SOLUTION CONCEPTS FOR MULTIPLE OBJECTIVE N-PERSON GAMES*

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Abstract

Multiple criteria decision problems with one decision-maker haven been recognised and discussed in the literature in optimisation theory, operations research and management science. Nevertheless a multiple criteria problem can naturally arise in decision situations involving conflict among n-persons and conflict among the criteria of each person. The corresponding concept with n-decision makers, namely multiple objective n-person games, has not been extensively explored.

In this paper we consider several approaches for solving normal form multiple objective n-person games. One of them is to find optimal response strategies. In non-cooperative games, characterised by strategic behaviour and individual rationality, given all other player's strategies, each player may choose a best response as an efficient solution of a vector maximisation problem. Also a best response can be established as a maximin solution. Another approach to solve these games is based on security levels. We present the concept of Pareto optimal security strategies for multiple objective n-person games and explore the relations with the optimal response strategies.

Keywords

Game theory, multicriteria games, Nash equilibrium, maximin strategy, Pareto-optimal security strategy.

1. Introduction

Multicriteria decision making and game theory have contributed important insights to several areas of social science. Multicriteria decision making has gained broad interest and has been extensively described in the literature on operations research and decision theory over the last two decades.

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On the other hand, game theory, born in 1944 with the publication of the book "Theory of Games and Economics Behavior" by John von Neumann and Oscar Morgenstern [14], studies the behavior of decision-makers whose decisions affect the others. In the last ten years recent advances in game theory have ensured its recognition as a key subject across a wide range of disciplines in the social science.

Both theories, multicriteria decision making and game theory, jointly applied yield to important and novel results in various areas of applications. However, multiple objective games that emerge whenever players in a game have multiple objectives or a vector payoff to optimize, have attracted limited attention in the game theory literature and have not yet been extensively explored.

The first publication on multiple objective games dates back to 1956 [1] and some important papers are [17, 6, 22, 10, 4, 2, 5, 9, 23].

Multicriteria games can naturally arise in decision situations involving conflict among n persons. In practical problems, it is typical that a player deals not with one, but with several criteria which he would like to satisfy, and there is not an explicitly given utility function.

As the methodology to solve multicriteria is based in solving multiple criteria problems we do not need to scalarize all the objectives in order to get a single value function. Also, it is interesting to analyze the possible extension of the results in classical game theory, because due to the additional difficulty of dealing with multiple criteria, many of the elegant and intuitively appealing theoretical results in scalar criterion games could not hold.

Multicriteria games can be both, cooperative and non-cooperative. In this paper we study the non-cooperative case, although we also consider some cooperation and partial cooperation situations.

Our main effort in this paper, has been devoted to show that, in multicriteria games, a single solution concept in terms of equilibrium points is not sufficient. For this reason, we propose and discuss different solution concepts associated with multicriteria games.

The paper is organized as follows. In section 3 we give the preliminary terminology used throughout the paper. In section 3 we analyze four different solution concepts and some examples are included to illustrate them. Finally a section devoted to conclusions and a list of references is offered.

2. Preliminaries

In this paper we consider a multiobjective n-person game in normal form defined as $\Gamma = \{N, X^i, u^i\}$ where $N = \{1, 2,..., n\}$ is the set of players. For each $i \in N$, X^i is player i's strategy set, which is assumed to be a non-empty subset in some finite-dimensional euclidean space ($X^i \subset \mathbb{R}^{L_i}$), $u^i : X = \prod_{i=1}^{n} X^i \rightarrow \mathbb{R}^{m_i}$ is player i's vector payoff which is a real m_i-dimensional vector function. A joint strategy is $x \in X = \prod_{i=1}^{n} X^i$, $x = \{x^1, x^2,..., x^n\}$ where $x^i \in X^i$, and the joint vector payoff is $u \in \mathbb{R}(N) = \prod_{i=1}^{n} \mathbb{R}^{m_i}$, $u = \{u^1, u^2,..., u^n\}$.

Notice that each player values a different number m_i of criteria. However, in order to simplify the notation, in some sections we will consider that $m_1 = m_2 = \ldots = m_n = m$.

Let $\mathcal{N}$ denote the set of all non-empty subsets of N, then each element of $\mathcal{N}$ represents a coalition of players. For each coalition $S \in \mathcal{N}$, let $|S|$ denote the number of elements in S. For each joint strategy $x = \{x^1, x^2, \ldots, x^n\} \in X$ let x_S denote the strategy of coalition S, x_{-S} the strategy of the players in the complementary coalition N-S. For the payoff vector associated, $u = \{u^1, u^2, \ldots, u^n\} \in \mathbb{R}(N)$, we denote by u_S and u_{-S} the projection of u on $\mathbb{R}(S)$ and $\mathbb{R}(-S)$ respectively:

$$x_S = \{x^i / i \in S\} \in X_S = \prod_{i \in S} X^i$$

$$x_{-S} = \{x^i / i \notin S\} \in X_{-S} = \prod_{i \notin S} X^i$$

$$u_S = \{u^i / i \in S\} \in \mathbb{R}(S) = \prod_{i \in S} \mathbb{R}^{m_i}$$

$$u_{-S} = \{u^i / i \notin S\} \in \mathbb{R}(-S) = \prod_{i \notin S} \mathbb{R}^{m_i}.$$

By $\mathbb{R}(S)_+$ we denote the positive orthant of $\mathbb{R}(S)$.

