\}.$$

Hence, $v_\lambda$ exists thanks to Propositions 6 and 7 and it defines a solution to $(9)_\lambda$.

To verify that it is a minimizer of $I_\lambda$ in $H_0^1(\Omega)$, by [4] it suffices to show that is a local minimizer in the $\mathcal{C}^1$ topology.

Taking $K > 0$ sufficiently large such that $s \mapsto \lambda\overline{q}(x,s) - b\overline{q}(x,s)^p + Ks$ be increasing in $[0, \max_{\overline{\Omega}} v_\lambda]$ and since $v_\lambda > W_{\lambda^*}$, we derive that

$$\begin{aligned}-\Delta(v_\lambda - W_{\lambda^*}) + K(v_\lambda - W_{\lambda^*}) = &(\lambda\overline{q}(x,v_\lambda) - b\overline{q}(x,v_\lambda)^p + Kv_\lambda)\\ &-(\lambda^*\overline{q}(x,W_{\lambda^*}) - b\overline{q}(x,W_{\lambda^*})^p + KW_{\lambda^*}) > 0.\end{aligned}$$

By the Strong Maximum Principle, it follows that $v_\lambda - W_{\lambda^*}$ lies in the interior of the positive cone of $\mathcal{C}_0^1(\overline{\Omega})$. Hence, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(v_\lambda) \subset \{u \in \mathcal{C}_0^1(\overline{\Omega}); u \geq W_{\lambda^*}\},$$

where $B_\varepsilon(v_\lambda)$ denote the open ball of radius $\varepsilon$ and center $v_\lambda$ in $\mathcal{C}^1$ topology.

Since $I_\lambda(v_\lambda)$ is the minimizer in $\{u \in H_0^1(\Omega); u \geq W_{\lambda^*}\}$, then it is also a local minimizer in $\mathcal{C}_0^1(\Omega)$.

The next result gives us a second local minimum of $I_\lambda$ in $H_0^1(\Omega)$.

**Proposition 9** *If $\lambda < \lambda_{a0}$, then the trivial solution $w \equiv 0$ is a local minimum of $I_\lambda$ on $H_0^1(\Omega)$ and is an isolated solution of $(9)_\lambda$.*

*Proof* We will consider two cases:

Case $\Omega_{a0} \neq \emptyset$

Fix $\varepsilon = \varepsilon(\lambda) > 0$ sufficiently small such that

$$1 - \varepsilon\frac{\lambda}{\lambda_1} - \frac{\lambda}{\lambda_{a0}} > 0.$$

Then, thanks to the properties of $\overline{q}$, we can get $C > 0$ and $1 < r < 2^*$ such that

$$\overline{q}(x,s) \leq q(x,s) \leq (\varepsilon + \mathcal{X}_{\Omega_{a0}}(x))s + Cs^r \quad \forall (x,s) \in \Omega \times [0,\infty).$$

Consequently,

$$\begin{aligned}I_\lambda(w) &\geq \frac{1}{2}\|w\|_{H_0^1}^2 - \frac{\lambda}{2}\int_\Omega (\varepsilon + \mathcal{X}_{\Omega_{a0}}(x))w^2 - \frac{C}{r+1}\int_\Omega w^{r+1}\\ &\geq \frac{1}{2}\left(1 - \varepsilon\frac{\lambda}{\lambda_1} - \frac{\lambda}{\lambda_{a0}}\right)\|w\|_{H_0^1}^2 - \frac{C}{\lambda_1(r+1)}\|w\|_{H_0^1}^{r+1}.\end{aligned}$$

Therefore, there exists $\delta > 0$ small such that

$$I_\lambda(w) \geq 0 \quad \forall w \in H_0^1(\Omega), \|w\|_{H_0^1} \leq \delta,$$

showing that $w \equiv 0$ is a local minimum of $I_\lambda$ in $H_0^1(\Omega)$.

To prove that 0 is isolated solution of (9) we argue by contradiction. Otherwise, there would be a sequence of positive solution $w_n$ such that $\|w_n\|_{H_0^1} \to 0$.