Recall that a partition of players $\{1, 2, \ldots, n\}$ in the game Γ is a collection of coalitions $\Delta = \{S_1, S_2, \ldots, S_k\}$ such that

$$\bigcup_{i=1}^{k} S_i = N$$

$$S_i \cap S_j = \varnothing \quad \forall i \neq j.$$

Example:

Consider the bicriteria three-person game represented in Table 1. Each player can play two different strategies, A and B, and the results of the game are evaluated in two different scenarios or with respect to two different criteria.

$$N = \{I, II, III\}$$

$$X^1 = \{IA, IB\} \quad X^2 = \{IIA, IIB\} \quad X^3 = \{IIIA, IIIB\}.$$

$$X = \prod_{i=1}^{3} X^i$$

In this example with finite joint strategy set the payoff functions are discrete and:

$$\forall x \in X \quad u^i(x) \in \mathbb{R}^2, \; u = \{u^1, u^2, u^3\} \text{ and so } u(x) \in \mathbb{R}^6$$

For instance, when player I plays IA, player II plays IIB and player III plays IIIA, that is to say, x = (IA, IIB, IIIA) the payoffs are:

$$u^1(x) = \begin{pmatrix} -50 \\ 50 \end{pmatrix} \quad u^2(x) = \begin{pmatrix} 0 \\ -100 \end{pmatrix} \quad u^3(x) = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$u(x) = (-50 \;\; 50 \;\; 0 \;\; -100 \;\; 50 \;\; 50)^t$$

This means that player I obtains -50 in his first criterion and 50 in his second criterion. Player II obtains 0 in his first criterion and -100 in his second criterion. Player III obtains 50 in his first criterion and 50 in his second criterion.

	IIIA			IIIB	
I \ II	IIA	IIB	I \ II	IIA	IIB
IA	(-50,-50,50) (0,0,0)	(-50,0,50) (50,-100,50)	IA	(-50,-50,0) (50,50,-100)	(100,0,0) (100,-50,-50)
IB	(0,-50,50) (0,50,50)	(0,0,100) (0,-50,-90)	IB	(0,100,0) (50,100,-50)	(0,0,0) (0,0,0)

Table 1 - Game payoffs

3. Solution Concepts

3.1 Nash equilibria

The concept of Nash equilibrium in conventional game theory can be extended to vector game theory. Each player i takes other's strategies as given and chooses a "best response" as an efficient solution of a vector maximization utility problem. Then there are a variery of solutions that can be chosen as the "best response": properly efficient solutions, [23], efficient solutions, [17,5], weakly efficient solutions, [12]. In this paper weakly efficient solutions are chosen as the best response. In what follows we will refer to weak efficiency as efficiency.

Definition 3.1 A joint strategy $\bar{x} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\} \in X$ is a Nash equilibrium of the multicriteria game $\Gamma = \{N, X^i, u^i\}$ if $\forall i \in N$, $\bar{x}^i$ is an efficient solution of the vector maximization problem:

$$\max_{x^i \in X^i} u^i(x^i, \bar{x}^{-i}) \text{ where } \bar{x}^{-i} = \{\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{i-1}, \bar{x}^{i+1}, \ldots, \bar{x}^n\} \quad (1)$$

This solution concept is a noncooperative solution characterized by the strategic behavior and the individual rationality. It requires that each player chooses a best response given all other players' strategies, and at the equilibrium no player has any incentive to deviate alone. However Nash equilibrium is a local solution. If there are several equilibria, the players have no compelling reason to choose among them.

Example (continued):

The Nash equilibria of the game whose payoffs are in Table 1 are the joint strategies:

x_1 = (IA, IIA, IIIA), x_2 = (IA, IIB, IIIA), x_3 = (IB, IIB, IIIA),

x_4 = (IB, IIA, IIIB), x_5 = (IB, IIA, IIIA)

For instance x_3 = (IB, IIB, IIIA)$\in X$ is a Nash equilibrium:

Player I obtains $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If the strategies of player II and player III are fixed, he can obtain $\begin{pmatrix} -50 \\ 50 \end{pmatrix}$ also, but this last payoff is not strictly better than the other.

Player II obtains $\begin{pmatrix} 0 \\ -50 \end{pmatrix}$. Assuming now I and III's strategies as given, he can also obtain $\begin{pmatrix} -50 \\ 50 \end{pmatrix}$, which is not strictly better than the other payoff.

Finally, III obtains $\begin{pmatrix} 100 \\ -90 \end{pmatrix}$. If I and II's strategies are fixed, he can obtain too $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but again, it is not strictly better than the other.

x = (IB, IIB, IIIB)∈ X is not a Nash equilibrium because if I and III's strategies are fixed, II obtains $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ but can also obtain $\begin{pmatrix} -100 \\ 100 \end{pmatrix}$. This payoff is strictly better than $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Consider that there is a partition of players, then we have coexistence of competition across coalition and cooperation within each coalition. In the Nash equilibrium each player takes all other's strategies as given, because no cooperation with other players is allowed.

This strategic behavior can be generalized to a group of players: each coalition takes all other coalitions' strategies as given, because no cooperation with other coalition is allowed. Thus given coalition $S \in \mathfrak{N}$ and given the complementary strategies $\overline{x}_{-S}$, the coalition S has to solve the vector maximization utility problem:

$$\max_{x_S \in X_S} \quad u_S(x_S, \overline{x}_{-S}) \tag{2}$$

where the complementary strategies x_{-S} are fixed parameters and there are $\sum_{i \in S} m_i$ objectives to be maximized. This leads to the following concept of S-efficiency:

Definition 3.2 For any coalition $S \in \mathfrak{N}$, a joint strategy $\overline{x} = \{\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n\} \in X$ is a S-efficient solution of the game $\Gamma = \{N, X^i, u^i\}$ if $\overline{x}_S$ is an efficient solution of the vector maximization problem (2).

Notice that a Nash equilibrium of the game Γ is a S-efficient solution of the game Γ for any coalition with only a player.

Suppose there is a partition $\Delta = \{S_1, S_2, \ldots, S_k\}$. If we consider a S-efficient solution for each $S \in \Delta$, we will have a hybrid solution concept between cooperative and non-cooperative solution concepts that we call a Nash equilibrium for the partition Δ:

Definition 3.3 For each partition of players $\Delta = \{S_1, S_2, \ldots, S_k\}$ in the game $\Gamma = \{N, X^i, u^i\}$ a joint strategy $\overline{x} = \{\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n\} \in X$ is a Nash equilibrium corresponding to Δ if $\forall S \in \Delta$, $\overline{x}_S$ is an efficient solution of the vector maximization problem (2) where $\overline{x}_S$ are the given strategies of the other players.

Notice that this kind of solutions generalizes the concept of Nash equilibrium because we obtain a Nash equilibrium when each coalition in the partition of the game Γ has only one player.

An existence theorem for these solutions is in [23].

Theorem 3.1 Given a partition of players $\Delta = \{S_1, S_2, \ldots, S_k\}$ in the game $\Gamma = \{N, X^i, u^i\}$ the set of the corresponding Nash equilibrium to Δ is non-empty if Γ satisfies:

1. For each player i, X^i is a closed bounded convex subset in $\mathbb{R}^{L_i}$.

2. For each coalition $S \in \Delta_j$: $j \in S$ are all continuous in $x = (x_S, x_{-S})$ and are quasiconcave in x_S.

Notice that this theorem generalizes the earlier existence theorems in [13], [17] and [5].

It is difficult to solve the multicriteria problems involed in the computation of Nash equilibria strategies. When more restricting conditions hold, such as concavity instead of quasiconcavity, it could be computed some of the solutions based in weighted criteria. Notice that a player has to know the other players' strategies to make his choice because of the local character of this kind of solutions.

3.2 Maximin Solution

In this section we analyze how a player or a coalition can choose the strategy without knowing the other player's choice. We will work from the point of view of a coalition $S \in \mathcal{N}$.

Players in S will try to maximize their payoffs. Being pessimist, for each strategy they can play, they consider the worst possible payoff. With this information they will choose the strategy with the better worst payoff. That is to say the maximin values of their utility function. Hence, for any $x_S \in X_S$, we will consider the efficient solutions of the vector minimization problem:

$$\min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) \tag{3}$$

The maximin values set for the function u_S is:

$$\max \bigcup_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S})$$

We call a maximin strategy of coalition S to a strategy attaining a maximin value.

As it is shown in [15], if the strategy set is a compact set, the maximin values set is non-empty.

Example (continued):

In order to find the maximin strategies for player I, we first consider the worst possible payoffs when he plays strategy IA (efficient for the vector minimization problem):

$$\min\left\{\begin{pmatrix}-50\\0\end{pmatrix}\begin{pmatrix}-50\\50\end{pmatrix}\begin{pmatrix}-50\\50\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix}\right\} = \left\{\begin{pmatrix}-50\\0\end{pmatrix}\begin{pmatrix}-50\\50\end{pmatrix}\right\}.$$

The worst possible payoffs when he plays strategy IB:

$$\min\left\{\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\50\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\50\end{pmatrix}\right\}.$$

Next we seek for the maximum values, that is to say:

$$\max\left\{\begin{pmatrix}-50\\0\end{pmatrix}\begin{pmatrix}-50\\50\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\50\end{pmatrix}\right\} = \left\{\begin{pmatrix}-50\\50\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\50\end{pmatrix}\right\}.$$

Hence, maximin values can be attained either with strategy IA or with strategy IB and both are maximin strategies.

Notice that player I does not know which of the minimal values he is going to get when playing a maximin strategy. Notice also that the valuation of each strategy, in general, is not a vector but a set of vectors, what makes difficult the task of comparing them.

Now, we are going to obtain the maximin strategies for the fixed coalition S = {1,3}. The joint strategies for coalition S are:

$$x_S^1 = (IA, IIIA),\ x_S^2 = (IA, IIIB),\ x_S^3 = (IB, IIIA),\ x_S^4 = (IB, IIIB)$$

The worst possible joint payoff for coalition S are:

$$\min_{x_{-S}\in X_{-S}} u_S(x_S^1, x_{-S}) = \min\left\{\begin{pmatrix}-50\\0\\50\\0\end{pmatrix}\begin{pmatrix}-50\\50\\50\\50\end{pmatrix}\right\} = \left\{\begin{pmatrix}-50\\0\\50\\0\end{pmatrix}\begin{pmatrix}-50\\50\\50\\50\end{pmatrix}\right\}$$

$$\min_{x_{-S}\in X_{-S}} u_S(x_S^2, x_{-S}) = \min\left\{\begin{pmatrix}-50\\50\\0\\-100\end{pmatrix}\begin{pmatrix}100\\100\\0\\-50\end{pmatrix}\right\} = \left\{\begin{pmatrix}-50\\50\\0\\-100\end{pmatrix}\begin{pmatrix}100\\100\\0\\-50\end{pmatrix}\right\}$$

$$\min_{x_{-S}\in X_{-S}} u_S(x_S^3, x_{-S}) = \min\left\{\begin{pmatrix}0\\0\\50\\50\end{pmatrix}\begin{pmatrix}0\\0\\100\\-90\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\\50\\50\end{pmatrix}\begin{pmatrix}0\\0\\100\\-90\end{pmatrix}\right\}$$

$$\min_{x_{-S}\in X_{-S}} u_S(x_S^4, x_{-S}) = \min\left\{\begin{pmatrix}0\\50\\0\\-50\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\50\\0\\-50\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}$$

The maximin values are obtained from the comparison of these sets of vectors.

Strictly competitive games are an interesting particular case. In these conditions coalition S plays against the other players that form the coalition -S, maximizing his utility function $u_S(x_S, x_{-S})$. The goal of -S will be to minimize the coalition S utility function. To this end, S will look for a maximin strategy whereas -S will look for a minimax strategy.