Therefore, we also have $\|w_n\|_0 \to 0$. By (11), for all $\delta > 0$, exists $n_\delta \in \mathbb{N}$ such that

$$\frac{q(x,w_n)}{w_n} - \mathcal{X}_{\Omega_{a0}} \le \delta \quad \forall n > n_\delta,\ x \in \Omega.$$

Consequently,

$$0 = \lambda_1 \left[ -\Delta - \lambda \frac{q(x,w_n)}{w_n} + b\frac{q(x,w_n)^p}{w_n}; \Omega \right] > \lambda_1[-\Delta - \lambda(\delta + \mathcal{X}_{\Omega_{a0}}); \Omega]$$

Taking $\delta \to 0$ we deduce that

$$0 \ge \lambda_1[-\Delta - \lambda \mathcal{X}_{\Omega_{a0}}; \Omega] = \mu(\lambda)$$

By the properties of $\mu$ (see Proposition 1), the above inequality provides us $\lambda \ge \lambda_{a0}$, which is a contradiction.

Case $\Omega_{a0} = \emptyset$

Similarly, using $q(x,s) \le s$, we have

$$\begin{aligned} I_\lambda(w) &\ge \frac{1}{2}\|w\|^2_{H^1_0} - \frac{\lambda}{2}\int_\Omega w^2 \\ &\ge \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|w\|^2_{H^1_0}. \end{aligned}$$

implying that 0 is a local minimum of $I_\lambda$ in $H^1_0(\Omega)$. Moreover, observing that $\mathcal{X}_{\Omega_{a0}} \equiv 0$, the same arguments of previous case can be applied to conclude that 0 is an isolated solution of (9).

Recall that, according to Definition II.12.2 in [12], for a convex and closed set $M \subset H^1_0(\Omega)$, a function $w \in H^1_0(\Omega)$ is a critical point of $I_\lambda$ on $M$ if

$$g(w) = \sup\{I'_\lambda(w)(w - v);\ v \in M,\ \|v - w\|_{H^1_0} \le 1\} = 0.$$

Taking

$$\mathcal{M} = \{w \in H^1_0(\Omega); 0 \le w(x) \le v_\lambda(x)\}.$$

Since $w \equiv 0$ and $v_\lambda$ are solutions of (9), then a critical point of $I_\lambda$ in $\mathcal{M}$ is also a critical of $I_\lambda$ in $H^1_0(\Omega)$. Let us show a Palais-Smale condition for the functional $I$ in $\mathcal{M}$.

**Proposition 10** *If $w_n$ is a sequence in $\mathcal{M}$ such that*

$$I_\lambda(w_n) \to c \quad \text{and} \quad g(w_n) \to 0,$$

*then $w_n$ possesses a strongly convergent subsequence in $H^1_0(\Omega)$.*

*Proof*

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega) \quad \text{and} \quad w_n(x) \to w(x) \text{ a.e. in } \Omega.$$

Once that $0 \leq w_n \leq v_\lambda$, we obtain $0 \leq w \leq v_\lambda$ and from Lebesgue's Dominated Convergence Theorem we get

$$\int_\Omega (\lambda \overline{q}(x, w_n) - b\overline{q}(x, w_n)^p)(w_n - w)dx \to 0.$$

Therefore,

$$\begin{aligned} g(w_n)\|w_n - w\|_{H_0^1} &\geq I'_\lambda(w_n)(w_n - w) \\ &= \int_\Omega \nabla w_n \nabla (w_n - w) + o(1) \\ &= \int_\Omega |\nabla (w_n - w)|^2 + o(1). \end{aligned}$$

Thus,

$$g(w_n) \geq \|w_n - w\|_{H_0^1} + o(1).$$

Passing to the limit $n \to \infty$ we deduce that $w_n \to w$ in $H_0^1(\Omega)$.

Finally, we are able to give the
Proof of Theorem 2 (c): Consider again the set

$$\mathcal{M} = \{w \in H_0^1(\Omega); 0 \leq w(x) \leq v_\lambda(x)\}.$$

where $v_\lambda$ is a solution that is a local minimum of $I_\lambda$ on $\mathcal{M}$ (according Proposition 8). Once that $I_\lambda$ satisfies the Palais-Smale condition in $\mathcal{M}$ (Proposition 10), we can apply the Theorem II.11.8 of [12] and deduce the following dichotomy: either

1. $I_\lambda$ has a critical point $w_\lambda$ in $\mathcal{M}$ which is not a local minimum; or
2. $I_\lambda(v_\lambda) = I_\lambda(0)$ and $v_\lambda$ and 0 may be connected in any neighborhood of the set of local minimal of $I_\lambda$ relative to $\mathcal{M}$, each of which satisfying $I_\lambda(w) = 0$

But, by Proposition 9, 0 is an isolated among the solution of $(9)_\lambda$, for all $\lambda \in (\lambda_1, \lambda_{a0})$. This excludes the possibility of the paragraph 2. occurs.

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