We say that $\bar{x}_S\in X_S$ is a "optimal response" strategy for coalition S against the given strategy $\bar{x}_{-S}\in X_{-S}$ of the other players if $\bar{x}_S$ is an efficient solution for the vector maximization problem:

$$\max_{x_S\in X_S} u_S(x_S, \bar{x}_{-S}) \tag{4}$$

Similarly, we say that $\bar{x}_{-S}\in X_{-S}$ is a "optimal response" strategy for coalition -S against the given strategy $\bar{x}_S\in X_S$ of the other players if $\bar{x}_{-S}$ is an efficient solution for the vector minimization problem:

$$\max_{x_{-S}\in X_{-S}} u_S(\bar{x}_S,x_{-S}) \tag{5}$$

The set of all optimal response strategies of a coalition S against the opponent's given strategy $\bar{x}_{-S} \in X_{-S}$ is defined by $R_S(\bar{x}_{-S})$ and the set of all optimal response strategies of a coalition -S against the opponent's given strategy $\bar{x}_S \in X_S$ is defined by $R_{-S}(\bar{x}_S)$. For convenience, we denote each optimal response set as follows:

$$D_1 = \{(x_S, x_{-S}) \in X / x_{-S} \in R_{-S}(x_S),\ x_S \in X_S\}$$
$$D_2 = \{(x_S, x_{-S}) \in X / x_S \in R_S(x_{-S}),\ x_{-S} \in X_{-S}\}.$$

As we defined in definition 3.3, a joint strategy $\bar{x} = (\bar{x}_S, \bar{x}_{-S}) \in X$ is said to be a Nash equilibrium of the game for the partition $\Delta = \{S, -S\}$ if:

$$\bar{x}_S \in R_S(\bar{x}_{-S}) \text{ and } \bar{x}_{-S} \in R_S(\bar{x}_{-S}) \Leftrightarrow \bar{x} \in D_1 \cap D_2 \Leftrightarrow$$
$$u_S(\bar{x}_S, \bar{x}_{-S}) \in \min_{x_{-S} \in X_{-S}} u_S(\bar{x}_S, x_{-S}) \cap \max_{x_S \in X_S} u_S(x_S, \bar{x}_{-S})$$

We call an equilibrium value or saddle value to $u_S(\bar{x}_S, x_{-S})$ where $(\bar{x}_S, \bar{x}_{-S}) \in X$ is an equilibrium strategy or generalized saddle point. For convenience we will denote the set of all saddle values of u_S by $SV(u_S)$.

The set of all maximin values of u_S and the set of all minimax values of u_S respectively are:

$$\max \bigcup_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) \equiv \max u_S\,(D_1)$$
$$\min \bigcup_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} u_S(x_S, x_{-S}) \equiv \min u_S\,(D_2).$$

Also we call a strategy $\bar{x}_S$ (respectively $\bar{x}_{-S}$) attaining a maximin value (respectively a minimax value) a maximin (respectively minmax) strategy.

The following theorems shows that under certain conditions, there exists at least an optimal response strategy for each coalition against and opponent's given strategy, and that there exist a maximin strategy and a minimax strategy. This results is an extension of Lema 5.5. given in [21].

Theorem 3.2 Given a coalition $S \in \mathfrak{N}$, let X_S and X_{-S} be nonempty compact subsets of $\mathbb{R}(S)$ and $\mathbb{R}(-S)$ respectively. If the vector-valued function u_S is continuous then for each $x_S^* \in X_S$ and $x_{-S}^* \in X_{-S}$:

$$\emptyset \neq \min_{x_{-S} \in X_{-S}} u_S(x_S^*, x_{-S}) \subset \max \bigcup_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) - \mathbb{R}(S)_+$$

$$\emptyset \neq \max_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}^*) \subset \min \bigcup_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} u_S(x_S, x_{-S}) + \mathbb{R}(S)_+.$$

Unfortunately, maximin strategies do not always provide equilibrium values, but may give equilibrium values in the sense of security levels when maximin strategies satisfy certain conditions. Thus, we note that the maximin and minimax values are upper and lower bounds of such equilibrium values, respectively.

Corollary 3.1 For a coalition $S \in \mathcal{N}$, if X_S and X_{-S} are non-empty compact subsets of $\mathbb{R}(S)$ and $\mathbb{R}(-S)$ and u_S is continuous then

$$SV(u_S) \subset \max \bigcup_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) - \mathbb{R}(S)_+$$

$$SV(u_S) \subset \min \bigcup_{x_{-S} \in X_S} \max_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) + \mathbb{R}(S)_+.$$

Hence, if the game S against -S has a joint equilibrium strategy $\bar{x} = (\bar{x}_S, \bar{x}_{-S}) \in X$, then there exist:

$$z_2 \in \max \bigcup_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) - \mathbb{R}^{S+}$$

$$z_1 \in \min \bigcup_{x_S \in X_S} \max_{x_{-S} \in X_{-S}} u_S(x_S, x_{-S}) + \mathbb{R}^{S+}$$

such that $u_S(\bar{x}_S, \bar{x}_{-S}) \leq z_2$ and $u_S(\bar{x}_S, \bar{x}_{-S}) \geq z_1$. We will say that the maximin inequality holds if $z_1 \leq z_2$. In [19], [20] and [21] we can find conditions under which this maximin inequality holds.

Example (continued): In the case that player II is interested only in minimizing the coalition $S = \{1,3\}$ joint payoff, he will seek for a minimax strategy.

$$\max_{x_S \in X_S} u_S(x_S, IIA) = \max\left\{\begin{pmatrix} -50 \\ 0 \\ 50 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 50 \\ 50 \end{pmatrix} \begin{pmatrix} -50 \\ 50 \\ 0 \\ -100 \end{pmatrix} \begin{pmatrix} 0 \\ 50 \\ 0 \\ -50 \end{pmatrix}\right\}$$

$$\max_{x_S \in X_S} u_S(x_S, IIB) = \max\left\{\begin{pmatrix} -50 \\ 50 \\ 50 \\ 50 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 100 \\ -90 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \\ 0 \\ -50 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Among these values player II will choose the minima.

3.3 Pareto-optimal security strategies (POSS)

We have seen in section 3.2 that the multicriteria extension of the concepts of maximin strategies is difficult and looses many of its interesting properties. Although, in some cases it is possible to establish the existence of such strategies, it is usually difficult to obtain them and in most cases the values provided are not unique. Thus when it is not possible to obtain maximin solutions, the concept of POSS becomes important in order to solve multicriteria games from a conservative point of view. This concept is independent of the notion of equilibrium, so that the opponents are only taken into account to establish the security levels for one's own payoff.

In this section we look for solutions to the n-person multiobjective game under the point of view of a coalition $S \in \mathcal{N}$. We assume that all the players value the same m criteria. Players in S will use the security level vector defined as follows:

Definition 3.4 The security level vector or guaranteed payoff vector to coalition $S \in \mathcal{N}$ is:

$$V_S : X_S \to \mathbb{R}(S)$$
$$V_S(x_S) = \left\{ \min_{x_{-S} \in X_{-S}} u_j^i(x_S, x_{-S}) : i \in S, \ j = 1, 2, \ldots, m \right\}.$$

For each $i \in S$

$$V_S(x_S)_i = \left\{ \min_{x_{-S} \in X_{-S}} u_j^i(x_S, x_{-S}) : j = 1, 2, \ldots, m \right\} \in \mathbb{R}^m$$

is player i's guaranteed vector payoff, given coalition's choice x_S. Then

$$V_j^i(x_S) = \min_{x_{-S} \in X_{-S}} u_j^i(x_S, x_{-S}) : j = 1, 2, \ldots, m.$$

Coalition $S \in \mathcal{N}$ will try to choose a strategy among all the possible such that the associated security vector be as best as possible:

Definition 3.5 A strategy $\bar{x}_S \in X_S$ is a POSS for coalition S if it is an efficient solution of the vector maximization problem:

$$\max_{x_S \in X_S} V_S(x_S). \tag{6}$$

Problem (6) is equivalent to:

$$\max \left\{ V_1^i(x_S), V_2^i(x_S), \ldots, V_m^i(x_S), i \in S \right\}$$
$$\text{s.t.: } x_S \in X_S$$

By scalarization we will characterize efficient solutions in the multiple objective problem. When payoff functions, u_i, are concave, the problem is convex. In this case we obtain (see [16]) the efficient solutions of the problem by solving the associated nonlinear scalar problem $P(\lambda)$:

$$\max \sum_{\substack{j=1 \\ i \in S}}^{m} \lambda_j^i V_j^i$$
$$\text{s.t. } x_S \in X_S$$

where

$$\lambda \in \Lambda^0 = \left\{ \lambda \in \mathbb{R}(S) / \sum_{\substack{j=1 \\ i \in S}}^{m} \lambda_j^i = 1, \ \lambda_j^i \geq 0, \ j = 1, \ldots, m, \ i \in S \right\}$$

Each component λ_j^i of parameter λ can be seen as the relative importance that the coalition S assigns to the scalar game with scalar payoff function u_j^i.

For strictly competitive games, the security level vector for the opponent coalition -S is:

$$V_{-S} : X_{-S} \to \mathbb{R}(-S)$$
$$V_{-S}(x_{-S}) = \left\{ \max_{x_S \in X_S} u_j^i(x_S, x_{-S}) \ i \in S, \ j = 1, 2, \ldots, m \right\}$$

The set of POSS for coalition -S is the set of efficient solutions of the vector minimization problem:

$$\min_{x_{-S}\in X_{-S}} V_{-S}(x_{-S}). \qquad (7)$$

Notice that if $(\bar{x}_S, \bar{x}_{-S})\in X$ is an equilibrium, then:

$$V_S(\bar{x}_S) \leq u_S(\bar{x}_S, \bar{x}_{-S}) \leq V_{-S}(\bar{x}_{-S})$$

Conversely if ($\bar{x}_S$ and $\bar{x}_{-S}$) are strategies such that $V_S(\bar{x}_S) = V_{-S}(\bar{x}_{-S})$, then $(\bar{x}_S, \bar{x}_{-S})$ is an equilibrium of the game.

In effect $\bar{x}_{-S}$ is a solution of problem (5) because if $\exists x^*_{-S}\in X_{-S}/u_S(\bar{x}_S, x^*_{-S}) \leq u_S(\bar{x}_S, \bar{x}_{-S})$ then:

$$V_S(\bar{x}_S) \leq u_S(\bar{x}_S, x^*_{-S}) \leq u_S(\bar{x}_S, \bar{x}_{-S}) \leq V_{-S}(\bar{x}_{-S}).$$

The inequalities above are really equalities; similarly we can see that $\bar{x}_S$ is a solution of problem (4).

A necessary and sufficient condition for $V_S(\bar{x}_S) = V_{-S}(\bar{x}_{-S})$ is that $\bar{x}_S$ and $\bar{x}_{-S}$ be ideal strategies for coalition S and coalition -S respectively; that is, $\bar{x}_S$ maximizes $V_S(x_S)$ simultaneously in all its components and $\bar{x}_{-S}$ minimizes $V_{-S}(x_{-S})$ simultaneously all its components. This result generalizes theorem 4.1 in [8] because $\forall j = 1, 2,\ldots, m$ each ideal strategy is an equilibrium for the scalar game with payoff function $u_j^i(x)$.

Example (continued):

To obtain the POSS for player I we compute the security level vectors:

$$V_1^1(IA) = \min\{-50,-50,-50,100\} = -50$$

$$V_2^1(IA) = \min\{0,50,50,100\} = 0$$

Then

$$V_1(IA) = \begin{pmatrix} -50 \\ 0 \end{pmatrix}$$

Analogously:

$$V_1(IB) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, in order to make the choice, player I has to compare two vectors.

The security level vectors for player III are:

$$V_3(IIIA) = \begin{pmatrix} 50 \\ -90 \end{pmatrix}$$

$$V_3(IIIB) = \begin{pmatrix} 0 \\ -100 \end{pmatrix}$$

We can also compute the POSS for any coalition. For coalition S = {1,3}, the security level vectors are:

$$V^1_{\{1,3\}_1}(IA,IIIA) = \min\{-50,-50\} = -50$$

$$V^1_{\{1,3\}_2}(IA,IIIA) = \min\{0,50\} = 0$$

Then

$$V^1_{\{1,3\}}(IA,IIIA) = \begin{pmatrix} -50 \\ 0 \end{pmatrix}.$$

Analogously $V^3_{\{1,3\}}(IA,IIIA) = \begin{pmatrix} 50 \\ 0 \end{pmatrix}$ and then $V_{\{1,3\}}(IA,IIIA) = \begin{pmatrix} -50 \\ 0 \\ 50 \\ 0 \end{pmatrix}$.

Similarly:

$$V_{\{1,3\}}(IA,IIIB) = \begin{pmatrix} -50 \\ 50 \\ 0 \\ -100 \end{pmatrix}, V_{\{1,3\}}(IB,IIIA) = \begin{pmatrix} 0 \\ 0 \\ 50 \\ -90 \end{pmatrix}, V_{\{1,3\}}(IB,IIIB) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -50 \end{pmatrix}.$$

Obtaining the optimal strategies in n-person games with multiple objectives is very difficult with any of the proposed solution concepts. Nevertheless, the set of POSS can be obtained imposing additional properties in the payoff functions of the game. This is the case of multiple objective n-person games with multidimensional-matrix payoff.

Let $P^i = \{e^i_1, e^i_2, \ldots, e^i_{L_i}\}$ denote the pure strategies of player i.

Let $A = [A^1, A^2, \ldots, A^n]$ be the payoff function where $A^i = [A^i(1), A^i(2), \ldots, A^i(m)]$ and $A^i(j)$ is a matrix with L_i rows (pure strategies of player i) and $\prod_{\substack{j=1 \\ j\neq i}}^{n} L_j$ columns (pure joint strategies of the players in $N\backslash\{i\}$).

Example (continued):

In this example there are two objective functions. Therefore, the payoff matrices for player 1 are:

$$A^1 = [A^1(1)\ A^1(2)] = \left[\begin{bmatrix} -50 & -50 & -50 & 100 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 50 & 50 & 100 \\ 0 & 0 & 50 & 0 \end{bmatrix} \right].$$

Let $S = \{i_1, i_2, \ldots, i_S\}$ be a coalition of players. The set of pure strategies for the coalition S is given by $P^S = \prod_{i\in S} P^i$ and its payoff function is a vector of matrices $A^S = [A^S(1), A^S(2), \ldots, A^S(m)]$ where each $A^S(j)$ is a matrix with $|P^S|$ rows and $|P^{-S}|$ columns whose elements are the payoffs received in the j-th objective by the players which belong to S:

$$A^S(j) = \left(a^{i_1}_{rt}(j), a^{i_2}_{rt}(j), \ldots, a^{i_{|S|}}_{rt}(j)\right)_{r\in P^S, t\in P^{-S}}$$

The following theorem reduces the computation of POSS, as defined in definition 3.5, to obtain the efficient solutions of a multiobjective linear problem.

Theorem 3.3 The set of POSS strategies and the corresponding security level vector for a coalition $S = \{i_1, i_2, \ldots, i_{|S|}\}$ in a multiple objective n-person game with multidimensional matrix payoff A, is given by the efficient solutions of the problem:

$$\max \quad \left(v_1^{i1}, v_2^{i1}, \dots, v_m^{i1}, \dots, v_1^{i|S|}, v_2^{i|S|}, \dots, v_m^{i|S|}\right)$$

$$\text{s.t.:} \quad x_S(A^S(1), A^S(2), \dots, A^S(m)) \geq \begin{bmatrix} v_1^{i1} \\ v_1^{i2} \\ \dots \\ v_1^{i|S|} \\ \dots \\ v_m^{i1} \\ v_m^{i2} \\ \dots \\ v_m^{i|S|} \end{bmatrix}.$$

$$x_S \in X_S$$

The proof runs analogously to theorem 3.1 in [7] applied to the amalgation of the game induced for the player in N\S (see [3] for the definition of amalgation of games).

Example (continued):

The security level vector and the set of POSS for the coalition $S = \{1\}$ is obtained solving the problem:

$$\begin{aligned} \max \quad & (v_1, v_2) \\ \text{s.t.:} \quad & -50x_1 \geq v_1 \\ & 100x_1 \geq v_1 \\ & 0 \geq v_2 \\ & 50x_1 \geq v_2 \\ & 50x_1 + 50x_2 \geq v_2 \\ & 100x_1 \geq v_2 \\ & x_1 + x_2 = 1 \\ & x_i \geq 0, \; i = 1, 2 \\ & v_i, \quad i = 1, 2. \end{aligned}$$

Using the software package ADBASE (see [18]), the POSS of this problem is:

$$\bar{x}_{\{1\}} = (0, 1), \; V_{\{1\}}(\bar{x}_{\{1\}}) = (0, 0).$$

Therefore, in this example there is a unique POSS of player I which is IB and the security levels are 0 in both payoff functions.

If we consider coalition $S = \{2\}$ the set of POSS is given by the convex hull of the strategies $\bar{x}_{\{2\}} = (1, 0)$ and $\bar{y}_{\{2\}} = (0, 1)$ whose associated security level vectors are $V_{\{2\}}(\bar{x}_{\{2\}}) = (0, -100)$ and $V_{\{2\}}(\bar{y}_{\{2\}}) = (-50, 0)$ respectively. Analogously, for the coalition $S = \{3\}$ we obtain the set of POSS as the convex hull of $\bar{x}_{\{3\}} = (13/25, 12/25)$ and $\bar{y}_{\{3\}} = (1, 0)$, and the corresponding security level vectors are $V_{\{3\}}(\bar{x}_{\{3\}}) = (658/25, -1684/25)$ and $V_{\{3\}}(\bar{y}_{\{3\}}) = (-50, -90)$.

Example: Consider a bicriteria three-person game in strategic form. The pure strategy sets for player I, player II and player III are:

$$P^1 = \{IA, IB, IC\}, \; P^2 = \{IIA, IIB, IIC\}, \; P^3 = \{IIIA, IIIB\},$$

respectively. The payoff matrices for this game are given in Table 2.

For the coalition S = {1, 3} the joint pure strategy set is:

$$P^S = \{(IA, IIIA), (IB, IIIA), (IC, IIIA), (IA, IIIB), (IB, IIIB), (IC, IIIB)\}$$

and $P^{-S} = \{IIA, IIB, IIC\}$.

The payoff matrix for the coalition S = {1, 3} is:

$$A^S = [A^S(1), A^S(2)]$$

with

$$A^S(1) = \begin{bmatrix} (1,-1) & (1,-1) & (1,-1) \\ (0,0) & (1,0) & (1,0) \\ (0,0) & (1,0) & (1,0) \\ (-2,1) & (0,1) & (0,1) \\ (0,2) & (-2,2) & (-2,2) \\ (1,-1) & (0,-1) & (0,-1) \end{bmatrix}$$

and

$$A^S(2) = \begin{bmatrix} (0,2) & (2,1) & (0,2) \\ (2,0) & (0,1) & (2,0) \\ (1,-1) & (0,0) & (1,-1) \\ (1,0) & (1,0) & (1,0) \\ (0,2) & (2,-1) & (0,2) \\ (0,0) & (-1,0) & (0,0) \end{bmatrix}$$

	IIIA			IIIB		
	IIA	IIB	IIC	IIA	IIB	IIC
IA	(1,2,-1) (0,1,2)	(1,0,-1) (2,0,1)	(1,1,-1) (0,-1,2)	(-2,0,1) (1,1,0)	(0,0,1) (1,0,0)	(0,-1,1) (1,-1,0)
IB	(0,1,0) (2,0,0)	(1,1,0) (0,1,1)	(1,-1,0) (2,-1,0)	(0,0,2) (0,1,2)	(-2,1,2) (2,0,-1)	(-2,-2,2) (0,0,2)
IC	(0,2,0) (1,0,-1)	(1,0,0) (0,1,0)	(1,2,0) (1,-1,-1)	(1,1,-1) (0,2,0)	(0,-1,-1) (-1,1,0)	(0,2,-1) (0,-1,0)

Table 2 - Payoffs matrices

In order to compute the POSS for the coalition S = {1, 3} and the security level vector associated, we solve the following multiobjective linear problem:

$$\begin{aligned}
\max\ & (v_1^1, v_2^1, v_1^3, v_2^3) \\
\text{s.t.:}\ & x_1 - 2x_4 + x_6 \geq v_1^1 \\
& x_1 + x_2 + x_3 - 2x_5 \geq v_1^1 \\
& 2x_2 + x_3 + x_4 \geq v_2^1 \\
& 2x_1 + x_4 + 2x_5 - x_6 \geq v_2^1 \\
& -x_1 + x_4 + 2x_5 - x_6 \geq v_1^3 \\
& 2x_1 - x_3 + 2x_5 \geq v_2^3 \\
& x_1 + x_2 - x_5 \geq v_2^3 \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\
& x_i \geq 0,\ \ i = 1, 2 \\
& v_j^i \ \ i = 1, 3 \ \ j = 1, 2
\end{aligned}$$

The extreme efficient solutions of this problem lead to the following POSS strategies and security level vectors:

Strategies $\overline{x}_S$	Security Levels $V_S(\overline{x}_S)$
(2/5,2/5,0,0,1/5,0)	(2/5,4/5,0,3/5)
(1/4,1/2,0,0,1/4,0)	(1/4,1,1/4,1/2)
(1/3,1/2,0,0,4/25,0)	(1/3,1,0,2/3)
(3/20,23/50,0,2/25,31/100,0)	(0,1,27/50,31/100)
(0,2/3,0,0,1/3,0)	(0,2/3,2/3,1/3)
(1/4,5/8,0,0,1/8,0)	(1/4,3/4,0,3/4)
(0,3/4,0,0,1/4,0)	(0,1/2,1/2,1/2)
(0,0,0,0,1,0)	(-2,0,2,-1)
(0,0,0,1/2,1/2,0)	(-1,1/2,3/2,-1/2)
(0,2/5,0,1/5,2/5,0)	(-2/5,1,1,0)
(1/2,1/2,0,0,0,0)	(1/2,1,-1/2,1)
(1,0,0,0,0,0)	(1,0,-1,1)

3.4 Core Solution

Security level vectors are used to define the core solution concept. A joint strategy is a core solution if it is stable in the sense that no coalition can guarantee a better result than the payoff obtained by the joint strategy.

Definition 3.6 A joint strategy $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \in X$ is a core solution of the n-person multiobjective game $\Gamma = \{N, X^i, u^i\}$ if:

$$\forall S \in \mathcal{N} : \Omega_S(\overline{x}) = \{x_S \in X_S / V_S(x_S) >> u_S(\overline{x})\} = \varnothing$$

Similarly, a vector $y \in \mathbb{R}(N)$ is a "core vector" (or in the core) of the game Γ if it is feasible and if:

$$\forall S \in \mathfrak{N} : \Phi_S(y) = \{x_S \in X_S / V_S(x_S) >> y_S\} = \emptyset$$

By feasible we mean $\exists x \in X/u(x) \geq y$

With >> we note strictly greater than componentwise.

In the following theorem we shoe a relation between core solution and POSS.

Theorem 3.4 For the game $\Gamma = \{N, X^i, u^i\}$, if the joint strategy $\bar{x}$ is such that $\bar{x}_S$ is POSS $\forall S \subset N$, then $\bar{x}$ is a core solution.

Proof: $\bar{x}_S$ is POSS for $S \subset N$, then there exists no $x_S \in X_S$ such that $v_S(x_S) >> v_S(\bar{x}_S) = \min_{x_{-S} \in X_{-S}} u_S(\bar{x}_S, x_{-S})$. But

$$u_S(\bar{x}) \geq \min_{x_{-S} \in X_{-S}} u_S(\bar{x}_S, x_{-S})$$

then there exists no $x_S \in X_S$ such that $v_S(x_S) >> u_S(\bar{x}_S)$. It follows that $\bar{x}$ is a core solution.

If there exists a fixed partition of the players $\Delta = \{S_1, S_2, \ldots, S_k\}$ then there will be partial cooperation among the members of each coalition and competence between the different coalitions. Then, the partition Δ will induce k parametric multiple objective games:

$$\Gamma_S(\bar{x}_{-S}) = \{S, X^i, u^i(x_S, \bar{x}_{-S})\} \quad \forall S = S_1, S_2, \ldots, S_k. \tag{8}$$

For the fixed parameter $\bar{x}_{-S}$, each $\Gamma_S(\bar{x}_{-S})$ has $|S|$ players with payoff functions $u^i(x_S, \bar{x}_{-S})$ and is simply a new multiple objective game. The concept of hybrid solution can now be given as:

Definition 3.7 For each partition of players $\Delta = \{S_1, S_2, \ldots, S_k\}$ in the n-person multiple objective game $\Gamma = \{N, X^i, u^i\}$, a joint strategy $\bar{x} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\} \in X$ is a core solution corresponding to the partition Δ if $\forall S \in \Delta$, $\bar{x}_S$ is a core solution of the corresponding parametric game: $\Gamma_S(\bar{x}_{-S}) = \{S, X^i, u^i(x_S, \bar{x}_{-S})\}$.

Notice that this definition generalizes the core solution concept given in definition 3.6, because a core solution is a core solution for the trivial partition $\Delta = \{N\}$.

In [23] an existence theorem of core solutions is established for any partition. The proof of this result suggests an algorithm to find a core solution to the multiple objective game based on Kakutani's fixed point theorem in [11].

Theorem 3.5 Given a partition of players $\Delta = \{S_1, S_2, \ldots, S_k\}$ in the multiple objective game $\Gamma = \{N, X^i, u^i\}$, the corresponding core solutions set is non-empty if Γ satisfies:

1. For each player i, X^i is a closed bounded convex subset in $\mathbb{R}^{L_i}$.
2. For each coalition $S \in \Delta$, $u^j(x)$, $j \in S$, are all continuous in $x = (x_S, x_{-S})$ and are quasiconcave in x_S.

4. Conclusions

Nash equilibrium has been the most frequently used concepts in the analysis of n-person non-cooperative games. However, the exhibits the same difficulties observed in bimatrix games both in the computation and use. These inconveniences may be avoided by using different solution concepts taken from the theory of matrix games. In this paper, we reviewed the concepts of Nash equilibria, maximin strategies and Pareto-optimal security strategies in the framework of n-person non-cooperative multiple objective games. New definitions were given and properties showing relationships were stated. Examples were included to show so the difficulty of the calculation of those strategies as their use. Finally, we proved the relationship between POSS and core solutions in a game with a given coalitional structure.

References

[1] Blackwell, O., An Analog of the Minimax Theorem for Vector Payoff, Pacific Journal of Mathematics 6 (1956) 1-8.
[2] Borm, P.E.M, Tijs, S.H. and Aarssen van Den, J.C.M., Pareto Equilibria in Multiobjective Games, in B. Fuchssteiner, T. Lengauer, H.J. Skaka (ed.) Methods of Operations Research 60 (1988) 313-323.
[3] Borm, P.E.M., Garcia-Jurado, I., Potters, J. and Tijs, S.H., An Amalgation of Games, European Journal of Operations Research 89 (1996) 570-580.
[4] Charnes, A., Cooper, W.W., Huang, Z.M. and Wei, Q.T., Multiple-Payoff Constrained n-Person Games over Cones, ccs Research Report 583, Center for Cybernetic Studies, University of Texas at Austin (1987).
[5] Charnes, A., Huang, Z.M., Rousseau, J.J. and Wei, Q.L., Cone Extremal Solutions of Multi-Payoff Games with Cross-Constrained Strategy Set, Optimization 21 (1990) 51-69.
[6] Contini, M.B., Olivetti, I.C. and Milano, C.S., A Decision Model under Uncertainty with Multiple Payoff, In A. Mensch (ed.) Theory of Games (1966) 50-63, New York, American Elsevier Pub. Co.
[7] Fernández, F.R. and Puerto, J., Vector Linear Programming in Zero-Sum Multicriteria Matrix Games, Journal Optimization and Applications 63 (1996) 167-189.
[8] Goshe, D. and Prassard, R., Solution Concept in Two-Person Multicriteria Games, Journal Optimization Theory and Applications 63 (1989) 167-189.
[9] Goshe, D., A Necessary and Sufficient Condition for Pareto-Optimal Security Strategies in Multicriteria Matrix Games, Journal Optimization Theory and Applications 68 (1991) 463-480.
[10] Hannan, E.L., On Games with Multiple Payoff, International Journal of Game Theory 11 (1982) 13-15.
[11] Kakutani, S., A Generalization of Brouwer's Fixed Point, Duke Mathematical Journal 8 (1941).
[12] Li, Z.F. and Wang, Y., Lagrange Multipliers and Saddle Points in Multiobjective Programming, Journal Application Mathematical Optimization 83 (1994) 63-81.
[13] Nash, J.F. Jr., Equilibrium in n-Person Games, Academy of Science 36 (1950) 48-49.
[14] Newumann von J. and Morgenstern, O., Theory of Games and Economics Behavior, University Press, Princeton, New Jersey (1944).
[15] Nieuwenhuis, J.W., Some Minimax Theorems in Vector-Valued Functions, Journal Optimization Theory and Applications 40 (1983) 463-475.
[16] Puerto, J. and Fernández, F.E., Solution Concepts Based on Security Levels in Constraint Multicriteria Concave-Convex Games, Opsearch 32 (1994) 16-30.
[17] Shapley, L.S., Equilibrium Points in Games with Vector Payoff, Naval Research Logistics Quarterly 6 (1959) 57-61.
[18] Steuer, R.E., Manual for the ADBASE, Multiple Objective Linear Programming Package, University of Georgia, Athens, Georgia (1995).
[19] Tanaka, T., Two Types of Minimax Theorems for Vector-Valued Functions, Journal Optimization Theory and Applications 68 (1991) 321-334.
[20] Tanaka, T., Minimax Theorems in Vector Optimization, Ph. D. Thesis, Niigata University, Niigata (1992).
[21] Tanaka, T., Generalized Quasiconvexities, Cone Saddle Points, and Minimax Theorem for Vector-V alued Functions, Journal Optimization Theory and Applications 81 (1994) 355-377.
[22] Zeleny, M., Games with Multiple Payoff, International Journal of Game Theory 4 (1975) 179-191.
[23] Zhao, J., The Equilibria of a Multiple Objective Game, International Journal of Game Theory 20 (1991) 171-182.